

LARGE DEVIATION PRINCIPLE FOR SOLUTIONS TO SDE DRIVEN BY MARTINGALE MEASURE

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ABSTRACT. We consider a type of large deviation principle(LDP) using Freidlin-Wentzell exponential estimates for the solutions to perturbed stochastic differential equations(SDEs) driven by Martingale measure(Gaussian noise). We are using exponential tail estimates and exit probability of a diffusion process. Referring to Freidlin- Wentzell inequality, we want to show another approach to get LDP for the solutions to SDEs.

1. Introduction

The large deviation principle(LDP) characterizes the limiting behavior of a family of probability measure $\{\mu_\epsilon\}$ on $(\mathcal{X}, \mathcal{B})$ as $\epsilon \rightarrow 0$ in terms of a rate function.

Recall that if $\{\mu_\epsilon\}$ is a family of probability measure on a Polish space \mathcal{P} with metric ρ , then μ_ϵ is said to have an LDP with rate functional $I : \mathcal{P} \rightarrow [0, \infty]$, which is a lower semi-continuous function such that the sets

$$\mathcal{K}(r) = \{x \in \mathcal{P} : I(x) \leq r\}, \quad r \geq 0$$

are compact and satisfies the following:

(L1) for each open set $O \subset \mathcal{P}$

$$\liminf_{\epsilon \rightarrow 0} 2\epsilon^2 \log \mu_\epsilon(O) \geq -\inf\{I(x) : x \in O\}$$

(L2) for each closed set $C \subset \mathcal{P}$

$$\limsup_{\epsilon \rightarrow 0} 2\epsilon^2 \log \mu_\epsilon(C) \leq -\inf\{I(x) : x \in C\}.$$

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Instead of directly trying to show the above inequalities, (L1) and (L2) we will follow a reformulated equivalent form of the so-called Freidlin-Wentzell exponential estimates. (see [7])

(L1') $\forall x \in \mathcal{P}, \forall \delta > 0$ and $\forall \gamma > 0 \exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon \{y : \rho(y, x) \leq \delta\} \geq \exp\left\{-\frac{I(x) + \gamma}{2\epsilon^2}\right\}$$

(L2') $\forall r, \forall \delta > 0$ and $\forall \gamma > 0 \exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon \{y : \rho(y, \mathcal{K}(r)) \geq \delta\} \leq \exp\left\{-\frac{r - \gamma}{2\epsilon^2}\right\}.$$

In this paper we consider a large deviation principle(LDP) for a mild solution to stochastic partial differential equations(SPDEs) driven by martingale measure. This extends the results proved by Freidlin and Wentzell [4] for diffusion processes, Peszat [7], and Cho [3].

Let L be the usual elliptic operator defined by

$$Lf = f_{xx} - f_t$$

and let $M(dt, dx)$ be a continuous orthogonal martingale measure defined on a probability space (Ω, \mathcal{F}, P) . For any $\epsilon > 0$, let $X^\epsilon(t, x)$ be the solution of the parabolic SPDE under the following conditions:

$$(1.1) \quad \begin{aligned} \partial_t X^\epsilon(t, x) &= LX^\epsilon(t, x) + b(t, x, X^\epsilon(t, x)) \\ &\quad + \epsilon \sigma(t, x, X^\epsilon(t, x))M(dt, dx) \\ X^\epsilon(0, \cdot) &= u_0(\cdot) \end{aligned}$$

for all $(t, x) \in [0, T] \times [0, 1]$ and $u_0 \in C_0^\infty$ with Dirichlet's or Neuman's boundary conditions. We also suppose that there exist positive constants C_1 and C_2 such that

$$(1.2) \quad |b(t, x, y) - b(t, x, z)| \leq C_1 |y - z|$$

$$(1.3) \quad |\sigma(t, x, y) - \sigma(t, x, z)| \leq C_2 |y - z|,$$

for all $(t, x) \in [0, T] \times [0, 1]$ and $y, z \in R$, and that there exists a constant C_3 such that for every $z \in R$,

$$(1.4) \quad \sup_{(t,x) \in [0,1] \times [0,T]} |\sigma(t, x, z) + b(t, x, z)| \leq C_3(1 + |z|).$$

Finally, we also assume that the function

$$(1.5) \quad (t, x, y) \longmapsto \sigma(t, x, y) \quad \text{is continuous.}$$

It is J. Walsh [9] who first gave a rigorous meaning to Eq. (1.1): that is $X^\epsilon(t, x)$ is the solution of the following evolution equation:

$$\begin{aligned} X^\epsilon(t, x) &= G_t(x, u_0) + \int_0^t \int_0^1 G_{t-s}(x, y) b(s, y, X^\epsilon(s, y)) ds dy \\ &+ \epsilon \int_0^t \int_0^1 G_{t-y}(x, y) \sigma(s, y, X^\epsilon(s, y)) M(ds, dy), \end{aligned}$$

where $G_t(x, u_0) = \int_0^1 G_t(x, y) u_0(y) dy$ for fixed $u_0 \in C[0, 1]$, and $G_t(\cdot, \cdot)$ is the Green kernel associated with the partial differential equation $\partial_t u = Lu$ and with the same boundary conditions as those of X^ϵ .

