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Rental Resource Management Model with Capacity Expansion and Return

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■ Abstract ■

We consider a rental company that dynamically manages its capacity level through capacity addition and return. While serving customer with its own capacity, the company expands its capacity by renting items from an outside source so that it can avoid lost opportunities of rental which occur when stock is not sufficient. If stock becomes sufficiently large enough to cope with demands, the company returns expanded capacity to the outside source. Formulating the model into a Markov decision problem, we identify an optimal capacity management policy which states when the company should expand its capacity and when it should return expanded capacity after capacity addition. Since it is intractable to analytically find the optimal capacity management policy and the optimal size of capacity expansion, we present a numerical procedure that finds these optimal values based on the value iteration method. Numerical analysis is implemented and we observe monotonic properties of the optimal performance measures by system parameters, which are meaningful in developing effective heuristic policies.

Keyword : Rental, Capacity Management, Reverse Logistics, Markov Decision Processes

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1. Introduction

This paper considers a rental company that dynamically controls its capacity level through capacity addition and return. While serving customer with its own capacity, the company expands its capacity by renting items from an outside source so that it can avoid lost opportunities of rental which occur when stock is not sufficient. If stock becomes sufficiently large enough to cope with demands, the company returns expanded capacity to the outside source.

Rental business is receiving growing attention in many business areas such as leisure and entertainment goods and office, medical, and house appliances. By its dynamics, items are returned after their rental so that they may be used repeatedly and rental periods are typically much shorter than the life of the items. Knowing how much rental capacity to maintain is critical to the rental business operation. Too much rental capacity reduces cash position of a rental company while too little rental capacity turns customers away to the rental company's competitor. Since demand and rental durations may be uncertain over any period of time, it is essential to develop effective policies for controlling the capacity.

The area of research relevant to this problem is the capacity management literature. Capacity expansion with growing demands over time has been studied in [2, 4, 21, 22, 25]. Research subject particularly relevant to our work is a capacity operation problem found in [1, 20, 24, 26]. Rocklin et al. [24] considered an optimal capacity expansion/contraction of a production/service facility under stochastic demands. They showed that the optimal policy is characterized by two threshold values such that at the beginning of each period of planning horizon, if the capacity level is below the lower (upper) limit, it is optimal to bring the capacity up (down) to that limit. Rajagopalan and Soteriou [20] studied a capacity expansion problem with features of capacity acquisition, disposal, and replacement using integer programming model. So and Tang [26] addressed a policy for dynamically adjusting operating capacity according to the system state to manage congestion in service systems and developed queueing models for analyzing its impacts on two types of service systems : a system with several operators to handle incoming calls over a number of lines and a system with multiple identical service counters in parallel to serve customers. Angelus and Porteus [1] addressed the problem of deciding how much capacity to have and how much to produce at each planning period in a produce-to-stock facility. Assuming that demands stochastically increase up to a peak and then decrease, they characterized the optimal capacity plan as a target interval policy similar to the one defined in [24] under instantaneous capacity additions or reductions. They also showed that in the case with carrying over unsold units, the optimal capacity target intervals can depend on the amount of initial inventory available at each period and it is a decreasing function of the initial inventory.

The other important area of research relevant to this problem is the literature regarding inventory management with product returns. Reusable products are taken back from the market after use and transformed into serviceable goods through remanufacturing processes or disposed depending on the quality standards. Since the remanufactured products may not be sufficient to cover demands on serviceable goods, stocks are replenished from outside suppliers. The readers are referred to [6] and [7] for the detailed literature review in this area. The product return models studied in the literature can be classified into several categories according to whether or not there exists an explicit distinction between old and new serviceable products, remanufacturing and replenishment lead times, fixed replenishment order costs, or dependency of demand and return processes. Those models can be also divided according to periodic review and continuous review and whether or not allowing for the disposal for the returned products.

The capacity management model in this paper has some features similar to those in the cash flow management in financial management decisions ([5, 15, 19]). The firm takes actions such as selling or buying securities to increase or decrease the daily cash level so that it can be brought to a suitable level. The cash flow management models above differs from our model in the sense that return and demand processes is not correlated, that is, cash inflow (deposit) and cash outflow (withdrawal, expenditure) are different streamlines. Finally, we conclude the relevant literature review by noting that the operational structure of the model presented in this paper can be viewed as a multi-echelon inventory system with lateral transshipment (see Grahovac and Chakravarty [8] for the detailed references in this area).

The prospect of the model presented in this paper raises important strategic issues with respect to the capacity management. More specifically, this paper addresses the following research questions : (1) When should the company schedule its capacity addition from the outside source? This decision is affected by the size of capacity in rent by customers as well as the size of capacity in stock the company holds. (2) After capacity addition, when should the company return expanded capacity to the outside source? (3) If the system parameters are changed, what are their impacts on the capacity addition and return decision?

