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IRREDUCIBILITY OF ARMA(p, q) PROCESS WITH MARKOV SWITCHING

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Abstract. We consider a autoregressive moving average process of order p and q with Markov switching coefficients and find sufficient conditions for irreducibility of the process. Identifying small sets is also examined.

1. Introduction

Recently time series models subject to Markov switching have attracted considerable attention in the econometric and statistical literature, for example, Hansen [9], McCulloch and Tsay [13], Holst et al [10], Hamilton [8], Yao and Attali [19], Francq and Zakoïan [5] etc. Markov switching process was first introduced by Hamilton [7] to analyze the rate of growth of US annual GNP series and has since been applied in variety of fields, especially for econometric series modeling (see, Lam [11], Evan and Lewis [4], Yang [17], Francq et al [6], Zhang and Stine [20], Yao [18] and references therein).

The two regime Markov switching AR(1) model is written as

$$y_t = \begin{cases} \phi_{0,1} + \phi_{1,1} y_{t-1} + e_t, & X_t = 1, \\ \phi_{0,2} + \phi_{1,2} y_{t-1} + e_t, & X_t = 2, \end{cases}$$

where X_t is an unobserved state variable that has the probability structure

$$P(X_t = 1 | X_{t-1} = 1) = p_{11}, P(X_t = 2 | X_{t-1} = 1) = p_{12}$$
$$P(X_t = 1 | X_{t-1} = 2) = p_{21}, P(X_t = 2 | X_{t-1} = 2) = p_{22}$$

with $p_{11}, p_{12}, p_{21}, p_{22} > 0$ and $p_{11} + p_{12} = 1, p_{21} + p_{22} = 1$.

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In this paper, we are interested in autoregressive moving average processes governed by a stationary Markov chain defined for integers $t \geq 1$, by

$$(1.1) \qquad y_t = \phi_0(X_t) + \sum_{i=1}^{p} \phi_i(X_t) y_{t-i} + \sum_{j=1}^{q} \theta_j(X_t) e_{t-j} + e_t,$$

where $p \geq 1, q \geq 0, \{X_t\}$ is a Markov chain with finite state space, and $\{e_t\}$ is a sequence of iid random variables.

When $q = 0$, we consider the process y_t given by

$$y_t = h_{X_t}(y_{t-1}, \ldots, y_{t-p}) + e_t,$$

where for each x, $h_x : R^p \to R$ is a Borel measurable function.

Recall that a Markov process with transition probability function $p(\cdot, \cdot)$ is said to be ϕ-irreducible with respect to a nontrivial measure ϕ, if for every A with $\phi(A) > 0$ one has

$$\sum_{n \geq 1} 2^{-n} p^{(n)}(x, A) > 0, \ \forall x.$$

A set B is said to be small with respect to ϕ if $\phi(B) > 0$, and for every A with $\phi(A) > 0$ there exists $j \geq 1$ such that

$$\inf_{x \in B} \sum_{n=1}^{j} p^{(n)}(x, A) > 0.$$

Throughout this paper, we assume that $\{X_t : t \geq 0\}$ is an irreducible, aperiodic Markov chain on a finite state space E with stationary n-step transition probability matrix $P^{(n)} = (p_{xy}^{(n)})$ and $\{e_t\}$ is a sequence of independent and identically distributed random variables with density f, continuous and positive everywhere. We assume that $\{e_t\}$ and $\{X_t\}$ are independent and initial conditions $\{y_0, y_{-1}, \ldots, y_{-p+1}, e_0, \ldots, e_{-q+1}\}$ are arbitrarily prescribable real-valued random variables independent of $\{e_t : t \geq 1\}$ and $\{X_t : t \geq 0\}$. Denote

$$(1.2) \qquad Y_t = (y_t, \ldots, y_{t-p+1}, e_t, \ldots, e_{t-q+1}), \quad W_t = (X_t, Y_t).$$

Then W_t is an aperiodic $E \times R^{p+q}$-valued Markov chain. Let the transition kernel for W_t be $\Pi(w, dw)$.

