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REMARKS ON CENTERED-LINDELÖF SPACES

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ABSTRACT. In this paper, we construct an example of a normal centered-Lindelöf space X such that $St\text{-}l(X) \geq \omega_1$ under $2^{\aleph_0} = 2^{\aleph_1}$.

1. Introduction

By a space, we mean a topological space. The purpose of this paper is to give an example stated in the abstract. In the rest of this section, we give definitions of terms which are used in the example. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $B \subseteq X$, let $St(B,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$.

DEFINITION 1.1 ([1, 2, 3]). A space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$.

In [2], a star-Lindelöf space is called $*$ Lindelöf and in [3], a star-Lindelöf space is called strongly star-Lindelöf.

Clearly, all Lindelöf spaces, all separable spaces and all countably compact spaces are star-Lindelöf (see [7]).

Recall that a family $\mathcal{U}$ of subsets of a space X is *centered* provided every finite subfamily of $\mathcal{U}$ has nonempty intersection, and $\mathcal{A}$ is called *σ-centered* if it is the union of countably many centered subfamilies.

DEFINITION 1.2 ([1, 8]). A space X is *centered-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a σ-centered subfamily $\mathcal{A}$ of $\mathcal{U}$ such that $\cup\mathcal{A} = X$.

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It is clear that every star-Lindelöf space is centered-Lindelöf. As a natural generalization of star-Lindelöfness, one can consider the following cardinal function:

DEFINITION 1.3 ([1, 7]). The *star-Lindelöf number* of the space X is

$$St\text{-}l(X) = \min\{\kappa : \forall \text{ open cover } \mathcal{U} \text{ of } X,\ \exists F \subseteq X \text{ such that } |F| \leq \kappa \text{ and } St(F,\mathcal{U}) = X\}.$$

Song [9] constructed an example of a Tychonoff centered-Lindelöf space X such that $St\text{-}l(X) \geq \kappa$ for every regular uncountable cardinal κ, which answered Bonanzinga and Matveev [1, Questions 2 and 3] and Matveev [7, Question 47]. Mat-veev [8] showed that $St\text{-}l(X) \leq \mathfrak{c}$ for a normal centered-Lindelöf space X, which answered Bonanzinga and Matveev [1, Question 2] and Matveev [7, Question 47]. Thus, it is natural for us to consider the following question:

QUESTION. *Is there a normal centered-Lindelöf space X such that $St\text{-}l(X) = \mathfrak{c}$?*

The purpose of this note is to construct an example of a normal centered-Lindelöf space X such that $St\text{-}l(X) \geq \omega_1$ under $2^{\aleph_0} = 2^{\aleph_1}$, which gives a partial answer to the above question.

Thorough this paper, the cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols that we do not define will be used as in [4].

2. An example on centered-Lindelöf spaces

In this section, we construct an example stated in the abstract by using the well-known example. But we included the original construction here for the sake of completeness. Recall that a family $\mathcal{U}$ of subsets of the set X is *independent* if for any collection $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_m$ of distinct elements of $\mathcal{U}$, $(\bigcap_{i\leq n} U_i) \cap (\bigcap_{j\leq m}(X \setminus V_j)) \neq \emptyset$.

EXAMPLE 2.1. (See Example E [10]) Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there is a separable normal T_1 space with an uncountable discrete closed subspace.

CONSTRUCTION. Let L be a set of cardinality $\aleph_1$ disjoint from ω. Let $\{F_\alpha : \alpha < \mathfrak{c}\}$ be an independent family of subsets of ω. Using $2^{\aleph_0} = 2^{\aleph_1}$, we may construct a complement-preserving map f from $\wp(L)$ onto $\{F_\alpha : \alpha < \mathfrak{c}\} \cup \{\omega \setminus F_\alpha : \alpha < \mathfrak{c}\}$. Let $X = L \cup \omega$ with a subbase φ for a topology defined by

(a) if $M \subseteq L$, then $M \cup f(M) \in \varphi$;
(b) if $n \in \omega$, then $\{n\} \in \varphi$;
(c) if $p \in X$, then $X \setminus \{p\} \in \varphi$.

The space X is a separable normal T_1 space with an uncountable discrete closed subspace.

Recall us recall from [5] that a space X is *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F, \mathcal{U}) = X$. It is not difficult to show that every countably compact space is starcompact(see [5]).