It is well known that for given $u_0 \in C[0, 1]$ the trajectories of the process, X^ϵ are known to be almost surely continuous in t and x .

Let M be a continuous martingale measure and $\pi(ds, dx)$ be its covariance measure. We assume that there exists a predictable process $h(s, x)$ such that

$$(1.6) \quad \pi(t, A) \leq \int_{A \times [0, t]} h(s, x) ds dx,$$

for every borel set A , and for $q' = \frac{p}{p-2}, 2 < p \leq 3$ and for some $p_0 \geq 0$,

$$E \int_0^T \left\| \frac{1}{1 + |x|^{p_0}} h(s, \cdot) \right\|_{q'} ds < \infty,$$

where $\|\cdot\|_{q'}$ is the usual $L_{q'}$ -norm.

REMARK. Condition(1.6) includes the case in which

$$M^2(t, A) - \int_{A \times [0, t]} h(s, x) ds dx$$

is a martingale. If $h(s, x) \equiv 1$ the associated martingale measure is a white noise based on Lebesgue measure. If $h(s, x) \equiv h(x)$ is non-random, the associated martingale measure having continuous sample paths is a Brownian motion with the variance $\int_A h(x) m(dx)$.

Let H denote the Cameron-Martin space associated with M , i.e.,

$$H = \{h(t, x) = \int_0^t \int_0^x \dot{h}(u, z) \pi(du, dz) : \dot{h} \in L_2\},$$

and let

$$\mathcal{H}_T = \{h \in H : \int_0^T \int_0^1 |\dot{h}(u, z)|^2 \pi(du, dz) < \infty\}.$$

We also let $h \in \mathcal{H}_T$ and z^h be defined by the solution of the following integral equation:

$$\begin{aligned} z^h(t, x) &= G_t(x, u_0) + \int_0^t \int_0^1 G_{t-s}(x, y) b(s, y, z^h(s, y)) ds dy \\ &\quad + \int_0^t \int_0^1 G_{t-u}(x, z) \sigma(s, y, z^h(s, y)) \dot{h}(s, y) \pi(ds, dy), \end{aligned}$$

for all $(t, x) \in [0, T] \times [0, 1]$.

Let $I : C[0, T] \times [0, 1] \rightarrow [0, \infty]$ be given by

$$I(z) = \inf\{|h|_{\mathcal{H}_T}^2 : z = z^h\},$$

where $|h|_{\mathcal{H}_T}^2 = \int_0^T \int_0^1 |\dot{h}(u, z)|^2 \pi(du, dz)$. This I is our candidate for the rate function.

Now, if $X^\epsilon(t, x)$ converges to a deterministic limit, $z^h(t, x)$, we want to know what the rate of this convergence is. This problem is related to the Freidlin-Wentzell theory for the analysis of dynamical systems and to the study of PDE with small perturbations.

2. Preliminaries

F. Chenal and A. Millet [2] show a LDP for the solutions to a parabolic SPDE perturbed by a small non-linear white noise. D. Marquez-Carreras and M. Sarra [6] prove a LDP for a perturbed stochastic heat equation driven by a Gaussian noise. Both of their proofs are based on a classical result given by Azencott [1].

Azencott's method is the following:

THEOREM [1]. Let $(E_i, d_i), i = 1, 2$ be two Polish spaces and $X_i^\epsilon : \Omega \rightarrow E_i, \epsilon > 0, i = 1, 2$ be two families of random variables. Suppose the following requirements:

1. $\{X_1^\epsilon > 0\}$ obeys a LDP with the rate function $I_1 : E_1 \rightarrow [0, \infty]$.
2. There exists a function $K : \{I_1 < \infty\} \rightarrow E_2$ such that for every $a < \infty$, the function $K : \{I_1 < a\} \rightarrow E_2$ is continuous.
3. For every $r, \rho, a > 0$, there exist $\theta > 0$ and $\epsilon_0 > 0$ such that, for $h \in E_1$ satisfying $I_1(h) \leq a$ and $\epsilon \leq \epsilon_0$, we have

$$P\{d_2(X_2^\epsilon, K(h)) \geq \rho, d_1(X_1^\epsilon, h) < \theta\} \leq \exp(-\frac{r}{\epsilon^2}).$$

Then, the family $\{X_2^\epsilon > 0\}$ obeys a LDP with the rate function

$$I_2(\phi) = \inf\{I_1(h) : K(h) = \phi\}.$$

If we deduce (from Schilder's theorem) that the family $\{\epsilon M : \epsilon > 0\}$ satisfies a LDP with a good rate function I and show the continuity of the skeleton, z , then we may have the LDP due to the above theorem. However, we are going to prove our LDP problem by showing (L1') and (L2'), which are the equivalent forms of (L1) and (L2).