The purpose of this paper is to find sufficient conditions under which the process W_t given in (1.2) is irreducible. Knowing this property holds may make it possible or easier to establish other properties such as stationarity, (geometric) ergodicity and existence of moments (see, Tweedie [16], Tong [15], An and Chen [1], Cline and Pu [3]). Those properties are important for further statistical analysis of the process. For some cases, it can be easily shown that all compact sets are small. Identifying small sets as well as irreducibility is critical part in proving stationarity or (geometric) ergodicity of the given Markov chain.

For terminologies and relevant results in Markov chain theory we refer to Meyn and Tweedie [14].

2. Main results

Consider the ARMA(p, q) model with Markov switching given by

$$y_t = \phi_0(X_t) + \sum_{i=1}^{p} \phi_i(X_t) y_{t-i} + \sum_{j=1}^{q} \theta_j(X_t) e_{t-j} + e_t, \tag{2.1}$$

where $\phi_i(x)$ and $\theta_j(x)$ ($i = 0, 1, \ldots, p$, $j = 1, 2, \ldots, q$, $x \in E$) are constants. Then $W_t = (X_t, Y_t) = (X_t, y_t, \ldots, y_{t-p+1}, e_t, \ldots, e_{t-q+1})$ is a Markov chain with state space $E \times R^{p+q}$ and assume $W_0 = (x_0, y)$. Let $Y_t(x_1, x_2, \ldots, x_t)$ denote Y_t given $X_1 = x_1, \ldots, X_t = x_t$.

For $t \geq p + q$, $Y_t(x_1, \ldots, x_t)$ can be rewritten as

$$Y_t(x_1, \ldots, x_t) = d + (e_{t-p-q+1}, \ldots, e_{t-1}, e_t) \cdot F^t_{t-p-q+2}, \tag{2.2}$$

where $d = d(y, \phi_i(x_l), \theta_j(x_l), i = 0, 1, \ldots, p, j = 1, \ldots, q, l = 1, \ldots, t, e_1, \ldots, e_{t-p-q})$ is a random vector in R^{p+q} and $F^t_{t-p-q+2} = F(\phi_i(x_l), \theta_j(x_l), i = 1, \ldots, p, j = 1, \ldots, q, l = t - p - q + 2, \ldots, t)$ is a $(p + q) \times (p + q)$ matrix.

THEOREM 2.1. *Suppose for each $z \in E$, $F^t_{t-p-q+2} = F(\phi_i(x_l), \theta_j(x_l), i = 1, \ldots, p, j = 1, \ldots, q, l = t - p - q + 2, \ldots, t)$ given in (2.2) is nonsingular for some finite sequence $x_1, \ldots, x_{t-1}, x_t$ such that $t \geq p + q$, $x_t = z$ and $p_{x_0x_1} \cdots p_{x_{t-1}x_t} > 0$. Then W_t is $(\nu \times \lambda)$-irreducible, where ν is a counting measure on E and λ is a Lebesgue measure on R^{p+q}. For $E' \subset E$ and compact subset C of R^{p+q}, $E' \times C$ is a small set.*

PROOF. Suppose that $(\nu \times \lambda)(E' \times A) > 0$, $E' \subset E$ and $A \in \mathcal{B}(R^{p+q})$. Since $\{X_t\}$ is aperiodic irreducible Markov chain with finite state space E, there exist $t \geq p + q$ such that $p^{(t)}_{x_0x_t} = P(X_t = x_t \mid X_0 = x_0) > 0$ and $x_t \in E'$ and hence, by assumption, we can find a finite sequence $x_1, x_2, \ldots, x_t$ such that $p_{x_0x_1} \cdots p_{x_{t-1}x_t} > 0$ and $F^t_{t-p-q+2}$ is nonsingular.

