Commun. Korean Math. Soc. 21 (2006), No. 3, pp. 515–526

VARIOUS INVERSE SHADOWING IN LINEAR DYNAMICAL SYSTEMS

TAEYOUNG CHOI AND KEONHEE LEE

ABSTRACT. In this paper, we give a characterization of hyperbolic linear dynamical systems via the notions of various inverse shadowing. More precisely it is proved that for a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, f has the $\mathcal{T}_h$-inverse($\mathcal{T}_h$-orbital inverse or $\mathcal{T}_h$-weak inverse) shadowing property if and only if the matrix A is hyperbolic.

1. Introduction

Consider a dynamical system generated by a homeomorphism f of a metric space X with a metric d. For a point $x \in X$, we denote by $O(x, f)$ its orbit in the system f; i.e., the set

$$O(x, f) = \{f^n(x) : n \in \mathbb{Z}\}.$$

We say that a sequence $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is a *δ-pseudo orbit* of f if the inequalities

$$d(f(x_n), x_{n+1}) < \delta, \ n \in \mathbb{Z}$$

hold. A δ-pseudo orbit is a natural model of computer output in a process of numerical investigation of the system f. In this case, the value δ measures errors of the method, round-off errors, etc.

Recall that f has the *shadowing property* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ we can find a point $y \in X$ with the property

$$d(f^n(y), x_n) < \varepsilon, \ n \in \mathbb{Z}.$$

Received October 5, 2005.

2000 Mathematics Subject Classification: Primary 37C50; Secondary 37D20.

Key words and phrases: inverse shadowing, weak inverse shadowing, orbital inverse shadowing, hyperbolicity.

This work was supported by the KRF Grant funded by the Korean Government (MOEHRD) (KRF-2005-070-C00015).

Of course, if f has the shadowing property formulated above, then the results of its numerical study with a proper accuracy reflect its qualitative structure.

Let $N(\varepsilon, A)$ be the ε-neighborhood of A. It is said that f has the *weak shadowing property* [resp. *orbital shadowing property*] if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_n\}$ of f we can find a point $y \in X$ with the property

$$\xi \subset N(\varepsilon, O(y, f)) \text{ [resp. } \xi \subset N(\varepsilon, O(y, f)) \text{ and } O(y, f) \subset N(\varepsilon, \xi)],$$

where d_H denotes the Hausdorff distance on the set of compact subsets of X. The weak shadowing property was introduced in [12] and the orbital shadowing property was introduced in [11].

Let $X^{\mathbb{Z}}$ be the space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a *δ-method* for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ denotes the 0th component of $\varphi(x)$. Then each $\varphi(x)$ is a δ-pseudo orbit of f through x. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. Say that φ is a *continuous δ-method* for f if the map φ is continuous. The set of all δ-methods [resp. continuous δ-methods] for f will be denoted by $\mathcal{T}_0(f, \delta)$ [resp. $\mathcal{T}_c(f, \delta)$]. If $g : X \to X$ is a homeomorphism with $d_\infty(f, g) < \delta$, where $d_\infty(f, g) = \sup_{x\in X}\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\}$, then g induces a continuous δ-method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in Z(X)$ with $d_\infty(f, g) < \delta$. We define $\mathcal{T}_\alpha(f)$ by

$$\mathcal{T}_\alpha(f) = \bigcup_{\delta>0} \mathcal{T}_\alpha(f, \delta),$$

where $\alpha = 0, c, h$. Clearly,

$$\mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

The concept of inverse shadowing for homeomorphisms as a "dual" notion of shadowing property was established by Corless and Pilyugin [2], and Kloeden *et al* [4, 5] redefined this property using the concept of a method. We say that f has the *$\mathcal{T}_\alpha$-inverse shadowing property*, for short IS_α, ($\alpha = 0, c, h$), if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f, \delta)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$d(f^n(x), \varphi(y)_n) < \varepsilon, \ n \in \mathbb{Z}.$$

Clearly we have the following relations among the various notions of inverse shadowing

$$\mathrm{IS}_0 \Rightarrow \mathrm{IS}_c \Rightarrow \mathrm{IS}_h.$$

When we study the inverse shadowing property in the qualitative theory of differentiable dynamical systems, an appropriate choice of the class of admissible pseudo orbits is crucial here ([2, 3, 5, 6, 10]). Moreover the inverse shadowing properties are not related to the shadowing property in general.