We briefly review the Green kernel. Recall that the Green kernel G associated with satisfies the following inequalities under the case of Dirichlet's condition: ($X^\epsilon(t, 0) = X^\epsilon(t, 1) = 0$) or Neumann's boundary conditions :($\partial_x X^\epsilon(t, 0) = \partial_x X^\epsilon(t, 1) = 0$). For all $x, y \in [0, 1]$ and $s, t \in [0, T]$,

$$(2.1) \quad |G_t(x, y)| \leq \frac{C}{\sqrt{t}} \exp(-\frac{(y-x)^2}{4t}),$$

$$(2.2) \quad \sup_{t \in [0, T]} \int_0^t \int_0^1 |G_u(x, z) - G_u(y, z)|^p dudz \leq C|x-y|^{3-p}, \quad \frac{3}{2} < p < 3$$

$$(2.3) \quad \sup_{x \in [0, 1]} \int_0^s \int_0^1 |G_{t-u}(x, z) - G_{s-u}(y, z)|^p dudz \leq C|t-s|^{\frac{3-p}{2}}, \quad 1 < p < 3$$

$$(2.4) \quad \sup_{x \in [0, 1]} \int_s^t \int_0^1 |G_u(x, z)|^p dudz \leq C|t-s|^{\frac{3-p}{2}} \leq C_T, \quad 1 < p < 3.$$

Also for convenience, we let $M(t, [0, x]) \equiv M(t, x)$ and $|\cdot|_\infty = \|\cdot\|$.

3. Lower bound

LEMMA 3.1. Assume that β is a predictable process and there exists a constant $l < \infty$ such that

$$\int_0^T \int_0^1 \beta^2(s, x) \pi(ds, dx) \leq l, \quad P - \text{almost surely.}$$

Then for all $r > 0$ one has

$$(3.1) \quad \mathbf{P}\left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \int_0^1 \beta(s, x) M(ds, dx) \right| \geq r \right\} \leq 3 \cdot \exp\left\{-\frac{r^2}{4l}\right\}.$$

PROOF. Let

$$(3.2) \quad Z(t) = \int_0^t \int_0^1 \beta(s, x) M(ds, dx)$$

For the following estimations let

$$(3.3) \quad \phi_\lambda(x) = (1 + \lambda x^2)^{\frac{1}{2}}, \quad \lambda, x \in R_+.$$

We again apply the Ito formula to (3.2).

$$(3.4) \quad \begin{aligned} & \phi_\lambda(z(t)) \\ &= 1 + \int_0^t \int_0^1 \phi'_\lambda(z(s)) \beta(s, x) M(ds, dx) \\ & \quad + \int_0^t \int_0^1 \frac{1}{2} \phi''_\lambda(z(s)) \beta(s)^2 \pi(ds, dx) \\ &= 1 + \int_0^t \int_0^1 \phi'_\lambda(z(s)) \beta(s, x) M(ds, dx) \\ & \quad - \frac{1}{2} \int_0^t \int_0^1 (\phi'_\lambda(z(s)) \beta(s, x))^2 \pi(ds, dx) \\ & \quad + \frac{1}{2} \int_0^t \int_0^1 (\phi'_\lambda(z(s)) \beta(s, x))^2 \pi(ds, dx) \\ & \quad + \frac{1}{2} \int_0^t \int_0^1 (\phi''_\lambda(z(s)) \beta^2(s, x)) \pi(ds, dx). \end{aligned}$$

Note that $\phi'_\lambda(x) = \frac{\lambda x}{\phi_\lambda(x)}$ and $\frac{1}{\phi_\lambda(x)} \leq 1$. For every $\lambda \in R_+$, let

$$(3.5) \quad \eta_t^\lambda \equiv \int_0^t \int_0^1 \phi'_\lambda(z(s))\beta(s, x)M(ds, dx) - \frac{1}{2} \int_0^t \int_0^1 (\phi'_\lambda(z(s))\beta(s, x))^2 \pi(ds, dx).$$

Then

$$(3.6) \quad \begin{aligned} \phi_\lambda(z(t)) &\leq 1 + \eta_t^\lambda + \frac{1}{2}\lambda \cdot l + \frac{1}{2} \int_0^t \int_0^1 \frac{\lambda \cdot \lambda x^2}{1 + \lambda x^2} \beta^2(s, x) \pi(ds, dx) \\ &\leq 1 + \eta_t^\lambda + \lambda \cdot l. \end{aligned}$$