Now

$$\begin{aligned}(2.3)\qquad & \Pi^{(t)}((x_0,y),E'\times A) = P(W_t\in E'\times A \mid W_0=(x_0,y)) \\ = & \sum_{x_t\in E'}\sum_{x_{t-1}\in E}\cdots\sum_{x_1\in E} p_{x_0x_1}p_{x_{t-1}x_t} \\ & \times P(Y_t(x_1,\dots,x_t)\in A \mid Y_0=y)\end{aligned}$$

and

$$\begin{aligned}(2.4)\qquad & P(Y_t(x_1,\dots,x_t)\in A \mid Y_0=y) \\ = & P(d+(e_{t-p-q+1},\dots,e_t)\cdot F^t_{t-p-q+2}\in A \mid Y_0=y)>0.\end{aligned}$$

Inequality in (2.4) follows from the fact that d and $(e_{t-p-q+1},\dots,e_t)$ are independent, $F^t_{t-p-q+2}$ is nonsingular, and therefore $\lambda((A-d){F^t_{t-p-q+2}}^{-1})>0$ if $\lambda(A)>0$. (2.3) and (2.4) show that for any initial $W_0=(x_0,y)$, $\Pi^{(t)}((x_0,y),E'\times A)>0$ for some $t\ge p+q$. Thus

$$\sum_{t\ge p+q} 2^{-t}\Pi^{(t)}((x_0,y),E'\times A)>0,\quad \forall(x_0,y)\in E\times R^{p+q}$$

and the $\nu\times\lambda$-irreducibility of W_t follows.

Now, for any compact subset C of R^{p+q},

$$\begin{aligned} & \inf_{y\in C} P(Y_t(x_1,\dots,x_t)\in A \mid Y_0=y) \\ (2.5)\qquad = & \inf_{y\in C} P(d+(e_{t-p-q+1},\dots,e_{t-1},e_t)\cdot F^t_{t-p-q+2}\in A)>0,\end{aligned}$$

since d is the only term that depends on y and is a linear sum of each coordinate of y. From (2.5), we may choose $j\ge 1$ so that

$$\inf_{(x_0,y)\in E\times C}\sum_{n=1}^{j} p^{(n)}((x_0,y),E'\times A)>0,$$

which implies that $E\times C$ is a small set. □

COROLLARY 2.1. *If F_1^{p+q-1} in eqn. (2.2) is nonsingular for each $(x_1, x_2,\dots,x_{p+q-1})\in E^{p+q-1}$, then W_t is $\nu\times\lambda$-irreducible.*

PROOF. By assumption on $\{X_t\}$, there exists an integer $N>0$ such that $p_{xz}^{(N+s)}>0$, $\forall x,z\in E, s\ge 0$. Suppose that $(\nu\times\lambda)(E'\times A)>0$. Then

$$(2.6)\qquad \Pi^{(N+p+q)}(w,E'\times A)>0,\quad \forall w$$

if for each $w_N\in E\times R^{p+q}$,

$$(2.7)\qquad P(W_{N+p+q}\in E'\times A \mid W_N=w_N)>0.$$

But (2.7) is true whenever

$$P(d + (e_{N+1}, \ldots, e_{N+p+q}) \cdot F_{N+2}^{N+p+q} \in A) > 0, \tag{2.8}$$

and inequality in (2.8) follows from the fact that F_{N+2}^{N+p+q} is nonsingular for any $(x_{N+2}, \ldots, x_{N+p+q}) \in E^{p+q-1}$ and d and $(e_{N+1}, \ldots, e_{N+p+q})$ are independent. (2.6) shows the irreducibility of W_t. □

EXAMPLE 1. (ARMA(1,1)-MS) Let $E = \{1, 2\}$, one step transition probability function $P = (p_{xy})_{x,y=1,2}$, and let $y_t = \phi_0(X_t) + \phi_1(X_t)y_{t-1} + \theta(X_t)e_{t-1} + e_t$ and $W_0 = (x_0, u, v)$. Then with probability $p = p_{x_0x_1}p_{x_1x_2}$,

$$\begin{pmatrix} y_2 \\ e_0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_1(x_2) + \theta(x_2) & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

where $a = \phi_0(x_2) + \phi_1(x_2)\phi_0(x_1) + \phi_1(x_2)\phi_1(x_1)u + \phi_1(x_2)\theta(x_2)v$ and $F_2^2 = \begin{pmatrix} \phi_1(x_2) + \theta(x_2) & 1 \\ 0 & 1 \end{pmatrix}^t$ (A^t denotes the transpose of a matrix A).