This paper deals with these issues in the context of the Markov model. Even though the Markov model may be restricted for modeling the real world rental problems, it can provide us with insights into the effective rental capacity management. As a starting point for the analysis, we restrict our attention to the company with a single type of capacity. There can be many ways to model rental periods and here we choose to represent it as an exponential random variable. The exponential rental period is appropriate when the period is random with a larger variance-to-mean ratio. In particular, if the rental period is flexible and extendable, it is known that the exponential distribution approximation is reasonable [29].

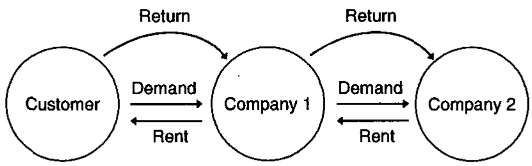
Our model differs from the inventory management literature with product returns in two aspects. First, all products issued in our model should return, which means that the return rate is determined by the number of products in rent by the customers. In contrast, product return models have partial returns from the products issued. Further, it is often assumed that the demand and return processes are independent. Second, we consider both capacity (serviceable products in stock) augmentation and reduction while product return models consider only capacity augmentation through stock replenishment or remanufacturing. Our model also differs from the capacity management literature above because they do not consider a situation where the product is returned after use.

The paper is organized as follows. In the next section, we provide a formulation of our model. Analysis of the optimal capacity management policy is given in Section 3. Section 4 presents a numerical procedure of finding the optimal performance and test results which exhibit a monotonicity of optimal performance with respect to system parameters. Finally we state our conclusions in the last section.

2. Problem Formulation

Consider a rental company, denoted by Company 1, that operates pieces of N rental items demanded by customers who arrive according to a Poisson process with rate λ. The durations of their rentals are independent, exponentially distributed random variables with mean μ^{-1}. We assume that rental items are not depreciated in time and thus all items are identical in terms of rental service. It is also assumed that each item rented by customer returns in a serviceable condition. Each arriving customer rents exactly one unit of item if items are in stock. If no items are available for rental, it is lost and charged at c_L. A revenue p is realized per unit of time during which each item is rented by customer while a holding cost c_h is incurred per unit of time during which each item is held in stock.

Whenever company 1 rents Q items from an outside source, denoted by company 2, a fixed setup cost, c_S, is incurred and a variable cost, c_k, is charged upon each item per unit of time during the rental period. Upon each instance company 1 returns Q items to Company 2, a fixed setup cost c_R is incurred. Delivery time from company 2 to company 1 is assumed to be negligible. [Figure 1] graphically illustrates a rental business process described above.



[Figure 1] A rental model with capacity addition and return

At each decision epoch, company 1 should decide whether or not to expand capacity by renting Q items from company 2, or whether or not to return Q items to company 2 after capacity addition. The set of decision epochs is the set of customer arrival epoch and customer return epoch. Without any loss of optimality, the class of admissible strategies is taken to be the set of non-anticipative, stationary, non-randomized, Markov policies that are based on perfect observations of the queue length processes.

The original problem is a continuous time Markov decision problem (MDP) where the sum of the transition rates at every state is bounded by $\gamma \equiv \lambda + (N+Q)\mu$. Let the profit at time $w \in R^+$ be discounted with a factor $e^{-\beta w}$. After following the uniformization process (Lipmann [13]), a continuous time MDP can be formulated with an equivalent discrete time MDP with a transition rate γ and a discount factor $\gamma/(\beta+\gamma)$. The essence of this uniformization process is to allow fictitious self-loop transitions for all the state that has a smaller transition rate than γ. By doing so, each state has the same transition rate equal to γ and the expected transition time is constant and equals $1/\gamma$. Without any loss of generality, we assume that $\beta+\gamma=1$. Hence, the discount factor of the discrete time MDP becomes γ.

A state is described by the vector (x_1,x_2,δ) where x_1 and x_2 denote the number of items in stock and the number of items rented by customer, respectively, and δ is an indicator variable. If $\delta=1$, company 1 is renting Q items whereas if $\delta=0$, it is not. The state space is denoted by Γ. At each decision epoch, there are *Rent* and *Do not rent* actions in state $(x_1,x_2,0)$, and *Return* and *Do not return* actions in state $(x_1,x_2,1)$. Note that $x_1+x_2=N+\delta Q$.

Either when $\delta=0$ and *Do not rent* action is applied or when $\delta=1$ and *Do not return* action is applied, (x_1,x_2,δ) is transited to (x_1-1,x_2+1,δ) and (x_1+1,x_2-1,δ) by a customer arrival and a customer return, respectively. If *Rent* action is applied to $(x_1,x_2,0)$, it is transited to $(x_1+Q-1,x_2+1,1)$ and $(x_1+Q+1,x_2-1,1)$ by a customer arrival and a customer return, respectively. When *Return* action is applied to $(x_1,x_2,1)$, it is transited to $(x_1-Q-1,x_2+1,0)$ and $(x_1-Q+1,x_2-1,0)$ by a customer arrival and a customer return, respectively.