EXAMPLE 1.1. [7] Consider the dynamical system f on the unit circle S^1 with coordinate $x \in [0,1)$, given by

$$f(x) = x + \frac{1}{2\pi}\sin(2\pi x).$$

Then it has the shadowing property. Therefore it has the $\mathcal{T}_c$-inverse shadowing property. But it does not have the $\mathcal{T}_0$-inverse shadowing property.

EXAMPLE 1.2. [8] Pseudo-Anosov maps on a compact surface have the $\mathcal{T}_h$-inverse shadowing property but it does not have the shadowing property.

EXAMPLE 1.3. [4] Let $\{0,1\}^{\mathbb{Z}}$ be the space of all two sided sequences $\mathbf{x} = \{\mathbf{x}_i; n \in \mathbb{Z}\}$ with elements $\mathbf{x}_i \in \{0,1\}$, endowed with a metric D defined by

$$D(\mathbf{x},\mathbf{y}) = \sup_{i\in\mathbb{Z}}\left\{\frac{|\mathbf{x}_i - \mathbf{y}_i|}{2^{|i|}}\right\},$$

where $\mathbf{x},\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$. We also write this space as $\sum_2$ to shorten the notation. Define a shift map $\sigma : \sum_2 \longrightarrow \sum_2$ by

$$\sigma(\mathbf{x})_i = \mathbf{x}_{i+1} \quad (i \in \mathbb{Z}),$$

where $\mathbf{x} \in \sum_2$. Then the shift homeomorphism σ is an expansive homeomorphism with the shadowing property, but it does not have the $\mathcal{T}_h$-inverse shadowing property.

Now we introduce the notion of weak [resp. orbital] inverse shadowing which is a "dual" notion of weak [resp. orbital] shadowing.

DEFINITION 1.4. We say that f has the *$\mathcal{T}_\alpha$-weak* [resp. *$\mathcal{T}_\alpha$-orbital*] *inverse shadowing property*, for short WIS_α [resp. OIS_α], $(\alpha = 0, c, h)$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f,\delta)$ and any point $x \in M$ there is a point $y \in M$ for which

$$\varphi(y) \subset N(\varepsilon, O(x,f)) \text{ [resp. } \xi \subset N(\varepsilon, O(y,f)) \text{ and } O(y,f) \subset N(\varepsilon,\xi)].$$

Clearly we have the following relations

$$\mathrm{WIS}_0 \Rightarrow \mathrm{WIS}_c \Rightarrow \mathrm{WIS}_h, \quad \mathrm{OIS}_0 \Rightarrow \mathrm{OIS}_c \Rightarrow \mathrm{OIS}_h,$$

and

$$\mathrm{ISP}_\alpha \Rightarrow \mathrm{OIS}_\alpha \Rightarrow \mathrm{WIS}_\alpha \quad (\alpha = 0, c, h).$$

REMARK 1.5. Suppose that $\mathcal{T}_a(f) \subset \mathcal{T}_b(f)$ for $a, b \in \{0, c, h\}$. If f has the $\mathcal{T}_b$-weak [resp. $\mathcal{T}_b$-orbital] inverse shadowing property then it has the $\mathcal{T}_a$-weak [resp. $\mathcal{T}_a$-orbital] inverse shadowing property. We can easily show that every irrational rotation f on the unit circle S^1 has the $\mathcal{T}_c$-weak (or $\mathcal{T}_h$-inverse) inverse shadowing property, but it does not have the $\mathcal{T}_c$-inverse (or $\mathcal{T}_h$-inverse) shadowing property. Furthermore we can show that every rational rotation on the unit circle has the $\mathcal{T}_c$-orbital inverse shadowing property, but it does not have the $\mathcal{T}_c$-weak inverse shadowing property. It can be checked that every shift homeomorphism does not have the $\mathcal{T}_c$-weak inverse shadowing property. Moreover Choi *et al.* [1] showed that the $\mathcal{T}_h$-weak inverse shadowing property is generic in the space of homeomorphisms on a compact metric space with the C^0 topology.

