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AN ALGORITHM FOR FINDING THE DISTANCE BETWEEN TWO ELLIPSES

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ABSTRACT. We are interested in the distance problem between two objects in three dimensional Euclidean space. There are many distance problems for various types of objects including line segments, boxes, polygons, circles, disks, etc. In this paper we present an iterative algorithm for finding the distance between two given ellipses. Numerical examples are given.

1. Introduction and preliminaries

The distance problem between two given objects in three dimensional space can be found often in computer-aided geometric design systems. Further, it is important to propose an efficient algorithm for finding the distance between two objects. There are many distance problems for various types of objects including line segments [5], boxes [6], polygons [8], circles [7], disks [1], etc. In the literature, many problems already have been studied and various numerical techniques to compute the optimal distance have been given. In this paper we consider the problem of finding the distance between two given ellipses. The representation of an ellipse in three dimensional space can be given by using a geometric transformation of a standard ellipse in the $xy-$plane. This may simplify the distance function between the two ellipses. Thus, our problem is reduced to the distance problem between one standard ellipse and the other ellipse. We can present an iterative algorithm which is mainly based on computing the distance between a given point and a standard ellipse.

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The standard form of the equation of an ellipse centered with origin in the $xy-$plane can be represented by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1.1}$$

where a and b are the semimajor axis and the semiminor axis respectively. Further, the parametric form of this standard ellipse can be given by

$$x=a\cos t, y=b\sin t, \tag{1.2}$$

where $-\pi \leq t \leq \pi$.

Also, one may characterize an ellipse in three dimensional Euclidean space by giving the three points c, p and q, where c is its center and p denotes one of the ends of the major axis and q denotes one of the ends of the minor axis. Let E be an arbitrary ellipse defined by its center $c=(c_1,c_2,c_3)^T$ and the two points $p=(p_1,p_2,p_3)^T$ and $q=(q_1,q_2,q_3)^T$. Then E can be represented by a geometric transformation(rotation and translation) of a standard ellipse.

Let us consider a relative standard ellipse E_s with respect to E:

$$E_s \;:\; \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1.3}$$

in its parametric form

$$x=a\cos t, y=b\sin t, \tag{1.4}$$

where

$$a=|p-c|=\sqrt{(p_1-c_1)^2+(p_2-c_2)^2+(p_3-c_3)^2}, \tag{1.5}$$

$$b=|q-c|=\sqrt{(q_1-c_1)^2+(q_2-c_2)^2+(q_3-c_3)^2}. \tag{1.6}$$

Then, by using a 3×3 rotation matrix $R=(r_{ij})$ given by

$$\begin{aligned}(r_{11},r_{21},r_{31})^T &= Re_1 = R(1,0,0)^T = \frac{p-c}{|p-c|},\\ (r_{12},r_{22},r_{32})^T &= Re_2 = R(0,1,0)^T = \frac{q-c}{|q-c|},\\ (r_{13},r_{23},r_{33})^T &= Re_3 = R(0,0,1)^T = \frac{(p-c)\times(q-c)}{|(p-c)\times(q-c)|}.\end{aligned} \tag{1.7}$$

the coordinate $\widetilde{w}=(\widetilde{x},\widetilde{y},\widetilde{z})^T$ of a point on E can be represented by

$$\widetilde{w}=Rw+c, \tag{1.8}$$

where the coordinate $w=(x,y,0)^T$ is the corresponding coordinate of a point of a standard ellipse E_s with respect to $\widetilde{w}$.

2. Distance between a point and a standard ellipse

Let there be a point $s = (s_1, s_2, s_3)^T$ and a standard ellipse defined by

$$E_s : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.1}$$

in its parametric form

$$x = a\cos t, y = b\sin t. \tag{2.2}$$

We consider the problem of computing the distance between s and E_s and finding the optimal point lying on E_s.

Suppose that $Q(s, E_s; t)$ is a function of a parameter t with respect to s and E_s defined by

$$Q(s, E_s; t) = (s_1 - a\cos t)^2 + (s_2 - b\sin t)^2. \tag{2.3}$$

Then, by finding the optimal point $\hat{w} = (\hat{x}, \hat{y}, 0)^T = (a\cos\hat{t}, b\sin\hat{t}, 0)^T$ on E_s such that

$$\begin{aligned} m = Q(s, E_s; \hat{t}) &= \min_{-\pi \le t \le \pi} Q(s, E_s; t) \\ &= \min_{-\pi \le t \le \pi} (s_1 - a\cos t)^2 + (s_2 - b\sin t)^2 \end{aligned} \tag{2.4}$$

we can compute the distance $d(s, E)$ given by $d(s, E) = \sqrt{m + s_3{}^2}$.