Let $Z_t^\lambda \equiv \exp\{\eta_t^\lambda\}$ for every real λ . Then Z_t^λ is a local martingale and it follows easily from the Ito formula that $EZ_t^\lambda = 1$. Hence using Doob's martingale inequality, we have

$$(3.7) \quad \begin{aligned} P\{ \sup_{0 \leq s \leq t} |z(s)| \geq r \} &\leq P\{ \sup_{0 \leq s \leq t} \phi_\lambda(z(s)) \geq (1 + \lambda r^2)^{\frac{1}{2}} \} \\ &\leq P\{ \sup_{0 \leq s \leq t} Z_s^\lambda \geq \exp\{(1 + \lambda r)^{\frac{1}{2}} - 1 - \lambda l\} \} \\ &\leq \exp\{-(1 + \lambda r^2)^{\frac{1}{2}} - 1 - \lambda l\} \\ &\leq \exp\{-(1 + \lambda r^2)^{\frac{1}{2}} + 1 + \lambda l\}. \end{aligned}$$

If $r^2 > 2l$ then taking $\lambda = \frac{r^4 - 4l^2}{4l^2 r^2}$ we obtain the desired bound (3.1). If $r^2 < 2l$, then $3 \exp(-\frac{r^2}{4l}) \geq 1$. Thus (3.1) holds for any $r > 0$. \square

LEMMA 3.2. *If there exists a constant $\eta < \infty$ such that*

$$\sup_{x \in [0,1]} \int_0^T \int_0^1 |G_{t-s}(x, y)\beta(s, y)|^2 \pi(ds, dy) < \eta, \quad P - \text{almost surely}$$

then

$$\begin{aligned} &P\{ \sup_{0 \leq t \leq T} \sup_{x \in [0,1]} | \int_0^t \int_0^1 G_{t-s}(x, y)\beta(s, y)M(ds, dy) | \geq \delta \} \\ &< 4T \exp\{-\frac{\delta^2}{9\eta}\}. \end{aligned}$$

PROOF. Let $\psi_t(s, x, y) = G_{t-s}(x, y)\beta(s, y)$ for $0 \leq s < t \leq T$ and 0 otherwise. Then by Lemma 3.1

$$\begin{aligned} &P\{\exp(\frac{1}{9\eta} \sup_{0 \leq t \leq T} \|\int_0^t \int_0^1 \psi_t(s, x, y)M(ds, dx)\|^2) \geq u\} \\ &\leq 3 \exp(-\frac{9}{4} \ln u) \leq 3u^{-\frac{9}{4}}. \end{aligned}$$

Hence

$$\begin{aligned} &E \exp\{\frac{1}{9\eta} \sup_{0 \leq t \leq T} \|\int_0^t \int_0^1 \psi_t(s, x, y)M(ds, dx)\|^2\} \\ &= \int_0^\infty P\{\exp(\frac{1}{9\eta} \sup_{0 \leq t \leq T} \|\int_0^t \int_0^1 \psi_t(s, x, y)M(ds, dx)\|^2) \geq u\} du \\ &\leq 1 + 3 \int_1^\infty u^{-\frac{9}{4}} du \leq 4 \\ &\times \int_0^T E \exp\{\frac{1}{9\eta} \sup_{0 \leq t \leq T} \|\int_0^t \int_0^1 \psi_t(s, x, y)M(ds, dx)\|^2\} dt \leq 4T. \end{aligned}$$

$$\begin{aligned} &P\{\sup_{0 \leq t \leq T} \|\int_0^t \int_0^1 G_{t-s}(x, y)\beta(s, y)M(ds, dy)\|^2 \geq \delta^2\} \\ &\leq P\{\sup_{0 \leq t \leq T} \exp(\frac{1}{9\eta} \|\int_0^t \int_0^1 \psi_t(s, x, y)M(ds, dx)\|^2) \geq \exp(\frac{\delta^2}{9\eta})\} \\ &\leq 4T \exp(-\frac{\delta^2}{9\eta}). \end{aligned}$$

□

THEOREM 3.3 (LOWER BOUND). We assume (1.2)-(1.5). $\forall T > 0, \forall l > 0, \forall \delta > 0, \forall \gamma > 0$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in [0, \epsilon_0]$ and $\forall h \in \mathcal{H}_T$ satisfying $|h|_{\mathcal{H}_T}^2 \leq l$, we have

$$(3.8) \quad \mathbf{P}\{\sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - z^h(t, \cdot)\|_\infty \leq \delta\} \geq \exp\{-\frac{|h|_{\mathcal{H}_T}^2 + \gamma}{2\epsilon^2}\}.$$

PROOF. Let $l > 0$, $r > 0$, $\delta > 0$ and $h \in \mathcal{H}_T$ satisfying $|h|_{\mathcal{H}_T}^2 \leq l$. Define

$$M^\epsilon(t, x) = M(t, x) - \frac{1}{\epsilon} \int_0^t \int_0^x \dot{h}(s, y) \pi(ds, dy).$$

According to the Girsanov theorem, for fixed x $M^\epsilon(\cdot, x)$ is a martingale process on the probability space $(\Omega, \mathcal{F}, P^\epsilon)$, where

$$\begin{aligned} \frac{dP}{dP^\epsilon} &= \exp\left\{-\frac{1}{\epsilon} \int_0^T \int_0^1 h(s, y) M(ds, dy) \right. \\ &\quad \left. - \frac{1}{2\epsilon^2} \int_0^T \int_0^1 \dot{h}^2(s, y) \pi(ds, dy)\right\}. \end{aligned}$$

Note that $P \ll P^\epsilon$.