2. Main theorem

Let A be a nonsingular matrix on $\mathbb{C}^n$. We consider the dynamical system $f(x) = Ax$ of $\mathbb{C}^n$. We say that the matrix A is called *hyperbolic* if the spectrum does not intersect the circle $\{\lambda : |\lambda| = 1\}$.

LEMMA 2.1. *Let (X, d) be a metric space. Assume that for two dynamical systems f and g on X there exists a homeomorphism h on X such that h and h^{-1} are Lipschitz, and $f \circ h = h \circ g$. Then f has the $\mathcal{T}_h$-weak inverse shadowing property [resp. $\mathcal{T}_h$-inverse shadowing property] if and only if g has the $\mathcal{T}_h$-weak inverse shadowing property [resp. $\mathcal{T}_h$-inverse shadowing property].*

PROOF. We prove the lemma only for the case of the $\mathcal{T}_h$-weak inverse shadowing property.

Assume that f has the $\mathcal{T}_h$-weak inverse shadowing property, and let $\varepsilon > 0$ be arbitrary. Find $\varepsilon_1 > 0$ such that the inequality $d(x, y) < \varepsilon_1$, $x, y \in X$, implies that $d(h^{-1}(x), h^{-1}(y)) < \varepsilon$. Take $\delta_1 > 0$ corresponding to ε_1 by the assumption of the $\mathcal{T}_h$-inverse shadowing property of f, and choose $\delta > 0$ such that $d(x, y) < \delta$ implies $d(h(x), h(y)) < \delta_1$.

Let $\widetilde{g}$ be a δ-perturbation of g, i.e., $d_\infty(\widetilde{g}, g) < \delta$, and let $x \in X$. Put $\widetilde{f} = h \circ \widetilde{g} \circ h^{-1}$. Then $d_\infty(h \circ \widetilde{g} \circ h^{-1}, h \circ g \circ h^{-1}) = d_\infty(\widetilde{f}, f) < \delta_1$. By the $\mathcal{T}_h$-inverse shadowing property of f, for the given $h(x)$, there exists a point $y \in X$ such that for any $k \in \mathbb{Z}$, we choose $n(k) \in \mathbb{Z}$ satisfying the inequality

$$d(\widetilde{f}^{\,n(k)}(y), f^k(h(x)) < \varepsilon_1.$$

Here we know that $f \circ h = h \circ g$ implies

$$h^{-1} \circ f^k = g^k \circ h^{-1} \quad \text{and} \quad h^{-1} \circ \widetilde{f}^k = \widetilde{g}^k \circ h^{-1} \quad \text{for any } k \in \mathbb{Z}.$$

This shows that for any $k \in \mathbb{Z}$, we can choose $n(k) \in \mathbb{Z}$ satisfying the inequality

$$d(\widetilde{g}^{\,n(k)}(h^{-1}(y)), g^k(x)) < \varepsilon, \quad k \in \mathbb{Z}.$$

This means that g has the $\mathcal{T}_h$-weak inverse shadowing property. □

LEMMA 2.2. *Let (X, d) be a metric space. If the dynamical system $f^m(x) = A^m x$ ($m \in \mathbb{N}$) on X has the $\mathcal{T}_h$-weak inverse shadowing property, then the dynamical system $f(x) = Ax$ on X has the $\mathcal{T}_h$-weak inverse shadowing property.*

PROOF. Assume that the dynamical system f^m has the $\mathcal{T}_h$-weak inverse shadowing property. Let $\varepsilon > 0$ be arbitrary and L be a Lipschitz constant of f. Take $0 < \varepsilon_1 < \min\{\frac{\varepsilon}{L^i \cdot m} \mid 1 \le i \le m\}$ such that

$$d(x, y) < \varepsilon_1 \;\Rightarrow\; d(f^i(x), f^i(y)) < \frac{\varepsilon}{m} \quad (1 \le i \le m).$$

Choose $\delta_1 > 0$ corresponding to ε_1 by the assumption of the $\mathcal{T}_h$-weak inverse shadowing property of f^m. Now we find $0 < \delta < \min\{\frac{\delta_1}{m}, \varepsilon_1\}$ such that

$$d_\infty(g, f) < \delta \;\Rightarrow\; d_\infty(g^i, f^i) < \frac{\delta_1}{m} \quad (1 \le i \le m).$$