The necessary condition $\dfrac{\partial Q}{\partial t} = 0$ for a minimum induces the following equation:

$$A\sin t - B\cos t + C\sin t\cos t = 0 \tag{2.5}$$

with $A = as_1$, $B = bs_2$ and $C = (b^2 - a^2)$.
Thus, if $C = 0$ in (2.5), then we can easily find $t = \hat{t}$ by

$$\hat{t} = \arctan\left(\frac{B}{A}\right). \tag{2.6}$$

In case of $C = c_2^2 - c_1^2 \neq 0$ the equation (2.5) induces the following two equations:

$$A\sin t - B\cos t + C\sin t\cos t = 0 \;\; (0 \le t \le \pi) \tag{2.7}$$

and

$$A\sin t + B\cos t + C\sin t\cos t = 0 \;\; (-\pi \le t \le 0). \tag{2.8}$$

Thus, if we let $v = \cos t$ in (2.7) and (2.8), then the corresponding values of $\sin t$ can be given by

$$\sin t = \begin{cases} \sqrt{1-v^2} & \text{if } \; 0 \le t \le \pi \\ -\sqrt{1-v^2} & \text{if } \; -\pi \le t \le 0, \end{cases}$$

and the equation (2.5) leads to the following quartic equation:

$$(2.9) \quad v^4 + 2\left(\frac{A}{C}\right)v^3 + \left(\frac{A^2+B^2-C^2}{C^2}\right)v^2 - 2\left(\frac{A}{C}\right)v - \left(\frac{A}{C}\right)^2 = 0.$$

By solving the equation (2.9) for v we have some solutions v_j $(j = 1, 2, \ldots, s)$ such that $-1 \le v_j \le 1$ and $1 \le s \le 4$, and get the value $\check{t}_j$ for each v_j such that $\cos \check{t}_j = v_j$ $(0 \le \check{t}_j \le \pi)$. Thus, t has two values $\mu_j^1 = \check{t}_j$ and $\mu_j^2 = -\check{t}_j$ for each $\check{t}_j$, and the corresponding values of $\sin t$ can be given by

$$(2.10) \quad \begin{aligned} \sin \mu_j^1 &= \sqrt{1-(v_j)^2}\,(j = 1, 2, \ldots, s), \\ \sin \mu_j^2 &= -\sqrt{1-(v_j)^2}\,(j = 1, 2, \ldots, s). \end{aligned}$$

In this case we can choose $t = \hat{t} = \mu_l^m$ for the solution of (2.5) such that

$$(2.11) \quad \begin{aligned} &(s_1 - a\cos\mu_l^m)^2 + (s_2 - b\sin\mu_l^m)^2 \\ &= \min_{\substack{i=1,2 \\ j=1,2,\ldots,s}} \left[(s_1 - a\cos\mu_j^i)^2 + (s_2 - b\sin\mu_j^i)^2\right]. \end{aligned}$$

Thus, we can find the optimal point $\hat{w} = (\hat{x}, \hat{y}, 0)^T = (a\cos\hat{t}, b\sin\hat{t}, 0)^T$ on E_s.

3. Distance between two ellipses

Let E^α and E^β be the two given ellipses in the three dimensional space. Suppose that E^α is defined by its center $c^\alpha = (c_1^\alpha, c_2^\alpha, c_3^\alpha)^T$ and the two points $p^\alpha = (p_1^\alpha, p_2^\alpha, p_3^\alpha)^T$ and $q^\alpha = (q_1^\alpha, q_2^\alpha, q_3^\alpha)^T$ and E^β is defined by $c^\beta = (c_1^\beta, c_2^\beta, c_3^\beta)^T$, $p^\beta = (p_1^\beta, p_2^\beta, p_3^\beta)^T$ and $q^\beta = (q_1^\beta, q_2^\beta, q_3^\beta)^T$. Then we can give the corresponding two standard ellipses E_s^α and E_s^β defined by

