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INTUITIONISTIC FUZZY SUBSEMIGROUPS AND SUBGROUPS ASSOCIATED BY INTUITIONISTIC FUZZY GRAPHS

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Abstract. The notion of intuitionistic fuzzy graphs is introduced. We show how to associate an intuitionistic fuzzy sub(semi)group with an intuitionistic fuzzy graph in a natural way.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [19] has achieved a great success in various fields. Out of several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [1, 2, 3] have been found to be highly useful to deal with vagueness. Gau and Buehrer [11] presented the concept of vague sets. But, Burillo and Bustince [7] showed that the notion of vague sets coincides with that of intuitionistic fuzzy sets. Szmidt and Kacprzyk [18] proposed a non-probabilistic-type entropy measure for intuitionistic fuzzy sets. De et al. [9] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Dengfeng and Chuntian [10] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, presented several new similarity measures for measuring the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measures and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems. Graph theory has numerous applications to problems in systems analysis, operations research, transportation, and economics. However, in many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be

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known exactly. In such cases, it is natural to deal with the uncertainty using fuzzy set theory. Many studies on fuzzy graphs are performed by Bhutani, Rosenfeld, Blue, Bush, Puckett, Mordeson, Nair, Peng, etc. (see [4, 5, 6, 13, 14, 15, 16]).

In this article, we introduce the notion of intuitionistic fuzzy graphs. We show how to associate an intuitionistic fuzzy sub(semi)group with an intuitionistic fuzzy graph in a natural way.

2. Preliminaries

A function $\mu : X \to [0,1]$ is called *a fuzzy set* in a set X. An *intuitionistic fuzzy set* (IFS, for short) in X is an expression α given by

$$\alpha = \{\langle x, \mu_\alpha(x), \gamma_\alpha(x)\rangle \mid x \in X\}$$

where the functions $\mu_\alpha : X \to [0,1]$ and $\gamma_\alpha : X \to [0,1]$ denote the degree of membership (namely $\mu_\alpha(x)$) and the degree of nonmembership (namely $\gamma_\alpha(x)$) of each element $x \in X$ to α, respectively, and

$$0 \le \mu_\alpha(x) + \gamma_\alpha(x) \le 1$$

for all $x \in X$. For the sake of simplicity, we shall use the notation $\alpha = (\mu_\alpha, \gamma_\alpha)$ instead of $\alpha = \{\langle x, \mu_\alpha(x), \gamma_\alpha(x)\rangle \mid x \in X\}$. Let $\{\alpha_i \mid i \in \Lambda\}$ be a family of IFSs in X. Then $\cap\alpha_i = (\mu_{\cap\alpha_i}, \gamma_{\cap\alpha_i})$, where $\mu_{\cap\alpha_i}(x) = \inf_{i\in\Lambda} \mu_{\alpha_i}(x)$ and $\gamma_{\cap\alpha_i}(x) = \sup_{i\in\Lambda} \gamma_{\alpha_i}(x)$. An IFS $\alpha = (\mu_\alpha, \gamma_\alpha)$ in a semigroup S is called an *intuitionistic fuzzy subsemigroup* of S [12] if it satisfies:

$$(\forall x, y \in S)\,(\mu_\alpha(xy) \ge \min\{\mu_\alpha(x), \mu_\alpha(y)\}, \gamma_\alpha(xy) \le \max\{\gamma_\alpha(x), \gamma_\alpha(y)\}).$$

An IFS $\alpha=(\mu_\alpha, \gamma_\alpha)$ in a group G is called an *intuitionistic fuzzy subgroup* of G [17] if it is an intuitionistic fuzzy subsemigroup of G and satisfies:

$$(\forall x \in G)\,(\mu_\alpha(x^{-1}) = \mu_\alpha(x),\ \gamma_\alpha(x^{-1}) = \gamma_\alpha(x)).$$

3. Intuitionistic fuzzy graphs

Let $G = (V, E)$ be a graph where V is the nonempty finite set of vertices of G and E is the set of edges of G. Let $\mathcal{P}(V)$ denote the power set of V and let

$$L := \{A \in \mathcal{P}(V) \mid |A| = 2\},$$

where $|A|$ denotes the cardinality of A. Then $E \subset L$.

DEFINITION 3.1. An *intuitionistic fuzzy graph* with underlying set V is defined to be an ordered pair (α, β) where $\alpha = (\mu_\alpha, \gamma_\alpha)$ is an IFS in V and $\beta = (\mu_\beta, \gamma_\beta)$ is an IFS in L such that $\mu_\beta(\{u,v\}) \leq \min\{\mu_\alpha(u), \mu_\alpha(v)\}$ and $\gamma_\beta(\{u,v\}) \geq \max\{\gamma_\alpha(u), \gamma_\alpha(v)\}$ for all $u, v \in V$.