EXAMPLE 2.2. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a normal centered-Lindelöf space X such that $St\text{-}l(X) \geq \omega_1$.

PROOF. Let $X = L \cup \omega$ be the same as in the construction of Example 2.1. Let

$$S(X, \omega) = L \cup (\omega_1 \times \omega)$$

and topologize $S(X, \omega)$ as follows: A basic neighborhood of $l \in L$ in $S(X, \omega)$ is a set of the from

$$G_{U,\alpha}(l) = (U \cap L) \cap ((\alpha, \omega_1) \times (U \cap \omega))$$

for a neighborhood U of l in X and for $\alpha < \omega_1$, and a basic neighborhood of $\langle \alpha, x \rangle \in \omega_1 \times \omega$ in $S(X, \omega)$ is a set of the form

$$G_V(\langle \alpha, x \rangle) = V \times \{x\},$$

where V is a neighborhood of α in ω_1.

First, we show that $S(X, \omega)$ is centered-Lindelöf. For this end, let $\mathcal{U}$ be an open cover of $S(X, \omega)$. For each $l \in L$, there exists $U_l \in \mathcal{U}$ such that $l \in U_l$, thus, we can find a $n_l \in \omega$ and $\beta_l < \omega_1$ such that

$$(\beta_l, \omega_1) \times \{n_l\} \subseteq U_l.$$

For each $n \in \omega$, if we put $L_n = \{l : n_l = n\}$ and $\mathcal{V}_n = \{U_l : l \in L_n\}$, then $\mathcal{V}_n$ is centered and $L_n \subseteq \cup \mathcal{V}_n$. If we put $\mathcal{V}' = \cup\{\mathcal{V}_n : n \in \omega\}$, then $\mathcal{V}'$ is σ-centered and $L \subseteq \cup\{\cup \mathcal{V}_n : n \in \omega\}$. On the other hand, for each

$n \in \omega$, since $\omega_1 \times \{n\}$ is countably compact, there exists a finite subset $F_n \subseteq \omega_1 \times \{n\}$ such that

$$\omega_1 \times \{n\} \subseteq St(F_n, \mathcal{U}),$$

since every countably compact space is starcompact. Let $F' = \bigcup\{F_n : n \in \omega\}$. Then, $\omega_1 \times \omega \subseteq St(F', \mathcal{U})$. If we put

$$\mathcal{V}'' = \cup\{\{U \in \mathcal{U} : x \in U\} : x \in F_n \text{ and } n \in \omega\},$$

then $\mathcal{V}''$ is σ-centered and $\omega_1 \times \omega \subseteq \cup\mathcal{V}''$. Consequently, if we put $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, then $\mathcal{V}$ is a σ-centered subcover of $\mathcal{U}$. Hence, $S(X, \omega)$ is centered-Lindelöf.

Next, we show that $St\text{-}l(S(X, \omega)) \geq \omega_1$. Since $|L| = \aleph_1$, we can enumerate L as $\{l_\alpha : \alpha < \omega_1\}$. Since $\{l_\alpha : \alpha < \omega_1\}$ is discrete closed in X, then, for each $\alpha < \omega_1$,there exists an open neighborhood V_α in X such that $V_\alpha \cap L = \{l_\alpha\}$. Let us consider the open cover

$$\mathcal{U} = \{G_{V_\alpha, \alpha}(l_\alpha) : \alpha < \omega_1\} \cup \{\omega_1 \times \omega\} \cup \{X \setminus \{l_\alpha : \alpha < \omega_1\}\}$$

of X. It remains to show that $St(B, \mathcal{U}) \neq X$ for every countable set B of $S(X, \omega)$. To show this, let B is a countable set in X. Since $B \cap L$ is countable, there exists $\beta' < \omega_1$ such that

$$B \cap \{l_\alpha : \alpha > \beta'\} = \emptyset.$$

On the other hand, since $B \cap (\omega_1 \times \omega)$ is countable, there exists $\beta'' < \omega_1$ such that

$$B \cap ((\omega_1, \beta'') \times \omega) = \emptyset.$$

If we pick $\alpha > \max\{\beta', \beta''\}$, then $l_\alpha \notin St(B, \mathcal{U})$, since $G_{V_\alpha, \alpha}(l_\alpha)$ is the only element of $\mathcal{U}$ containing l_α and $G_{V_\alpha, \alpha}(l_\alpha) \cap B = \emptyset$, which shows that $St\text{-}l(S(X, \omega)) \geq \omega_1$.