Let g be a δ-perturbation of f, i.e., $d_\infty(g, f) < \delta$, and let $x \in X$. Then g^m be a δ_1-perturbation of f^m. By the $\mathcal{T}_h$-weak inverse shadowing property of f^m, there exists $y \in X$ such that for any $k \in \mathbb{Z}$, we choose $n(k) \in \mathbb{Z}$ satisfying the inequality

$$d((f^m)^{n(k)}(x), (g^m)^k(y)) < \varepsilon_1.$$

Then for any $k \in \mathbb{Z}$ and $0 \le j \le m$,

$$d(f^{m \cdot n(k)+j}(x), g^{m \cdot k+j}(y)) < \varepsilon.$$

Hence we can easily show that for any $l \in \mathbb{Z}$, we choose $t(l) \in \mathbb{Z}$ satisfying the inequalities

$$d(f^{t(l)}(x), g^l(y)) < \varepsilon, \quad l \in \mathbb{Z}.$$

This means that f has the $\mathcal{T}_h$-weak inverse shadowing property. $\square$

LEMMA 2.3. [9] *Let A be a hyperbolic matrix on $\mathbb{C}^n$. Then there exists $C > 0$, a natural number m, $0 < \lambda < 1$, invariant linear subspaces $S(p)$ and $U(p)$ of $T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$ such that*

1. $T_p\mathbb{C}^n = S(p) \oplus U(p)$;
2. $|A^{mk}(v)| < C\lambda^k|v|, \quad v \in S(p),\ k \geq 0$;
3. $|A^{-mk}(v)| < C\lambda^{-k}|v|, \quad v \in U(p),\ k < 0$;
4. *If $P(p)$ and $Q(p)$ are the projectors in $T_p\mathbb{C}^n$ onto $S(p)$ parallel to $U(p)$ and onto $U(p)$ parallel to $S(P)$ with the property $P(p) + Q(p) = I(p)$, then*

$$||P(p)|| \text{ and } ||Q(p)|| \leq C.$$

LEMMA 2.4. [9] *Let A be a non-hyperbolic matrix, and λ be an eigenvalue of A with $|\lambda| = 1$. Then there exists a nonsingular matrix T such that $J = T^{-1}AT$ is a Jordan form of A and the matrix J has the form*

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is the nonsingular $m \times m$ complex matrix with the form

$$\begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda \end{pmatrix}$$

LEMMA 2.5. [Schaduer-Tychonoff Theorem] *Let Λ be a closed, convex set in a Banach space and $f : \Lambda \to \Lambda$ a continuous function. If $\overline{f(\Lambda)}$ is compact, then f has a fixed point.*

THEOREM 2.6. *For a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the following conditions are mutually equivalent:*

1. *f has the $\mathcal{T}_h$-inverse shadowing property,*
2. *f has the $\mathcal{T}_h$-orbital inverse shadowing property,*
3. *f has the $\mathcal{T}_h$-weak inverse shadowing property,*
4. *The matrix A is hyperbolic.*

PROOF. By the definition, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. We prove that (3) $\Rightarrow$ (4) and that (4) $\Rightarrow$ (1).

Proof of (3) $\Rightarrow$ (4) : Assume that f has the $\mathcal{T}_h$-weak inverse shadowing property. To obtain a contradiction, assume that the matrix A has an eigenvalue λ such that $|\lambda|$=1. Lemma 2.4 shows that there exists a nonsingular matrix T such that $J = T^{-1}AT$ is a Jordan form of A and the matrix J has the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is the nonsingular $m \times m$ complex matrix with the form

$$\begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda \end{pmatrix}$$

Then, for the dynamical system $g(x) = J(x)$ and the homeomorphism $h(x) = T(x)$, the equality $f \circ h = h \circ g$ holds. Since the homeomorphisms h and h^{-1} are Lipschitz in $\mathbb{C}^n$, Lemma 2.1 implies that g has the $\mathcal{T}_h$-weak inverse shadowing property. Let $\delta > 0$ corresponding to $\varepsilon = 1$ by the definition of the $\mathcal{T}_h$-weak inverse shadowing property of g. Denote by x_i the i-th component of a vector $x \in \mathbb{C}^n$. We fix a point $w \in \mathbb{C}^n$ with $|w_1| = 3$ and construct a δ-perturbation $\widetilde{g}$ of g as follows :

$$\widetilde{g}(x_1, \dots, x_n) = \left(\lambda x_1\left(1 + \frac{\delta}{2|x_1|}\right), (Jx)_2, \dots, (Jx)_n\right).$$