Now, consider the following:

$$\begin{aligned} &|X^\epsilon(t, x) - z^h(t, x)| \\ &\leq C \int_0^t \int_0^1 |G_{t-s}(x, y)(X^\epsilon(s, y)) - z^h(s, y)| ds dy \\ &\quad + \epsilon \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(X^\epsilon(s, y)) M^\epsilon(ds, dy) \right| \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) |X^\epsilon(s, y) - z^h(s, y)| |\dot{h}(s, y)| \pi(ds, dy). \end{aligned}$$

By (1.2)-(1.5) and the Gronwall inequality, there exists a constant K_1 such that

$$\begin{aligned} &\sup_{0 \leq t \leq T} |X^\epsilon(t, x) - z^h(t, x)| \\ &\leq K_1 \sup_{0 \leq t \leq T} \left\| \epsilon \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(X^\epsilon(s, y)) M^\epsilon(ds, dy) \right\|. \end{aligned}$$

Let $\tilde{l} \in (0, l)$, and

$$\begin{aligned} \mathcal{G}(\epsilon, t, x) &= \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(X^\epsilon(s, y)) M^\epsilon(ds, dy) \\ \mathcal{A}(\epsilon) &= \left\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - z^h(t, \cdot)\| \leq \delta \right\} \\ \mathcal{B}(\epsilon) &= \left\{ \left\| \epsilon \int_0^T \int_0^1 \dot{h}(s, y) M^\epsilon(ds, dy) \right\| \leq \frac{\tilde{l}}{2} \right\}. \end{aligned}$$

Then

$$\mathbf{P}\{\mathcal{A}(\epsilon)\} = E^\epsilon\left[\frac{dP}{dP^\epsilon} : \mathcal{A}(\epsilon)\right] \geq \exp\left\{-\frac{|h|_{\mathcal{H}}^2 + \tilde{l}}{2\epsilon^2}\right\} P^\epsilon\{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\}.$$

We also want to show that $P^\epsilon\{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\} \rightarrow 1$ uniformly with respect to x and h on bounded sets. Now,

$$\begin{aligned} & P^\epsilon\{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\} \\ & \geq 1 - P^\epsilon\left\{\sup_{0 \leq t \leq T} \|\mathcal{G}(\epsilon, t, \cdot)\| > \frac{\delta}{K_1\epsilon}\right\} - P^\epsilon\{\mathcal{B}^c(\epsilon)\}. \end{aligned}$$

Note that by condition (1.4) and (2.4), there exists a constant l_0 such that

$$\sup_{x \in [0,1]} \int_0^T \int_0^1 |G_{t-s}(x, y)\sigma(X^\epsilon(s, y))|^2 \pi(ds, dy) < l_0.$$

Lemma 3.2 shows that

$$(3.9) \quad P^\epsilon\left\{\sup_{0 \leq t \leq T} \|\mathcal{G}(\epsilon, t, \cdot)\| > \frac{\delta}{K_1\epsilon}\right\} \leq 3 \exp\left\{-\frac{\delta^2}{4K_1^2\epsilon^2 l_0}\right\} \rightarrow 0$$

and $P^\epsilon\{\mathcal{B}^c(\epsilon)\} \rightarrow 0$ as $\epsilon \rightarrow 0$ by Lemma 3.1. Hence, we get the lower bound. □

4. Upper bound

For convenience, let

$$I^\epsilon(t, x) \equiv \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(s, y, X^\epsilon(s, y))M(ds, dx).$$

LEMMA 4.1. $\forall r >, \exists M > 0$ such that $\forall \epsilon \in (0, 1]$ we have

$$(4.1) \quad \mathbf{P}\left\{\sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot)\| \geq M\right\} \leq \exp\left\{-\frac{r}{\epsilon^2}\right\}.$$

PROOF. From (1.2)-(1.3), (2.4) and Gronwall's inequality, there exist $C_1, C_2 > 0$ such that $\forall \epsilon$ and $x \in B(x_0, R)$ (for fixed x_0),

$$\sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot)\|^2 \leq C_1 + C_2 \sup_{0 \leq t \leq T} \|\epsilon I^\epsilon(t, \cdot)\|^2.$$

Let $M > 0$ be such that $M^2 > C_1$. By Lemma 3.2

$$\begin{aligned} & P\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot)\| \geq M \} \\ & \leq P\{ \sup_{0 \leq t \leq T} \|I^\epsilon(t, \cdot)\| \geq (\frac{M^2 - C_1}{C_2})^{\frac{1}{2}} \} \\ & \leq \exp\{-\frac{1}{\epsilon^2} (\frac{M^2 - C_1}{9C_2\eta} - \log 4T)\} \\ & \leq \exp(-\frac{r}{\epsilon^2}), \end{aligned}$$

where r is chosen satisfying $r = \frac{M^2 - C_1}{9C_2\eta} - \log 4T$. □

The proofs of the following two propositions are simple modifications of Proposition 6.2 and Proposition 6.1 in [7] respectively in our setting. For the compactness of this paper we omit them.