EXAMPLE 3.2. (1) Let $V = \{a, b\}$ and

$$\alpha = \left\langle V, \left(\tfrac{a}{0.5}, \tfrac{b}{0.2}\right), \left(\tfrac{a}{0.3}, \tfrac{b}{0.7}\right)\right\rangle, \ \beta = \left\langle L, \tfrac{\{a,b\}}{0.1}, \tfrac{\{a,b\}}{0.8}\right\rangle.$$

Then (α, β) is an intuitionistic fuzzy graph.

(2) Let $V = \{a, b, c\}$ and

$$\alpha = \left\langle V, \left(\tfrac{a}{0.8}, \tfrac{b}{1}, \tfrac{c}{0.5}\right), \left(\tfrac{a}{0.1}, \tfrac{b}{0}, \tfrac{c}{0.3}\right)\right\rangle,$$

$$\beta = \left\langle L, \left(\tfrac{\{a,b\}}{0.7}, \tfrac{\{b,c\}}{0.5}, \tfrac{\{a,c\}}{0.2}\right), \left(\tfrac{\{a,b\}}{0.2}, \tfrac{\{b,c\}}{0.4}, \tfrac{\{a,c\}}{0.5}\right)\right\rangle.$$

Then (α, β) is an intuitionistic fuzzy graph. If we also define

$$\alpha = \left\langle V, \left(\tfrac{a}{0.25}, \tfrac{b}{0.5}, \tfrac{c}{0.75}\right), \left(\tfrac{a}{0.7}, \tfrac{b}{0.5}, \tfrac{c}{0.2}\right)\right\rangle,$$

$$\beta = \left\langle L, \left(\tfrac{\{a,b\}}{0.25}, \tfrac{\{b,c\}}{0.06}, \tfrac{\{a,c\}}{0.25}\right), \left(\tfrac{\{a,b\}}{0.7}, \tfrac{\{b,c\}}{0.9}, \tfrac{\{a,c\}}{0.7}\right)\right\rangle,$$

then (α, β) is an intuitionistic fuzzy graph.

(3) Let $V = \{a, b, c\}$ and

$$\alpha = \left\langle V, \left(\tfrac{a}{0.3}, \tfrac{b}{0.2}, \tfrac{c}{0.6}\right), \left(\tfrac{a}{0.5}, \tfrac{b}{0.6}, \tfrac{c}{0.2}\right)\right\rangle,$$

$$\beta = \left\langle L, \left(\tfrac{\{a,b\}}{0.1}, \tfrac{\{b,c\}}{m}, \tfrac{\{a,c\}}{0.2}\right), \left(\tfrac{\{a,b\}}{0.7}, \tfrac{\{b,c\}}{n}, \tfrac{\{a,c\}}{0.6}\right)\right\rangle$$

where $m, n \in [0,1]$ with $m + n \leq 1$. If $m > 0.2$ or $n < 0.6$, then (α, β) is not an intuitionistic fuzzy graph.

PROPOSITION 3.3. *If $\{(\alpha_i, \beta_i) \mid i \in \Lambda\}$ is a family of intuitionistic fuzzy graphs with the underlying set V, then so is $(\cap\alpha_i, \cap\beta_i)$.*

PROOF. For any $x, y \in V$, we have

$$\begin{aligned}\mu_{\cap\beta_i}(\{x,y\}) &= \inf_{i\in\Lambda} \mu_{\beta_i}(\{x,y\}) \leq \inf_{i\in\Lambda} \min\{\mu_{\alpha_i}(x), \mu_{\alpha_i}(y)\} \\ &= \min\{\inf_{i\in\Lambda} \mu_{\alpha_i}(x), \inf_{i\in\Lambda} \mu_{\alpha_i}(y)\} = \min\{\mu_{\cap\alpha_i}(x), \mu_{\cap\alpha_i}(y)\}\end{aligned}$$

$$\begin{aligned}\gamma_{\cap\beta_i}(\{x,y\}) &= \sup_{i\in\Lambda} \gamma_{\beta_i}(\{x,y\}) \geq \sup_{i\in\Lambda} \max\{\gamma_{\alpha_i}(x), \gamma_{\alpha_i}(y)\} \\ &= \max\{\sup_{i\in\Lambda} \gamma_{\alpha_i}(x), \sup_{i\in\Lambda} \gamma_{\alpha_i}(y)\} = \max\{\gamma_{\cap\alpha_i}(x), \gamma_{\cap\alpha_i}(y)\}.\end{aligned}$$

Hence $(\cap\alpha_i, \cap\beta_i)$ is an intuitionistic fuzzy graph with the underlying set V. □

