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GENERALIZATIONS OF GAUSS'S SECOND SUMMATION THEOREM AND BAILEY'S FORMULA FOR THE SERIES ${}_2F_1(1/2)$

Arjun K. Rathie, Yong Sup Kim, and Junesang Choi

Abstract. We aim mainly at presenting two generalizations of the well-known Gauss's second summation theorem and Bailey's formula for the series ${}_2F_1(1/2)$. An interesting transformation formula for ${}_pF_q$ is obtained by combining our two main results. Relevant connections of some special cases of our main results with those given here or elsewhere are also pointed out.

1. Introduction and preliminaries

We start with Kummer's theorem [2]

$$(1.1)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ 1+a-b \end{matrix}\,\middle|\, -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+a-b)}{2^a\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b\right)},$$

Gauss's second summation theorem [2]

$$(1.2)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) \end{matrix}\,\middle|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)},$$

and Bailey's formula [2]

$$(1.3)\qquad {}_2F_1\left[\begin{matrix} a, & 1-a \\ & c \end{matrix}\,\middle|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}.$$

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As Bailey pointed out in his tract [2], the summation theorems (1.2) and (1.3) can be obtained from the following result [2]

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, \frac{1}{2}\right] = 2^a\, {}_2F_1\left[\begin{matrix} a,\ c-b \\ c \end{matrix} \,\middle|\, -1\right] \tag{1.4} $$

by taking $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$ and $b = 1 - a$ respectively and using (1.1).

In 1996, Lavoie, Grondin and Rathie [3] generalized the above summation theorems (1.1) to (1.3) and obtained the explicit expressions for

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ 1+a-b+i \end{matrix} \,\middle|\, -1\right], \tag{1.5} $$

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\middle|\, \frac{1}{2}\right], \tag{1.6} $$

and

$$ {}_2F_1\left[\begin{matrix} a,\ 1-a+i \\ c \end{matrix} \,\middle|\, \frac{1}{2}\right] \tag{1.7} $$

for $i = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5$.

In 1927, Whipple [5] generalized the Kummer's theorem (1.1) in the form

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, -1\right] \\ &= \frac{\Gamma(c)\,\Gamma\left(\frac{1}{2}b + \frac{1}{2}c\right)}{\Gamma(b+c)\,\Gamma\left(\frac{1}{2}c - \frac{1}{2}b\right)} \\ &\quad \times {}_3F_2\left[\begin{matrix} b,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(b+c-a+1) \\ b+c-a,\ \frac{1}{2}(b+c+1) \end{matrix} \,\middle|\, 1\right] \end{aligned} \tag{1.8} $$

and in 1929, Bailey [1] generalized the Kummer's theorem (1.1) in the form

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, -1\right] \\ &= \frac{\Gamma\left(\frac{1}{2}(b+c+1)\right)\,\Gamma\left(\frac{1}{2}(a-b+c+1)\right)}{\Gamma\left(\frac{1}{2}(c-b+1)\right)\,\Gamma\left(\frac{1}{2}(a+b+c+1)\right)} \\ &\quad \times {}_7F_6\left[\begin{matrix} \frac{1}{2}b,\ \frac{1}{2}b+\frac{1}{2},\ \frac{1}{2}a,\ \frac{1}{2}(b+c-1),\ \frac{1}{4}(b+c+3), \\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{4}(b+c-1),\ \frac{1}{4}(a+b+c+3), \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \frac{1}{4}(b+c-a-1),\ \frac{1}{4}(b+c-a+1) \\ \frac{1}{4}(a+b+c+1),\ \frac{1}{2}(1-a+b+c) \end{matrix} \,\middle|\, 1\right]. \end{aligned} \tag{1.9} $$

The authors aim mainly at providing generalizations of (1.2) and (1.3) by using the results (1.8) and (1.9). An interesting transformation formula for $_pF_q$ is obtained by just combining their main results. Relevant connections of some special cases of their main results with those given here or elsewhere are also pointed out.