$$(3.1) \quad \begin{aligned} E_s^\alpha \;&:\; w^\alpha = (a^\alpha \cos t^\alpha, b^\alpha \sin t^\alpha, 0)^T, \qquad -\pi \le t^\alpha \le \pi \\ E_s^\beta \;&:\; w^\beta = (a^\beta \cos t^\beta, b^\beta \sin t^\beta, 0)^T, \qquad -\pi \le t^\beta \le \pi, \end{aligned}$$

where

$$(3.2) \qquad a^\alpha = |p^\alpha - c^\alpha| = \sqrt{(p_1^\alpha - c_1^\alpha)^2 + (p_2^\alpha - c_2^\alpha)^2 + (p_3^\alpha - c_3^\alpha)^2},$$

$$(3.3) \qquad b^\alpha = |q^\alpha - c^\alpha| = \sqrt{(q_1^\alpha - c_1^\alpha)^2 + (q_2^\alpha - c_2^\alpha)^2 + (q_3^\alpha - c_3^\alpha)^2},$$

$$(3.4) \qquad a^\beta = |p^\beta - c^\beta| = \sqrt{(p_1^\beta - c_1^\beta)^2 + (p_2^\beta - c_2^\beta)^2 + (p_3^\beta - c_3^\beta)^2},$$

$$(3.5) \qquad b^\beta = |q^\beta - c^\beta| = \sqrt{(q_1^\beta - c_1^\beta)^2 + (q_2^\beta - c_2^\beta)^2 + (q_3^\beta - c_3^\beta)^2}.$$

Also, the coordinate $\widetilde{w^\alpha} = \left(\widetilde{x^\alpha}, \widetilde{y^\alpha}, \widetilde{z^\alpha}\right)^T$ of a point of E^α and the coordinate $\widetilde{w^\beta} = \left(\widetilde{x^\beta}, \widetilde{y^\beta}, \widetilde{z^\beta}\right)^T$ of a point of E^β can be represented by

$$(3.6) \qquad \begin{aligned} &E^\alpha \ : \ \widetilde{w^\alpha} = R^\alpha w^\alpha + c^\alpha \\ &= R^\alpha(a^\alpha \cos t^\alpha, b^\alpha \sin t^\alpha, 0)^T + (c_1^\alpha, c_2^\alpha, c_3^\alpha)^T, \ -\pi \le t^\alpha \le \pi \\ &E^\beta \ : \ \widetilde{w^\beta} = R^\beta w^\beta + c^\beta \\ &= R^\beta(a^\beta \cos t^\beta, b^\beta \sin t^\beta, 0)^T + (c_1^\beta, c_2^\beta, c_3^\beta)^T, \ -\pi \le t^\beta \le \pi, \end{aligned}$$

where the two 3×3 rotation matrices $R^\alpha = (r_{ij}^\alpha)$ and $R^\beta = (r_{ij}^\beta)$ are given by the following:
For $k = \alpha$ or β

$$(3.7) \qquad \begin{aligned} (r_{11}^k, r_{21}^k, r_{31}^k)^T &= R^k e_1 = R^k(1,0,0)^T = \frac{p^k - c^k}{|p^k - c^k|}, \\ (r_{12}^k, r_{22}^k, r_{32}^k)^T &= R^k e_2 = R^k(0,1,0)^T = \frac{q^k - c^k}{|q^k - c^k|}, \\ (r_{13^k}, r_{23}^k, r_{33}^k)^T &= R^k e_3 = R^k(0,0,1)^T = \frac{(p^k - c^k) \times (q^k - c^k)}{|(p^k - c^k) \times (q^k - c^k)|}. \end{aligned}$$