DEFINITION 3.4. Let (α, β) be an intuitionistic fuzzy graph with the underlying set V. A map $\Phi : V \to V$ is called an *intuitionistic morphism* of (α, β) if it satisfies:

$$(1) \qquad \begin{aligned} &(\forall x, y \in V)\,(\mu_\beta(\{\Phi(x), \Phi(y)\}) \geq \mu_\beta(\{x, y\})), \\ &(\forall x, y \in V)\,(\gamma_\beta(\{\Phi(x), \Phi(y)\}) \leq \gamma_\beta(\{x, y\})), \end{aligned}$$

$$(2) \qquad (\forall x \in V)\,(\mu_\alpha(\Phi(x)) \geq \mu_\alpha(x),\ \ \gamma_\alpha(\Phi(x)) \leq \gamma_\alpha(x)).$$

Given an intuitionistic fuzzy graph (α, β) with the underlying set V, let $\mathfrak{M}(\alpha, \beta)$ be the set of all intuitionistic morphisms of (α, β). Let $e : V \to V$ be defined by $e(x) = x$ for all $x \in V$. Obviously $e \in \mathfrak{M}(\alpha, \beta)$.

THEOREM 3.5. *Given an intuitionistic fuzzy graph (α, β) with the underlying set V, $(\mathfrak{M}(\alpha, \beta), \circ)$ is a semigroup with the identity element e.*

PROOF. Let $\Phi, \Psi \in \mathfrak{M}(\alpha, \beta)$ and $x, y \in V$. Then

$$\begin{aligned} &\mu_\beta(\{(\Phi \circ \Psi)(x), (\Phi \circ \Psi)(y)\}) = \mu_\beta(\{\Phi(\Psi(x)), \Phi(\Psi(y))\}) \\ &\quad \geq \mu_\beta(\{\Psi(x), \Psi(y)\}) \geq \mu_\beta(\{x, y\}), \end{aligned}$$

$$\begin{aligned} &\gamma_\beta(\{(\Phi \circ \Psi)(x), (\Phi \circ \Psi)(y)\}) = \gamma_\beta(\{\Phi(\Psi(x)), \Phi(\Psi(y))\}) \\ &\quad \leq \gamma_\beta(\{\Psi(x), \Psi(y)\}) \leq \gamma_\beta(\{x, y\}), \end{aligned}$$

$$\mu_\alpha((\Phi \circ \Psi)(x)) = \mu_\alpha(\Phi(\Psi(x))) \geq \mu_\alpha(\Psi(x)) \geq \mu_\alpha(x),$$

$$\gamma_\alpha((\Phi \circ \Psi)(x)) = \gamma_\alpha(\Phi(\Psi(x))) \leq \gamma_\alpha(\Psi(x)) \leq \gamma_\alpha(x).$$

Therefore $\Phi \circ \Psi \in \mathfrak{M}(\alpha, \beta)$. Obviously $\mathfrak{M}(\alpha, \beta)$ satisfies the associativity under the operation $\circ$, and $\Phi \circ e = \Phi = e \circ \Phi$. Hence $(\mathfrak{M}(\alpha, \beta), \circ)$ is a semigroup with the identity element e. □

We show how to associate an intuitionistic fuzzy subsemigroup with an intuitionistic fuzzy graph in a natural way.

PROPOSITION 3.6. *Let (α, β) be an intuitionistic fuzzy graph with the underlying set V and let $f = (\mu_f, \gamma_f)$ be an IFS in $\mathfrak{M}(\alpha, \beta)$ defined by*

$$(3) \qquad \begin{aligned} \mu_f(\Phi) &= \sup\{\mu_\beta(\{\Phi(x), \Phi(y)\}) \mid x, y \in V\}, \\ \gamma_f(\Phi) &= \inf\{\gamma_\beta(\{\Phi(x), \Phi(y)\}) \mid x, y \in V\} \end{aligned}$$

for all $\Phi \in \mathfrak{M}(\alpha, \beta)$. Then $f = (\mu_f, \gamma_f)$ is an intuitionistic fuzzy subsemigroup of $\mathfrak{M}(\alpha, \beta)$.