Let $y = (y_1, \dots, y_n)$ be an arbitrary vector in $\mathbb{C}^n$. Since for $k \to \infty$, $(\widetilde{g}(y)_1)^k$ leaves on the 1-neighborhood of $S_3 = \{x_1 \in \mathbb{C} : |x_1| = 3\}$, there exists $k(y) \in \mathbb{N}$ such that $(\widetilde{g}(y)_1)^{k(y)}$ leaves on 1-neighborhood of S_3. This means that $\widetilde{g}^{k(y)}(y)$ leaves on 1-neighborhood of $O(w, g)$. Hence we show that g does not have the $\mathcal{T}_h$-weak inverse shadowing property, and so the contradiction completes the proof.

Proof of (4) $\Rightarrow$ (1) : Assume that the matrix A is hyperbolic. It suffices to show that $f(x) = Ax$ has the Lipschitz $\mathcal{T}_h$-inverse shadowing property, i.e., there exist positive numbers δ_0 and L such that for if g is a δ-perturbation of f with $\delta < \delta_0$, then for any $p \in \mathbb{C}^n$ there exists a point $x_0 \in \mathbb{C}^n$ satisfying the inequalities

$$|g^k(x_0) - f^k(p)| < L\delta, \quad k \in \mathbb{Z}.$$

Denote by $S(p)$ the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues λ_j of A such that $|\lambda_j| < 1$, and by $U(p)$ the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues λ_j of A such that $|\lambda_j| > 1$. By Lemma 2.3, there exist $C > 0$, a natural number m,

$0 < \lambda < 1$, invariant linear subspaces $S(p)$ and $U(p)$ of $T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$ such that

(a1) $T_p\mathbb{C}^n = S(p) \oplus U(p)$;
(a2) $|A^{mk}(v)| < C\lambda^k|v|, \quad v \in S(p),\ k \geq 0$;
(a3) $|A^{-mk}(v)| < C\lambda^{-k}|v|, \quad v \in U(p),\ k < 0$;
(a4) If $P(p)$ and $Q(p)$ are the projectors in $T_p\mathbb{C}^n$ onto $S(p)$ parallel to $U(p)$ and onto $U(p)$ parallel to $S(P)$ with the property $P(p) + Q(p) = I(p)$, then

$$||P(p)||\ ,\ ||Q(p)|| \leq C.$$

By Lemma 2.2, it is enough to show that $f^m(x) = A^m(x)$ has the $\mathcal{T}_h$-inverse shadowing property. To simplify the notations, we assume that the inequalities (a2) and (a3) hold with $m \doteq 1$ (another possibility holds similarly.)

Fix a point $p \in \mathbb{C}^n$ and identify the tangent space $T_p\mathbb{C}^n$ with the linear space of $\mathbb{C}^n$. For a point $x \in \mathbb{C}^n$, we define a mapping $a_p : \mathbb{C}^n \to T_p\mathbb{C}^n$ by $a_p(x) = (x-p)_p$. It is easy to see that the following statements hold:

(b1) the mapping $a_p : \mathbb{C}^n \to T_p\mathbb{C}^n$ is continuous ;
(b2) $|a_p(x) - a_p(y)| \leq |x - y|$ for $x, y \in \mathbb{C}^n$;
(b3) there exists a positive number r'(independent of p) such that a_p is a diffeomorphism of the set

$$B_{r'}(p) = \{x \in \mathbb{C}^n : |x - p| < r'\}$$

onto its image for which $Da_p(p) = I$ and

$$|a_p^{-1}(v) - a_p^{-1}(v')| \leq 2|v - v'| \text{ for } v, v' \in a_p(B_{r'}(p)). \tag{2.1}$$

In formula (2.1) and below, for $v \in a_p(B_{r'}(p))$, we denote by $a_p^{-1}(v)$ the unique point $x \in B_{r'}(p)$ such that $a_p(x) = v$.