Finally, we prove that $S(X, \omega)$ is normal. Let E and F be disjoint closed subsets of $S(X, \omega)$. If we put $E_L = E \cap L$, $F_L = F \cap L$ and $E_n = E \cap (\omega_1 \times \{n\})$, $F_n = F \cap (\omega_1 \times \{n\})$ for each $n \in \omega$. Since $\omega_1 \times \{n\}$ is homeomorphic to ω_1, there exist disjoint clopen subsets E'_n and F'_n in $\omega_1 \times \{n\}$ such that $E_n \subseteq E'_n$ and $F_n \subseteq F'_n$ for each $n \in \omega$. Since E_n and F_n can not be cofinal in $\omega_1 \times \{n\}$, thus we may pick E'_n and F'_n by the following way

(1) E'_n is cofinal in $\omega_1 \times \{n\}$ if and only if E_n is cofinal in $\omega_1 \times \{n\}$

and

(2) F'_n is cofinal in $\omega_1 \times \{n\}$ if and only if F_n is cofinal in $\omega_1 \times \{n\}$.

for each $n \in \omega$. Let

$$\overline{E} = E_L \cup \bigcup_{n\in\omega} E'_n \text{ and } \overline{F} = F_L \cup \bigcup_{n\in\omega} F'_n.$$

Then, $E \subseteq \overline{E}$, $F \subseteq \overline{F}$ and $\overline{E} \cap \overline{F} = \emptyset$. It follows from (1), (2) and the construction of $S(X,\omega)$ that $\overline{E}$ and $\overline{F}$ are closed in $S(X,\omega)$. Without loss of generality, we can assume that $E = \overline{E}$ and $F = \overline{F}$. Since E_L and F_L are disjoint closed subsets of X, then there exist disjoint open sets U_E and U_F in X such that $E_L \subseteq U_E$ and $F_L \subseteq U_F$. Let

$$V_E = (U_E \cap E) \cup \bigcup_{n\in U_E\cap\omega} \omega_1 \times \{n\} \text{ and } V_F = (U_F \cap F) \cup \bigcup_{n\in U_F\cap\omega} \omega_1 \times \{n\}.$$

Then, V_E and V_F are disjoint open subsets in $S(X,\omega)$ and $E_L \subseteq V_E$ and $F_L \subseteq V_F$. If we put

$$W_E = E \cup (V_E \setminus F) \text{ and } W_F = F \cup (V_F \setminus E).$$

Then, W_E and W_F are disjoint open subsets in $S(X,\omega)$ such that $E \subseteq W_E$ and $F \subseteq W_F$, which completes the proof. □

REMARK. It is well-known that $2^{\aleph_0} = 2^{\aleph_1}$ implies negation of CH. Thus Example 2.2 gives a partial answer to the above question. But the author does not know if there exists a ZFC counter example to the question.

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References

[1] M. Bonanzinga and M. V. Matveev, *Centered-Lindelöfness versus star-Lindelöfness*, Comment. Math. Univ. Carolin **41(1)** (2000), 111–122.

[2] Mu-Min Dai, *A topological space cardinality inequality involving the * Lindelöf number*, Acta.Math.Sinica(Chinese) **26** (1983), 731–735.

[3] E. K. van Douwn, G. M. Reed, A. W. Roscoe and I. J. Tree, *Star covering properties*, Topology Appl **39** (1991), 71–103.

[4] R. Engelking, *General Topology, Revised and completed edition*, Heldermann Verlag, Berlin, 1989.

[5] W. M. Fleischman, *A new extension of countable compactness*, Fund. Math **67** (1970), 1–9.

[6] W. Just, M. V. Matveev and P. Szeptycki, *Some results on property (a)*, Topology Appl **100** (2000), 67–83.

[7] M. V. Matveev, *A survey on star-covering properties*, Topological Atlas, vol. No 330, 1998.

[8] ———, *How weak is weak extent?*, Topology Appl **119** (2002), 229–232.

[9] Y.-K. Song, *Spaces with large star-Lindelöf number and large extent*, Houston J. Math **29 No 2** (2003), 345–352.

[10] F. D. Tall, *Normality verse Collectionwise Normality*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 685–732.

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