PROPOSITION 4.2. $\forall r > 0, \forall \delta > 0 \exists b > 0$ and $\exists \epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0]$ we have

$$(4.2) \quad \mathbf{P}\{ \sup_{0 \leq t \leq T} \|\epsilon I^\epsilon(t, \cdot)\| \geq \delta, \sup_{0 \leq t \leq T} \|\epsilon M(t, \cdot)\| \leq b \} \leq \exp\{-\frac{r}{\epsilon^2}\}.$$

To prove (4.2), X^ϵ is being approximated by the simpler process X_n^ϵ which is defined to be stopped on small intervals.

For $n \in \mathbb{N}$ and $k = 0, \dots, 2^n - 1$ set $t_{n,k} = k \cdot 2^{-n}T$ and $d_{n,k} = [t_{n,k}, t_{n,k+1})$. Let $\pi_n(t) = t_{n,k}$ for $t \in d_{n,k}$. Finally, let

$$X_n^\epsilon(t, x) \equiv \int_0^1 G_{t-\pi_n(t)}(x, y) X^\epsilon(\pi_n(t), y) dy.$$

PROPOSITION 4.3. $\forall r > 0, \forall \delta > 0 \exists n_0, \forall n \geq n_0$, and $\exists \epsilon_n > 0$ such that $\forall \epsilon \in (0, \epsilon_n]$ we have

$$\mathbf{P}\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - X_n^\epsilon(t, \cdot)\| \geq \delta \} \leq \exp\{-\frac{r}{\epsilon^2}\}.$$

PROPOSITION 4.4. Under the assumption (1.2)-(1.5), $\forall r > 0, \forall \delta > 0, \forall h \in \mathcal{H}_T$, there exists $\epsilon_0 > 0$ and $b > 0$ such that $\forall \epsilon \in (0, \epsilon_0]$ we have

$$(4.3) \quad \begin{aligned} & \mathbf{P}\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - z^h(t, \cdot)\| \geq \delta, \\ & \sup_{0 \leq t \leq T} \|\epsilon M(t, \cdot) - \int_0^t \int_0^\cdot \dot{h}(s, y) \pi(ds, dy)\| \leq b\} \\ & \leq \exp\{-\frac{r}{\epsilon^2}\}. \end{aligned}$$

PROOF. Let

$$(4.4) \quad \begin{aligned} \mathcal{N}(\epsilon, b) \equiv & \{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, x) - z^h(t, x)\| \geq \delta, \\ & \sup_{0 \leq t \leq T} \|\epsilon M(t, \cdot) - \int_0^t \int_0^\cdot \dot{h}(s, y) \pi(ds, dy)\| \leq b\} \end{aligned}$$

and $h \in \mathcal{H}_T$. Let W^ϵ and P^ϵ be defined as in the proof of Theorem 3.4. For $\lambda > 0$ set

$$\mathcal{M}(\epsilon, \lambda) = \{ \int_0^T \int_0^1 \dot{h}(t, x) M(ds, dx) \geq -\frac{\lambda}{\epsilon} \}.$$

Obviously, for an arbitrary $\lambda > 0$, we have

$$(4.5) \quad \mathbf{P}\{\mathcal{N}(\epsilon, b)\} \leq \mathbf{P}\{\mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda)\} + \mathbf{P}\{\mathcal{M}^c(\epsilon, \lambda)\}.$$

By Theorem 3.1, we have

$$(4.6) \quad \mathbf{P}\{\mathcal{M}^c(\epsilon, \lambda)\} \leq 3 \exp\{-\frac{\lambda^2}{4\epsilon^2 |h|_{\mathcal{H}_T}^2}\}.$$

Note that

$$(4.7) \quad \begin{aligned} \mathbf{P}\{\mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda)\} &= E^\epsilon[\frac{dP}{dP^\epsilon} : \mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda)] \\ &\leq \exp\{\frac{\lambda}{\epsilon^2} + \frac{|h|_{\mathcal{H}_T}^2}{2\epsilon^2}\} \mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\}. \end{aligned}$$

We have

$$\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\} = \mathbf{P}\left\{\sup_{0 \leq t \leq T} \|\tilde{X}^\epsilon(t, \cdot) - z^h(t, \cdot)\| > \delta, \|\epsilon M(t, \cdot)\| \leq b\right\},$$

where $\tilde{X}(t, x)$ is the solution of the SDE

$$\begin{aligned} \partial_t \tilde{X}^\epsilon(t, x) &= L\tilde{X}^\epsilon(t, x) + b(t, x, \tilde{X}^\epsilon(t, x)) + \epsilon \sigma(t, x, \tilde{X}^\epsilon(t, x))M(dt, dx) \\ X^\epsilon(0, \cdot) &= u_0(\cdot). \end{aligned}$$