2. Generalizations of (1.2) and (1.3)

The following two generalizations of the results (1.2) and (1.3) will be established:

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\middle|\frac{1}{2}\right]\\ &= \frac{2^a\,\Gamma(c)\,\Gamma\left(c-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma(2c-b)}\\ &\quad\times {}_3F_2\left[\begin{matrix}c-b,\ c-\frac{1}{2}b-\frac{1}{2}a,\ c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\\ c-\frac{1}{2}b+\frac{1}{2},\ 2c-a-b\end{matrix}\middle|1\right]\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\middle|\frac{1}{2}\right]\\ &= \frac{2^a\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}\\ &\quad\times {}_7F_6\left[\begin{matrix}c-\frac{1}{2}b-\frac{1}{2},\ \frac{1}{2}c-\frac{1}{4}b+\frac{3}{4},\ \frac{1}{2}c-\frac{1}{2}b,\ \frac{1}{2}c-\frac{1}{2}b+\frac{1}{2},\\ \frac{1}{2}c-\frac{1}{4}b-\frac{1}{4},\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+\frac{1}{4}a-\frac{1}{4}b+\frac{3}{4},\end{matrix}\right.\\ &\qquad\qquad\left.\begin{matrix}\frac{1}{2}c-\frac{1}{4}a-\frac{1}{4}b-\frac{1}{4},\ \frac{1}{2}c-\frac{1}{4}a-\frac{1}{4}b+\frac{1}{4},\ \frac{1}{2}a\\ \frac{1}{2}c+\frac{1}{4}a-\frac{1}{4}b+\frac{1}{4},\ c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\end{matrix}\middle|1\right].\end{aligned}\tag{2.2}$$

PROOF OF (2.1) AND (2.2). For convenience, we rewrite (1.4) here:

$$ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\middle|\frac{1}{2}\right] = 2^a\,{}_2F_1\left[\begin{matrix}a,\ c-b\\ c\end{matrix}\middle|-1\right],\tag{2.3}$$

which, given in Bailey's tract [2], can be obtained from the following result:

$$\begin{aligned}&(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\middle|-\frac{z}{1-z}\right]\\ &= {}_2F_1\left[\begin{matrix}a,\ c-b\\ c\end{matrix}\middle|z\right]\quad\left(|z|<1;\ \Re(z)<\frac{1}{2}\right)\end{aligned}\tag{2.4}$$

by taking $z \to -1$.

The derivations of (2.1) and (2.2) are straightforward. In fact, if we use the result (1.8) by replacing b by $c-b$ in the right-hand side of (2.3) or (1.4), we get (2.1). Similarly, if we use the result (1.9) by replacing b by $c-b$ in the right-hand side of (2.3), we get (2.2).

Furthermore we shall present another method of proof of the result (2.3). For this, recall the integral representation for ${}_2F_1$ [2]:

$$(2.5)\quad {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt,$$

which, upon taking $z=-1$ and replacing b by $c-b$, yields

$$(2.6)\quad \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ c-b \\ c \end{matrix}\,\middle|\, -1\right] \\ &= \frac{\Gamma(c)}{\Gamma(c-b)\,\Gamma(b)} \int_0^1 t^{c-b-1}\,(1-t)^{b-1}\,(1+t)^{-a}\,dt. \end{aligned}$$

On the other hand, we put $z=1/2$ in (2.5) to get

$$(2.7)\quad {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, \frac{1}{2}\right] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\left(1-\frac{t}{2}\right)^{-a} dt,$$

which, upon taking $1-t=u$ and replacing u by t in the resulting identity, becomes

$$(2.8)\quad {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, \frac{1}{2}\right] = \frac{2^a\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{c-b-1}\,(1-t)^{b-1}\,(1+t)^{-a}\,dt.$$

Now it is easy to get (2.3) by combining (2.6) and (2.8).

By equating (2.1) and (2.2), and using a well-known transformation formula of ${}_3F_2$ (see [2, p. 14, Eq. (1)]), we get the following transformation formula:

$$(2.9)\quad \begin{aligned} &{}_3F_2\left[\begin{matrix} c-a,\ \frac{1}{2}b,\ \frac{1}{2}b+\frac{1}{2} \\ c-\frac{1}{2}a,\ c-\frac{1}{2}a+\frac{1}{2} \end{matrix}\,\middle|\, 1\right] \\ &= \frac{\Gamma(c-b)\,\Gamma(2c-b)\,\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)}{\Gamma(c)\,\Gamma(2c-a-b)\,\Gamma\left(c-\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)} \\ &\quad\times {}_7F_6\left[\begin{matrix} c-\frac{1}{2}b-\frac{1}{2},\ \frac{1}{2}c-\frac{1}{4}b+\frac{3}{4},\ \frac{1}{2}c-\frac{1}{2}b,\ \frac{1}{2}c-\frac{1}{2}b+\frac{1}{2}, \\ \frac{1}{2}c-\frac{1}{4}b-\frac{1}{4},\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+\frac{1}{4}a-\frac{1}{4}b+\frac{3}{4}, \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \frac{1}{2}c-\frac{1}{4}a-\frac{1}{4}b-\frac{1}{4},\ \frac{1}{2}c-\frac{1}{4}a-\frac{1}{4}b+\frac{1}{4},\ \frac{1}{2}a \\ \frac{1}{2}c+\frac{1}{4}a-\frac{1}{4}b+\frac{1}{4},\ c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,\middle|\, 1\right]. \end{aligned}$$