PROOF. For any $\Phi, \Psi \in \mathfrak{M}(\alpha, \beta)$ we have

$$\begin{aligned}\mu_f(\Phi \circ \Psi) &= \sup\{\mu_\beta(\{(\Phi \circ \Psi)(x), (\Phi \circ \Psi)(y)\}) \mid x, y \in V\}\\ &= \sup\{\mu_\beta(\{\Phi(\Psi(x)), \Phi(\Psi(y))\}) \mid x, y \in V\}\\ &\geq \sup\{\mu_\beta(\{\Psi(x), \Psi(y)\}) \mid x, y \in V\}\\ &= \mu_f(\Psi),\end{aligned}$$

$$\begin{aligned}\gamma_f(\Phi \circ \Psi) &= \inf\{\gamma_\beta(\{(\Phi \circ \Psi)(x), (\Phi \circ \Psi)(y)\}) \mid x, y \in V\}\\ &= \inf\{\gamma_\beta(\{\Phi(\Psi(x)), \Phi(\Psi(y))\}) \mid x, y \in V\}\\ &\leq \inf\{\gamma_\beta(\{\Psi(x), \Psi(y)\}) \mid x, y \in V\}\\ &= \gamma_f(\Psi).\end{aligned}$$

Thus $\mu_f(\Phi \circ \Psi) \geq \min\{\mu_f(\Phi), \mu_f(\Psi)\}$ and $\gamma_f(\Phi \circ \Psi) \leq \max\{\gamma_f(\Phi), \gamma_f(\Psi)\}$. Therefore $f = (\mu_f, \gamma_f)$ is an intuitionistic fuzzy subsemigroup of $\mathfrak{M}(\alpha, \beta)$. □

DEFINITION 3.7. Let (α, β) be an intuitionistic fuzzy graph with the underlying set V. A one-to-one and onto map $\Phi : V \to V$ is called an *intuitionistic automorphism* of (α, β) if it satisfies:

$$(4) \qquad \begin{aligned}&(\forall x, y \in V)\,(\mu_\beta(\{\Phi(x), \Phi(y)\}) = \mu_\beta(\{x, y\})),\\ &(\forall x, y \in V)\,(\gamma_\beta(\{\Phi(x), \Phi(y)\}) = \gamma_\beta(\{x, y\})),\end{aligned}$$

$$(5) \qquad (\forall x \in V)\,(\mu_\alpha(\Phi(x)) = \mu_\alpha(x),\ \gamma_\alpha(\Phi(x)) = \gamma_\alpha(x)).$$

Given an intuitionistic fuzzy graph (α, β) with the underlying set V, let $\mathfrak{A}(\alpha, \beta)$ denote the set of all intuitionistic automorphisms of (α, β). Using the similar way to the proof Theorem 3.5, we have the following theorem.

THEOREM 3.8. *Let (α, β) be an intuitionistic fuzzy graph with the underlying set V. Then $(\mathfrak{A}(\alpha, \beta), \circ)$ is a group.*

We show how to associate an intuitionistic fuzzy subgroup with an intuitionistic fuzzy graph.

THEOREM 3.9. *Let (α, β) be an intuitionistic fuzzy graph with the underlying set V. Define an IFS $g = (\mu_g, \gamma_g)$ in $\mathfrak{A}(\alpha, \beta)$ by*

$$(6) \qquad \begin{aligned}\mu_g(\Phi) &= \sup\{\mu_\beta(\{\Phi(x), \Phi(y)\}) \mid x, y \in V\},\\ \gamma_g(\Phi) &= \inf\{\gamma_\beta(\{\Phi(x), \Phi(y)\}) \mid x, y \in V\}\end{aligned}$$

for all $\Phi \in \mathfrak{A}(\alpha, \beta)$. Then $g = (\mu_g, \gamma_g)$ is an intuitionistic fuzzy subgroup of $\mathfrak{A}(\alpha, \beta)$.

PROOF. We note that the definition of $g = (\mu_g, \gamma_g)$ given by (6) is identical to the definition of $f = (\mu_f, \gamma_f)$ given by (3) in the construction of an intuitionistic fuzzy subsemigroup. In the present situation, since $\Phi \in \mathfrak{A}(\alpha, \beta)$ it follows by using (4) that the equation (6) can be expressed as:

$$\begin{aligned} \mu_g(\Phi) &= \sup\{\mu_\beta(\{x, y\}) \mid x, y \in V\}, \\ \gamma_g(\Phi) &= \inf\{\gamma_\beta(\{x, y\}) \mid x, y \in V\}. \end{aligned} \tag{7}$$

From (7) it is clear that if $\Phi, \Psi \in \mathfrak{A}(\alpha, \beta)$ then

$$\begin{aligned} \mu_g(\Phi \circ \Psi) &\geq \min\{\mu_g(\Phi), \mu_g(\Psi)\}, \\ \gamma_g(\Phi \circ \Psi) &\leq \max\{\gamma_g(\Phi), \gamma_g(\Psi)\}, \end{aligned}$$

$\mu_g(\Phi^{-1}) = \mu_g(\Phi)$, and $\gamma_g(\Phi^{-1}) = \gamma_g(\Phi)$. Hence $g = (\mu_g, \gamma_g)$ is an intuitionistic fuzzy subgroup of $\mathfrak{A}(\alpha, \beta)$. □

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