Using the same argument as in the proof of Theorem 3.2 we get

$$\begin{aligned} (4.8) \quad & \mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\} \\ & \leq \mathbf{P}\left\{\sup_{0 \leq t \leq T} \left\|\epsilon \int_0^t \int_0^1 G_{t-s}(\cdot, y) \sigma(\tilde{X}^\epsilon(s, y)) M^\epsilon(ds, dy)\right\| \geq \frac{\delta}{C_1}, \right. \\ & \quad \left. \sup_{0 \leq t \leq T} \|\epsilon M(t, \cdot)\| \leq b\right\} \\ & \leq \mathbf{P}\left\{\sup_{0 \leq t \leq T} \left\|\int_0^t \int_0^1 G_{t-s}(\cdot, y) \sigma(\tilde{X}^\epsilon(s, y)) M^\epsilon(ds, dy)\right\| \geq \frac{\delta}{\epsilon C_1}\right\} \\ & \leq 3 \exp\left\{-\frac{\delta^2}{4\epsilon^2 C_1^2 l_0}\right\}, \end{aligned}$$

where C_1 is a constant and l_0 is a constant defined as (3.9). Hence for each $\tilde{a} > a$ we can find b such that

$$(4.9) \quad \mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\} \leq \exp\left\{-\frac{\tilde{a}}{\epsilon^2}\right\}.$$

Combining (4.5) to (4.6), we have

$$\mathbf{P}\{\mathcal{N}(\epsilon, b)\} \leq 3 \exp\left\{-\frac{\lambda^2}{4\epsilon^2 |h|_{\mathcal{H}_T}^2}\right\} + \exp\left\{\frac{2\lambda + |h|_{\mathcal{H}_T}^2 - 2\tilde{a}}{2\epsilon^2}\right\}.$$

If we first choose λ and then \tilde{a} we get (4.4). □

Let

$$(4.10) \quad \mathcal{H}_T^r = \{h \in H : |h|_{\mathcal{H}_T}^2 \leq r\}.$$

PROPOSITION 4.5. $\forall T > 0, \forall r > 0, \forall \delta > 0, \forall \gamma > 0$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0]$ we have

$$(4.11) \quad \mathbf{P}\{\text{dist}_{C_{[0,T] \times [0,1]}}(\epsilon M, \mathcal{H}_T^r) \geq \delta\} \leq \exp\left\{-\frac{r - \gamma}{2\epsilon^2}\right\},$$

where $\text{dist}_{C_{[0,T] \times [0,1]}}(\epsilon M, \mathcal{H}_T^r)$ means the distance of ϵM to the space, \mathcal{H}_T^r in the metric of $C_{[0,T] \times [0,1]}$.

PROOF. As before, let

$$\frac{dP}{dP^\epsilon} = \exp\left\{-\frac{1}{\epsilon} \int_0^T \int_0^1 1 M(ds, dy) - \frac{1}{2\epsilon^2} \int_0^T \int_0^1 \dot{h}^2(s, y)\pi(ds, dy)\right\}.$$

Also let

$$C_r(\epsilon) = \left\{ \inf_{h \in \mathcal{H}_T^r} \left| \epsilon M(T, 1) - \int_0^T \int_0^1 \dot{h}(s, y)\pi(ds, dy) \right| \geq \delta, |h|_{\mathcal{H}_T}^2 \leq r \right\}.$$

Then

$$\begin{aligned} P\{C_r(\epsilon)\} &= E^\epsilon \left\{ \frac{dP}{dP^\epsilon} : C_r(\epsilon) \right\} \\ &\leq \exp\left\{-\frac{r - \gamma}{2\epsilon^2}\right\}. \end{aligned}$$

□

THEOREM 4.6 (UPPER BOUND). Under the condition (1.2)-(1.5), $\forall T > 0, \forall y \in R, \forall \delta > 0, \forall R > 0, \forall r \geq 0, \forall \gamma > 0$, there exists $\epsilon_0 > 0$ such that $\forall x \in R : |x - y| \leq R$ and $\forall \epsilon \in (0, \epsilon_0]$ we have

$$\mathbf{P}\{\text{dist}_{C_{[0,T] \times [0,1]}}(X^\epsilon(\cdot, x), \mathcal{K}(r)) \geq \delta\} \leq \exp\left\{-\frac{r - \gamma}{2\epsilon^2}\right\}.$$

PROOF. By Proposition 4.4, $\forall h \in \mathcal{H}_T$, there exist $b_h > 0$ and $\epsilon_h > 0$ such that $\forall \epsilon \in (0, \epsilon_h]$ and $\forall x \in B(x_0, R)$,

$$(4.12) \quad \begin{aligned} &P\left\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - z^h(t, \cdot)\| \geq \delta, \right. \\ &\left. \sup_{0 \leq t \leq T} \left\| \epsilon M(t, \cdot) - \int_0^t \int_0^1 \dot{h}(s, y)\pi(ds, dy) \right\| \leq b_h \right\} \\ &\leq \exp\left\{-\frac{a}{\epsilon^2}\right\}. \end{aligned}$$