□

3. Special cases

Some special cases of our main results (2.1) and (2.2) are shown to be connected, implicitly or explicitly, with some known results given here or elsewhere.

(1) In (2.1), if we take $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$ and then apply the well-known Gauss's theorem [2]:

$$ (3.1) \qquad {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b) > 0) $$

to the resulting equation, we get, after a little simplification, the result (1.2).

(2) In (2.1), if we take $b = 1 - a$ and use the Watson's theorem [2]:

$$ (3.2) \qquad \begin{aligned} &{}_3F_2\left[\begin{matrix} a, \; b, \; c \\ \frac{1}{2}(a+b+1), \; 2c \end{matrix} \,\middle|\, 1\right] \quad (\Re(2c-a-b) > -1) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \end{aligned} $$

we get the result (1.3).

(3) In (2.1), if we take $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}$, for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, we get eleven results which are seen to be equivalent to the known results (1.6) obtained by Lavoie, Grondin and Rathie [3].

(4) In (2.1), if we take $b = 1 - a + i$, for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, we also get eleven results which are seen to be equivalent to the known results (1.7) obtained by Lavoie, Grondin and Rathie [3].

(5) In (2.2), if we take $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$, we immediately get (1.2), since one member of the numerator parameters, i.e., $\frac{1}{2}c - \frac{1}{4}a - \frac{1}{4}b - \frac{1}{4}$ in ${}_7F_6$ becomes zero.

(6) In (2.2), if we take $b = 1 - a$, we, after a little simplification, get

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\ 1-a \\ c \end{matrix} \,\middle|\, \frac{1}{2}\right] \\
(3.3)\quad &= \frac{2^a\,\Gamma\left(c+\frac{1}{2}a\right)}{\Gamma(c+a)\,\Gamma\left(1-\frac{1}{2}a\right)} \\
&\times {}_5F_4\left[\begin{matrix} c+\frac{1}{2}a-1,\ \frac{1}{2}c+\frac{1}{4}a+\frac{1}{2},\ \frac{1}{2}c+\frac{1}{2}a-\frac{1}{2},\ \frac{1}{2}c-\frac{1}{2},\ \frac{1}{2}a \\ \frac{1}{2}c+\frac{1}{4}a-\frac{1}{2},\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+\frac{1}{2}a+\frac{1}{2},\ c \end{matrix} \,\middle|\, 1\right].
\end{aligned}
$$

In (3.3), if we use the known result [2]

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} a,\ \frac{1}{2}a+1,\ b,\ c,\ d \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix} \,\middle|\, 1\right] \\
(3.4)\quad &= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a-b-c)\,\Gamma(1+a-c-d)\,\Gamma(1+a-b-d)\,\Gamma(1+a)}
\end{aligned}
$$

and use the duplication formula for the Gamma function (see, *e.g*, [4, p. 7, Eq. (49)]), we get, after a little simplification, the result (1.3).

(7) In (2.2), if we take $c = \frac{1}{2}a + \frac{1}{6} + \frac{1}{2}i + \frac{1}{2}$ and $b = 1 - a + i$, then, for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get two identities (each one containing eleven summation formulas) which are seen to be equivalent to the known results (1.6) and (1.7), respectively, obtained by Lavoie, Grondin and Rathie [3].

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Arjun K. Rathie
Department of Mathematics
Govt. Sujangarh College Distt. Churu
Rajasthan State, India
E-mail: akrathie@rediffmail.com

Yong Sup Kim
Department of Mathematics
WonKwang University
Iksan 570-749, Korea
E-mail: yspkim@wonkwang.ac.kr

Junesang Choi
Department of Mathematics
College of Natural Sciences, Dongguk University
Kyongju 780-714, Republic of Korea
E-mail: junesang@mail.dongguk.ac.kr