Take

$$L = 4L_0 + 1,$$

where $L_0 = C^2\frac{1+\lambda}{1-\lambda}$. For $r > 0$, denote $W_r(p) = \{v \in T_p\mathbb{C}^n : |v| \leq r\}$. It is easy to see that we can choose a positive number $r < r'$ (where r' is from the property (b3) of the mappings a_p) such that, for any $p \in \mathbb{C}^n$, the inclusions $W_r(p) \subset a_p(B_{r'}(p))$ hold, hence the mappings

$$F_p = a_{f(p)} \circ f \circ a_p^{-1}$$

are defined on $W_r(p)$. We assume that, for the chosen r, any mapping F_p can be represented as

$$F_p(v) = A(v) + G(v), \tag{2.2}$$

where

$$(2.3) \qquad |G(v)| \le \frac{1}{2L_0} \quad \text{for} \quad v \in W_r(p).$$

We take

$$\delta < \delta_0 = \frac{r}{2L_0}$$

and fix a δ-perturbation g of f, i.e., $d_\infty(g, f) < \delta$, and $p \in \mathbb{C}^n$. We denote $p_k = f^k(p)$ and $g_k = g$. We introduce the following mappings defined for $v \in W_r(p_k)$; G_k are the mappings in the representation (2.2) for the points p_k,

$$\Phi_k = a_{p_{k+1}} \circ f \circ a_{p_k}^{-1} \quad \text{and} \quad \Psi_k = a_{p_{k+1}} \circ g_k \circ a_{p_k}^{-1}.$$

Let E be the space of sequences

$$V = \{v_k \in T_{p_k}\mathbb{C}^n : k \in \mathbb{Z}\}$$

such that $||V||_\infty = \sup_{|k|<\infty}|v_k| \le 2L_0\delta$.

For a natural number m, we introduce the space E_m of sequences

$$V = \{v_k \in T_{p_k}\mathbb{C}^n : |k| \le m\}$$

with the norm

$$||V||_m = \max_{|k| \le m}|v_k| \le 2L_0\delta.$$

Denote by π_m and $\pi_m^l, m \le l$, the natural projectors of E to E_m and of E_l to E_m, respectively. For a sequence $V \in E$, let $Z(V) = \{z_k(V)\}$, where

$$z_{k+1}(V) = G_k(v_k) + \Psi_k(v_k) - \Phi_k(v_k).$$

Since $|f(x) - g_k(x)| < \delta$ for all x and k, and $v_k \in W_r(p_k)$ by the definition of the space E and by our choice of δ, it follows from (b2) and (2.3) that

$$(2.4) \qquad ||Z(V)||_\infty < \frac{1}{2L_0}||V||_\infty + d.$$

Define an operator R on the space E as follows : $R(V) = \{w_k\}$, where

$$(2.5) \qquad w_k = \sum_{i=-\infty}^{k} A^{k-i}(p_i)P(p_i)z_i(V) - \sum_{i=k+1}^{\infty} A^{k-i}(p_i)Q(p_i)z_i(V).$$

The inequalities (a2)-(a4) show that

$$||R(V)||_\infty \le L_0||Z(V)||_\infty,$$

hence it follows from (2.4) that R maps E into itself.

Now it suffices to show that the operator R has a fixed point in E. Consider the space E with the topology of uniform convergence on compact subsets of Z. For a natural number m, we define the operator $R_m : E \to E_m$ by

$$R_m(V) = \{w_k : |k| \le m\},$$

where

$$w_k = \sum_{i=-m}^{k} A^{k-i} P(p_i) z_i(V) - \sum_{i=k+1}^{m} A^{k-i} Q(p_i) z_i(V).$$

Since the values $z_k(V), |k| \le m$, are determined by the values $v_k, |k| \le m+1$, each operator R_m is continuous.