If we sum over all $h \in \mathcal{H}_T$ satisfying $|h|_{\mathcal{H}_T}^2 \leq r$,

$$\mathcal{H}_T^r \subset \bigcup \left\{ u \in C_{[0,T] \times [0,1]} : \sup_{0 \leq t \leq T} \left\| u(t, \cdot) - \int_0^t \int_0^{\cdot} \dot{h}(s, y) \pi(ds, dy) \right\| \leq b_h \right\}.$$

Since \mathcal{H}_T^r is compact, we may find $\{h_1, \dots, h_l\}$, where each of h_i satisfies $|h_i|_{\mathcal{H}_T}^2 \leq r$ such that

$$\begin{aligned} \mathcal{H}_T^r \subset \bigcup_{i=1}^l \left\{ u \in C_{[0,T] \times [0,1]} : \sup_{0 \leq t \leq T} \left\| u(t, \cdot) \right. \right. \\ \left. \left. - \int_0^t \int_0^{\cdot} \dot{h}(s, y) \pi(ds, dy) \right\| \leq b_{h_i} \right\} \equiv \tilde{\mathcal{H}}. \end{aligned}$$

(4.13)

$$\begin{aligned} & P\{\text{dist}_{C_{[0,T] \times [0,1]}}(X^\epsilon(\cdot, \cdot), \mathcal{K}(r)) \geq \delta\} \\ & \leq P\{\text{dist}_{C_{[0,T] \times [0,1]}}(X^\epsilon(\cdot, \cdot), \mathcal{K}(r)) \geq \delta, \epsilon M \in \tilde{\mathcal{H}}\} + P\{\epsilon M \notin \tilde{\mathcal{H}}\}. \end{aligned}$$

Let $\epsilon_1 = \min\{\epsilon_{h_1}, \dots, \epsilon_{h_l}\}$. From (4.12) for $\epsilon \in (0, \epsilon_1]$ we have

(4.14)

$$\begin{aligned} & P\{\text{dist}_{C_{[0,T] \times [0,1]}}(X^\epsilon(\cdot, \cdot), \mathcal{K}(r)) \geq \delta, \epsilon M \in \tilde{\mathcal{H}}\} \\ & \leq \sum_{i=1}^l P\left\{ \sup_{0 \leq t \leq T} \|X^\epsilon(t, \cdot) - z^{h_i}(t, \cdot)\| \geq \delta, \right. \\ & \quad \left. \sup_{0 \leq t \leq T} \left\| \epsilon M(t, \cdot) - \int_0^t \int_0^{\cdot} \dot{h}(s, y) \pi(ds, dy) \right\| \leq b_{h_i} \right\} \\ & \leq l \cdot \exp\left\{-\frac{a}{\epsilon^2}\right\}. \end{aligned}$$

Since \mathcal{H}_T^r is compact, there exists $\tilde{\delta} > 0$ such that

$$\{u \in C_{[0,T] \times [0,1]} : \text{dist}_{C_{[0,T] \times [0,1]}}(\epsilon M, \mathcal{H}_T^r) < \tilde{\delta}\} \subset \tilde{\mathcal{H}}.$$

By Proposition 4.5, there exists $\epsilon_2 > 0$ such that $\forall \epsilon \in [0, \epsilon_2]$,

$$\begin{aligned} (4.15) \quad & P\{\epsilon M \notin \tilde{\mathcal{H}}\} \leq P\{\text{dist}_{C_{[0,T] \times [0,1]}}(\epsilon M, \mathcal{H}_T^r) \geq \tilde{\delta}\} \\ & \leq \exp\left\{\frac{r - \tilde{\gamma}}{2\epsilon^2}\right\}. \end{aligned}$$

Let $\epsilon_3 = \epsilon_1 \wedge \epsilon_2$. (4.13)-(4.15) imply that for all $\epsilon \in (0, \epsilon_3]$ and $x \in \mathcal{B}(x_0, R)$, letting $2a > r - \tilde{\gamma}$,

$$\begin{aligned} & P\{\text{dist}_{C_{[0,T] \times [0,1]}}(X^\epsilon(\cdot, \cdot), \mathcal{K}(r)) \geq \delta\} \\ & \leq l \exp\left\{-\frac{a}{\epsilon^2}\right\} + \exp\left\{-\frac{r - \tilde{\gamma}}{2\epsilon^2}\right\} \\ & \leq \exp\left\{-\frac{r - \tilde{\gamma}}{2\epsilon^2}\right\}, \end{aligned}$$

for all $\epsilon \in (0, \epsilon_0]$ and $\forall x \in B(x_0, R)$. □

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