Furthermore, since Euclidean metric is invariant under $R^\alpha = (r_{ij}^\alpha)$ and $R^\beta = (r_{ij}^\beta)$ the distance between E^α and E^β can be given by

$$\begin{aligned} d(E^\alpha, E^\beta) &= \min_{\substack{u \in E^\alpha \\ v \in E^\beta}} |u - v| \\ &= \min_{\substack{-\pi \le t^\alpha \le \pi \\ -\pi \le t^\beta \le \pi}} |(R^\alpha w^\alpha + c^\alpha) - (R^\beta w^\beta + c^\beta)| \end{aligned}$$

$$= \min_{\substack{-\pi\le t^\alpha\le\pi \\ -\pi\le t^\beta\le\pi}} \left|R^\alpha\left(w^\alpha - (R^\alpha)^{-1}R^\beta w^\beta - (R^\alpha)^{-1}(c^\beta - c^\alpha)\right)\right|$$
$$= \min_{\substack{-\pi\le t^\alpha\le\pi \\ -\pi\le t^\beta\le\pi}} \left|w^\alpha - (R^\alpha)^{-1}\left(R^\beta w^\beta + (c^\beta - c^\alpha)\right)\right|$$

or

$$d(E^\alpha, E^\beta) = \min_{\substack{u\in E^\alpha \\ v\in E^\beta}} |u - v| \tag{3.9}$$
$$= \min_{\substack{-\pi\le t^\beta\le\pi \\ -\pi\le t^\alpha\le\pi}} \left|w^\beta - (R^\beta)^{-1}\left(R^\alpha w^\alpha + (c^\alpha - c^\beta)\right)\right|.$$

We now describe an iteration algorithm for finding the optimal points $\widetilde{w^\alpha} = \left(\widetilde{x^\alpha}, \widetilde{y^\alpha}, \widetilde{z^\alpha}\right)^T$ and $\widetilde{w^\beta} = \left(\widetilde{x^\beta}, \widetilde{y^\beta}, \widetilde{z^\beta}\right)^T$ lying on E^α and E^β respectively. Our algorithm is mainly based on computing the distance between a point and a standard ellipse.

Algorithm:

Step 0. Compute $a^\alpha, b^\alpha, a^\beta, b^\beta$ from (3.2), (3.3), (3.4) and (3.5). Define the rotation matrices R^α and R^β and their inverse matrices $(R^\alpha)^{-1}$ and $(R^\beta)^{-1}$ from (3.7). Set $j := 0$.

Step 1. Let t_0^α be given as an initial value for t^α. Then, from (3.1) and (3.6) we can initialize $w_0^\alpha = (x_0^\alpha, y_0^\alpha, z_0^\alpha)^T = (a^\alpha\cos t_0^\alpha, b^\alpha \sin t_0^\alpha, 0)^T$ and $\widetilde{w_0^\alpha} = \left(\widetilde{x_0^\alpha}, \widetilde{y_0^\alpha}, \widetilde{z_0^\alpha}\right)^T = R^\alpha w_0^\alpha + c^\alpha$.

We consider the following minimization problem:

To find $w_0^\beta = (a^\beta\cos t_0^\beta, b^\beta\sin t_0^\beta, 0)^T$ and $\widetilde{w_0^\beta} = \left(\widetilde{x_0^\beta}, \widetilde{y_0^\beta}, \widetilde{z_0^\beta}\right)^T = R^\beta w_0^\beta + c^\beta$ such that

$$\left|\widetilde{w_0^\beta} - \widetilde{w_0^\alpha}\right| = \min_{w^\beta\in E_s^\beta} \left|w^\beta - (R^\beta)^{-1}\left(R^\alpha w_0^\alpha + (c^\alpha - c^\beta)\right)\right|. \tag{3.10}$$

Thus, let $s = (R^\beta)^{-1}\left(R^\alpha w_0^\alpha + (c^\alpha - c^\beta)\right)$ and $E_s = E_s^\beta$ with $a = a^\beta$ and $b = b^\beta$ in (2.3). Then we can find the optimal point $\widehat{w} = (\widehat{x}, \widehat{y}, 0) = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ on E_s^β such that

$$Q(s, E_s; \widehat{t}) = \min_{-\pi\le t\le\pi} Q(s, E_s; t). \tag{3.11}$$

Set $w_0^\beta = (a\cos t_0^\beta, b\sin t_0^\beta, 0)^T = \widehat{w} = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ and $\widetilde{w_0^\beta} = R^\beta w_0^\beta + c^\beta$.