If $\phi_1(1) + \theta(1) \neq 0$ and $\phi_1(2) + \theta(2) \neq 0$, then F_2^2 is nonsingular for any $x_2 \in E$, and the irreducibility of W_t follows from Corollary 2.1.

In fact, $p_{xy}^{(2)} > 0$ for each $x, y \in E$ implies that $\phi_1(1) + \theta(1) \neq 0$ or $\phi_1(2) + \theta(2) \neq 0$ is sufficient for irreducibility of W_t.

EXAMPLE 2. (ARMA(2,2)-MS) Let $E = \{1, 2, \ldots, m\}$ and y_t be given by

$$y_t = \phi_o(X_t) + \phi_1(X_t)y_{t-1} + \phi_2(X_t)y_{t-2} + \theta_1(X_t)e_{t-1} + \theta_2(X_t)e_{t-2} + e_t.$$

Then

$$(y_4, y_3, e_4, e_3) = d + (e_1, e_2, e_3, e_4) \cdot F_2^4$$

and

$$\begin{aligned} \det(F_2^4) &= (\phi_1(x_4)\phi_1(x_3)f_{12} + \phi_1(x_4)f_{23} + \phi_2(x_4)f_{12})f_{13} \\ &\quad -(\phi_1(x_4)f_{13} + f_{24})(\phi_1(x_3)f_{12} + f_{23}), \end{aligned}$$

where $f_{ij} = \phi_i(x_j) + \theta_i(x_j), i = 1, 2, j = 2, 3, 4$. If $\det(F_2^4) \neq 0$ for any $(x_2, x_3, x_4) \in E^3$, then irreducibility of W_t is obtained.

Procedure to prove the irreducibility of ARMA(p,q)-MS model with large $p > 1$ and $q > 1$ is entirely analogous, but involves messier notation.

Theorem 2.1 can be easily extended to multivariate ARMA(p, q) with Markov switching coefficient models.

For nonlinear ARMA(p, q)-MS model with $p > 1$ and $q > 1$ such as threshold ARMA(p, q)-MS model, it is not an easy job to determine whether the process is irreducible. Even for TARMA(p,q) model (without Markov switching), showing irreducibility is very awkward.

Now, we study the irreducibility of nonlinear AR(p) model with Markov switching coefficients. This model includes TAR(p)-MS, momentum TAR(p)-MS.

Let y_t be generated by

$$y_t = h_{X_t}(y_{t-1}, \ldots, y_{t-p}) + e_t, \tag{2.9}$$

where $\{h_x, x \in E\}$ is a family of Borel measurable functions on R^p to R. For fixed $x_1, x_2, \ldots, x_p$, $u = (u_p, \ldots, u_1), v = (v_p, \ldots, v_1)$, define

$$\begin{aligned} &g(u, v \mid x_1, \ldots, x_p) \\ = \;& f(v_1 - h_{x_1}(u_p, \ldots, u_1)\Pi_{i=2}^p f(v_i - h_{x_i}(v_{i-1}, \ldots, v_1, u_p, \ldots, u_i)). \end{aligned} \tag{2.10}$$

THEOREM 2.2. *Consider a Markov chain* $W_t = (X_t, Y_t) = (X_t, y_t, \ldots, y_{t-p+1})$ *obtained by (2.9). If* $\forall x \in E$, h_x *is bounded on compacts, then* W_t *is* $\nu \times \lambda$*-irreducible and for every compact set* $C \in R^p$, $E \times C$ *is a small set.*

PROOF. Since the density function f is continuous and positive everywhere, we have that if $\lambda(A) > 0$ and C is a compact subset of R^p, then we have that (see, Bhattacharya and Lee [2])

$$\int_A g(u, v \mid x_1, \ldots, x_p) d\lambda(v) > 0 \tag{2.11}$$

and

$$\inf_{u \in C} \int_A g(u, v \mid x_1, \ldots, x_p) d\lambda(v) > 0. \tag{2.12}$$