The operator $\pi_m R$ maps a sequence $V \in E$ to the sequence $\{w_k : |k| \le m\}$, where the w_k are given by formula (2.5). Fix a number $l > m$ and consider the operator $\pi_m^l R_l$ mapping a sequence $V \in E$ to the sequence $\{w'_k : |k| \le m\}$, where

$$w'_k = \sum_{i=-l}^{k} A^{k-i} P(p_i) z_i(V) - \sum_{i=k+1}^{l} A^{k-i} Q(p_i) z_i(V).$$

Let us estimate

$$\begin{aligned} & ||\pi_m R(V) - \pi_m^l R_l(V)||_m \\ &= \max\nolimits_{|k|\le m} |w_k - w'_k| \\ &\le 2L_0 C^2 d \max\nolimits_{|k|\le m} \left(\sum_{i=-\infty}^{-l-1} \lambda^{k-i} + \sum_{i=l+1}^{\infty} \lambda^{i-k} \right) \\ &\le \frac{4L_0 C^2 d \lambda^{1-m}}{1-\lambda} \lambda^l . \end{aligned}$$

This estimate implies that the operator $\pi_m R$ is the uniform limit (as $l \to \infty$) of the continuous operators $\pi_m^l R_l$, hence the operator $\pi_m R$ is continuous. It follows from our choice of topology of the space E that the operator R is continuous. It is easy to see that the image $R(E)$ is relatively compact in E. Since R maps E into itself, Lemma 2.5 implies the existence of a fixed point of R in E.

If $V = R(V)$ for some $V \in E$, then

$$\begin{aligned} v_{k+1} &= Av_k + z_{k+1}(V) \\ &= Av_k + G_k(v_k) + \Psi_k(v_k) - \Phi_k(v_k), \end{aligned}$$

i.e., $v_{k+1} = \Psi_k(v_k)$. This means that, for the sequence of points $\{x_k = a_{p_k}^{-1}(v_k)\}$, the equalities $x_{k+1} = g_k(x_k)$ hold. The inclusion $V \in E$ and the property (b3) of the mappings a_p imply the inequalities

$$|g^k(x_0) - f^k(p)| = |x_k - p_k| = |a_{p_k}^{-1}(v_k) - a_{p_k}^{-1}(0_{p_k})| \\ \leq |v_k - 0_{p_k}| \leq 4L_0\delta < L\delta.$$

Therefore, f has the Lipschitz $\mathcal{T}_h$-inverse shadowing property, and so completes the proof. □

REMARK 2.7. Remark 2.1 in [11] and Theorem 3.2.1 in [9] say that, for a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the following conditions are mutually equivalent:

1. f has the shadowing property,
2. f has the orbital shadowing property,
3. f has the weak shadowing property,
4. the matrix A is hyperbolic.

References

[1] T. Choi, S. Kim and K. Lee, *Weak inverse shadowing and genericity*, Bull. Korean Math. Soc. **43** (2006), 43–52.

[2] R. Corless and S. Pilyugin, *Approximate and real trajectories for generic dynamical systems*, J. Math. Anal. Appl. **189** (1995), 409–423.

[3] P. Diamond, K. Lee and Y. Han, *Bishadowing and hyperbolicity*, International Journal of Bifurcations and Chaos **12** (2002), 1779–1788.

[4] P. Kloeden and J. Ombach, *Hyperbolic homeomorphisms and bishadowing*, Ann. Pol. Math **45**, (1997), 171–177.

[5] P. Kloeden, J. Ombach and A. Porkrovskii, *Continuous and inverse shadowing*, Funct. Diff. Equ. **6** (1999), 137–153.

[6] K. Lee, *Continuous inverse shadowing and hyperbolicity*, Bull. Austral. Math. Soc. **67** (2003), 15–26.

[7] K. Lee and J. Park, *Inverse shadowing of circle maps*, Bull. Austral. Math. Soc. **69** (2004), 353–359.

[8] J. Lewowicz, *Persistence in expansive systems*, Ergodic Theory Dynam. Systems **3** (1983), 567–578.

[9] S. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Math. **1706**, Springer-Verlag, Berlin, 1999, 182–185.

[10] ———, *Inverse shadowing by continuous methods*, Discrete and Continuous Dynamical Systems **8** (2002), 29–38.

[11] S. Pilyugin, A. Rodionova and K. Sakai, *Orbital and weak shadowing properties*, Discrete and Continuous Dynamical Systems **9** (2003), 287–308.

[12] O. B. Plamenevskaya, *Weak shadowing for two-dimensional diffeomorphisms*, Vestnik St. Petersburg Univ. Math. **31** (1998), 49–56.

Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail: shadowcty@hanmail.net
khlee@math.cnu.ac.kr