Step 2. Find $w^\alpha_{j+1} = (x^\alpha_{j+1}, y^\alpha_{j+1}, z^\alpha_{j+1})^T = (a^\alpha \cos t^\alpha_{j+1}, b^\alpha \sin t^\alpha_{j+1}, 0)^T$ and $\widetilde{w^\alpha_{j+1}} = \left(\widetilde{x^\alpha_{j+1}}, \widetilde{y^\alpha_{j+1}}, \widetilde{z^\alpha_{j+1}}\right)^T = R^\alpha w^\alpha_{j+1} + c^\alpha$ such that

$$\left|\widetilde{w^\alpha_{j+1}} - \widetilde{w^\beta_{j+1}}\right| = \min_{w^\alpha \in E^\alpha_s} \left|w^\alpha - (R^\alpha)^{-1}\left(R^\beta w^\beta_j + (c^\beta - c^\alpha)\right)\right|. \tag{3.12}$$

That is, let $s = (R^\alpha)^{-1}\left(R^\beta w^\beta_j + (c^\beta - c^\alpha)\right)$ and $E_s = E^\alpha_s$ with $a = a^\alpha$ and $b = b^\alpha$ in (2.3). Then, the optimal point $\widehat{w} = (\widehat{x}, \widehat{y}, 0) = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ on E^α_s such that

$$Q(s, E_s; \widehat{t}) = \min_{-\pi \le t \le \pi} Q(s, E_s; t). \tag{3.13}$$

Set $w^\alpha_{j+1} = (a\cos t^\alpha_{j+1}, b\sin t^\alpha_{j+1}, 0)^T = \widehat{w} = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ and $\widetilde{w^\alpha_{j+1}} = R^\alpha w^\alpha_{j+1} + c^\alpha$. Further, find $w^\beta_{j+1} = (a^\beta \cos t^\beta_{j+1}, b^\beta \sin t^\beta_{j+1}, 0)^T$ and $\widetilde{w^\beta_{j+1}} = \left(\widetilde{x^\beta_{j+1}}, \widetilde{y^\beta_{j+1}}, \widetilde{z^\beta_{j+1}}\right)^T = R^\beta w^\beta_{j+1} + c^\beta$ such that

$$\left|\widetilde{w^\beta_{j+1}} - \widetilde{w^\alpha_{j+1}}\right| = \min_{w^\beta \in E^\beta_s} \left|w^\beta - (R^\beta)^{-1}\left(R^\alpha w^\alpha_j + (c^\alpha - c^\beta)\right)\right|. \tag{3.14}$$

In other words, let $s = (R^\beta)^{-1}\left(R^\alpha w^\alpha_j + (c^\alpha - c^\beta)\right)$ and $E_s = E^\beta_s$ with $a = a^\beta$ and $b = b^\beta$ in (2.3). Then we obtain the optimal point $\widehat{w} = (\widehat{x}, \widehat{y}, 0) = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ on E^β_s such that

$$Q(s, E_s; \widehat{t}) = \min_{-\pi \le t \le \pi} Q(s, E_s; t). \tag{3.15}$$

Set $w^\beta_{j+1} = \widehat{w} = (a\cos\widehat{t}, b\sin\widehat{t}, 0)^T$ and $\widetilde{w^\beta_{j+1}} = R^\beta w^\beta_{j+1} + c^\beta$. Define a distance function U between two optimal points $\widetilde{w^\alpha_k}$ and $\widetilde{w^\beta_k}$ on E^α and E^β respectively, given by $U(k) = \left|\widetilde{w^\alpha_k} - \widetilde{w^\beta_k}\right|$ for $k = 1, 2, \ldots$. And if $U(j+1) < U(j)$, then we set $j := j + 1$ and go back to **step 1**.

Here, we see that for $k = 0, 1, 2, \ldots$ two optimal points $\widetilde{w^\alpha_k}$ and $\widetilde{w^\beta_k}$ can be found after k iterations with an initial value $\widetilde{w^\alpha_0}$. Further, though the proof is not given the convergence of our algorithm may be guaranteed by a descent property : $U(k+1) \le U(k)$ for $k = 0, 1, 2, \ldots$. Thus, two sequences $\left(\widetilde{w^\alpha_n}\right)$ and $\left(\widetilde{w^\beta_n}\right)$ of optimal points converge to the global optimal points $\widetilde{w^\alpha}$ and $\widetilde{w^\beta}$ respectively, and the optimal distance between two ellipses E^α and E^β is given by $d(E^\alpha, E^\beta) = \left|\widetilde{w^\alpha} - \widetilde{w^\beta}\right|$.