For any $E' \subset E$, choose $x_1, x_2, \ldots, x_t$ in E such that $t \geq p$, $p_{x_0 x_1} \cdots p_{x_{t-1} x_t} > 0$ and $x_t \in E'$. Now

$$\begin{aligned} &\Pi^{(t)}((x_0, u)E' \times A) \\ = &\sum_{x_t \in E'} \sum_{x_{t-1} \in E} \cdots \sum_{x_1 \in E} p_{x_0 x_1} \cdots p_{x_{t-1} x_t} \\ &\times P(Y_t(x_1, \ldots, x_t) \in A \mid W_0 = (x_0, u)). \end{aligned} \tag{2.13}$$

But, from (2.11)

$$\begin{aligned} &P(Y_t(x_1, \ldots, x_t) \in A \mid Y_{t-p} = w) \\ = &\int_A g(w, v \mid x_{t-p+1} \ldots, x_t) > 0, \ \forall w \end{aligned} \tag{2.14}$$

Combining (2.11)-(2.14), we have that $P(Y_t(x_1,\dots,x_t)\in A \mid Y_0=u)>0$ and $\inf_{u\in C} P(Y_t(x_1,\dots,x_t)\in A \,|Y_0=u)>0$, from which the conclusions follow. □

EXAMPLE 3. (TAR(1)-MS) Consider the threshold AR(1)-MS process given by

$$y_t=\alpha_0(X_t)+\alpha_1(X_t)y_{t-1}^{+}+\alpha_2(X_t)y_{t-1}^{-}+e_t$$

where $h_x(y)=\alpha_0(x)+\alpha_1(x)y^+ +\alpha_2(x)y^-, y^+=\max\{0,y\}, y^-=\max\{0, -y\}, x\in E, y\in R$ is not continuous but bounded on compacts. Therefore $W_t=(X_t,y_t)$ is $\nu\times\lambda$-irreducible.

REMARK. The proof of irreducibility of ARCH type-MS process is much simpler. Let the model y_t be defined by

$$y_t=h_{X_t}(y_{t-1},\dots,y_{t-p})e_t,$$

where $h_x, x\in E$ is measurable and bounded on compacts. If we assume that $h_x(y_1,\dots,y_p)>0$ for all $(y_1,\dots,y_p)\in R^p$, $x\in E$, then $W_t=(X_t,y_t,\dots,y_{t-p+1})$ is $\nu\times\lambda$-irreducible. In this case, $g(u,v|x_1,\dots,x_p)$ in (2.10) is defined by

$$\begin{aligned}&g(u,v|x_1,\dots,x_p)\\&=f(h_{x_1}^{-1}(u_1,\dots,u_p)v_1)\Pi_{i=2}^{p}f(h_{x_i}^{-1}(v_{i-1},\dots,v_1,u_p,\dots,u_i)v_i).\end{aligned}$$

EXAMPLE 4. (TARCH(p)-MS) Consider the threshold autoregressive conditional heteroskedastic model of order p with Markov switching (TARCH(p)-MS) obtained by

$$\varepsilon_t=e_t h_t^{1/2} \tag{2.15}$$

$$h_t=\sum_{i=1}^{2}(\alpha_{i0}(X_t)+\sum_{j=1}^{p}\alpha_{ij}(X_t)\varepsilon_{t-j}^2)I_{it}, \tag{2.16}$$

where $\alpha_{i0}(x)>0, \alpha_{ij}(x)\ge 0, x\in E, i=1,2, j=1,\dots,p$, $I_{1t}=I(\varepsilon_{t-1}\ge 0)$, $I_{2t}=1-I_{1t}$ ($I(B)$ denotes the indicator function of B). Combining (2.15) and (2.16), we have that

$$\varepsilon_t=(\sum_{i=1}^{2}(\alpha_{i0}(X_t)+\sum_{j=1}^{p}\alpha_{ij}(X_t)\varepsilon_{t-j}^2)I_{it})^{1/2}e_t.$$

Since $h_{X_t}(\varepsilon_{t-1},\dots,\varepsilon_{t-p}) = (\sum_{i=1}^{2}(\alpha_{i0}(X_t)+\sum_{j=1}^{p}\alpha_{ij}(X_t)\varepsilon_{t-j})I_{it})^{1/2}$ is bounded on compacts, $\nu\times\lambda$-irreducibility of $W_t=(X_t,\varepsilon_t,\varepsilon_{t-1},\dots, \varepsilon_{t-p+1})$ can be obtained.

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