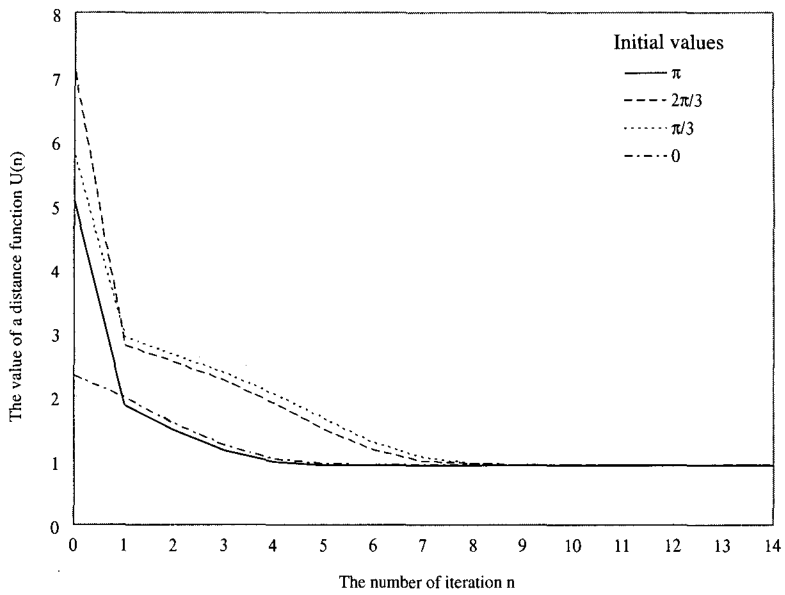

FIGURE 1. Convergence of the distance function for different initial values.

Finally, to test our algorithm we give an example for finding the distance between two ellipses E^α and E^β in the three dimensional space. We can see the convergence of the corresponding distance function $U(k)$ for each initial value t_0^α. In fact, it is certain that each function $U(k)$ converges to the distance between two given ellipses E^α and E^β which is the global minimum.

EXAMPLE. Let there be two ellipses E^α and E^β. E^α is characterized by its center $c^\alpha = (c_1^\alpha, c_2^\alpha, c_3^\alpha)^T = (1,1,1)^T$ and the two points $p^\alpha = (p_1^\alpha, p_2^\alpha, p_3^\alpha)^T = (2,3,2)^T$ and $q^\alpha = (q_1^\alpha, q_2^\alpha, q_3^\alpha)^T = (5,2,-5)^T$. Also, E^β is defined by $c^\beta = (c_1^\beta, c_2^\beta, c_3^\beta)^T = (3,5,5)^T$, $p^\beta = (p_1^\beta, p_2^\beta, p_3^\beta)^T = (1,2,5)^T$ and $q^\beta = (q_1^\beta, q_2^\beta, q_3^\beta)^T = (6,3,2)^T$. We employ our algorithm for finding the distance between two ellipses E^α and E^β by using four different values $t_0^\alpha = \pi$, $t_0^\alpha = \frac{2\pi}{3}$, $t_0^\alpha = \frac{\pi}{3}$ and $t_0^\alpha = 0$ as initial values for t^α . Then we have the corresponding four initial optimal points $\widetilde{w_0^\alpha} = (0,-1,0)^T$, $\widetilde{w_0^\alpha} = (3.9641, 0.8660, -4.6962)^T$, $\widetilde{w_0^\alpha} = (4.9641, 2.8660, -3.6962)^T$ and $\widetilde{w_0^\alpha} = (2,3,2)^T$. Moreover, using these initial optimal points we find the same optimal points $\widetilde{w^\alpha} = (-0.0827, 2.2560, 4.8051)^T$ and $\widetilde{w^\beta} = (0.6134, 2.2973, 5.4048)^T$ in each case. We can obtain them after the same 13 iterations for $t_0^\alpha = \frac{2\pi}{3}$ and $t_0^\alpha = \frac{\pi}{3}$. Also the optimal points can be found after 10 iterations for each of $t_0^\alpha = \pi$ and $t_0^\alpha = 0$. The distance is given by $d(E^\alpha, E^\beta) = |\widetilde{w^\alpha} - \widetilde{w^\beta}| = 0.9198$. Further, in Figure 1 we see the convergence of the distance function $U(k)$ to the optimal distance between two ellipses in each case.

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