The goal of this paper is to find a rental capacity addition and return strategy that maximizes company 1's profit subject to the costs of stock holding, customer losing, and capacity addition/return. Let $J(x_1,x_2,\delta)$ be the optimal expected discounted profit over an infinite horizon when the initial state is given by (x_1,x_2,δ). We first define the one stage expected discounted profit in state (x_1,x_2,δ) which is given by

$$g(x_1,x_2,\delta)=px_2-c_hx_1-c_kQ1\{\delta=1\}-\lambda c_L\,1\{x_1=0\}$$

where the indicator function $1a$ is 1 if a is true, otherwise, 0. Note that γ is the discount factor for the discrete time MDP and the expected transition time is $1/\gamma$. In $(x_1,x_2,1)$, a cost of renting Q items is incurred at a rate of c_kQ during the expected transition time while in $(0,x_2,\delta)$, a cost of losing customers is incurred at a rate of λc_L during one stage. Denote

$$D(x_1,x_2)=(x_1-1,x_2+1)\text{ if } x_1>0\,;\,(x_1,x_2)\text{ otherwise,}$$
$$I(x_1,x_2)=(x_1+1,x_2-1)\text{ if } x_2>0\,;\,(x_1,x_2)\text{ otherwise.}$$

Since the expected discounted profit during the expected transition time is bounded, the optimal total discounted profit function J can be shown to satisfy the following optimality equation (Bellman's Equation) (see Proposition 2, Ch.5 of Bertsekas [3]) :

$$J(x_1,x_2,\delta)=\max\{T_UJ(x_1,x_2,\delta),T_RJ(x_1,x_2,\delta)\},\ \ \delta=0,1$$

where

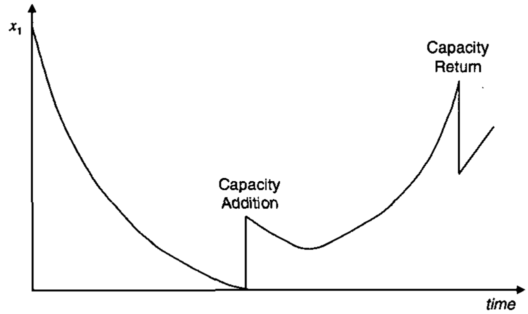
$$\begin{aligned}T_UJ(x_1,x_2,\delta)&=g(x_1,x_2,\delta)+\lambda J(D(x_1,x_2,\delta))\\&\quad+x_2\mu J(I(x_1,x_2,\delta))\\&\quad+(N+Q-x_2)\mu J(x_1,x_2,\delta),\\T_RJ(x_1,x_2,0)&=-c_S+T_UJ(x_1+Q,x_2,1),\\T_RJ(x_1,x_2,1)&=-c_R+T_UJ(x_1-Q,x_2,0).\end{aligned}$$

T_U and T_R respectively are value iteration operators corresponding to *Do not rent* and *Rent* when $\delta=0$, and *Do not return* and *Return* when $\delta=1$.

3. Optimal Capacity Expansion and Return Policy

In this section, we characterize the optimal properties of the capacity expansion and return policy. [Figure 2] shows a typical sample path of stock depletion at company 1 in time. Since the stock level is decreased by customer demand and increased by customer return, the net stock depletion rate can have both positive and negative slopes depending on the status of demand and return rates.

We start with introducing the following lemma. Its proof directly follows from the definition of value function T_U and T_R. The first property of Lemma 1 says if it is optimal to rent Q items from in state $(x_1, x_2, 0)$, then the return action in $(x_1+Q, x_2, 1)$ is not optimal. Similarly, the second property states that if it is optimal to return Q items in state $(x_1, x_2, 1)$, company 1 should not expand its capacity in state $(x_1-Q, x_2, 0)$.



[Figure 2] Stock flow at company 1 in time

Lemma 1.

(i) If $J(x_1, x_2, 0) = T_R J(x_1, x_2, 0)$, then

$$J(x_1+Q, x_2, 1) = T_U J(x_1+Q, x_2, 1).$$

(ii) If $J(x_1, x_2, 1) = T_R J(x_1, x_2, 1)$,, then

$$J(x_1-Q, x_2, 0) = T_U J(x_1-Q, x_2, 0).$$

The problem presented here falls in the category of the optimal control of finite queueing system. When queueing capacities are unlimited, the optimality of a monotonic control policy has been well established from the properties of value function such as convexity and submodularity (supermodularity). When the queue capacity constraint is introduced, however, the analysis becomes complex and few results on the optimal control have been reported. Our numerical investigation indicates that $J(x_1, x_2, \delta)$ is neither convex nor submodular with respect to x_1 and x_2. These results keep us from applying general results about minimization of submodular functions (see [27]) to our problem to structuralize an optimal policy.

In this paper, we show that there exists an optimal policy with threshold property, provided that the following relationship holds among cost and revenue parameters :

Assumption P.

$$c_h + p \geq \lambda c_L \geq c_S + (c_h + c_k)Q/(1-\gamma). \qquad (1)$$

The company earns p per unit time if there is at least one item in rent and pays c_h per unit time if there is at least one item in stock. Hence, $c_h + p$ can be interpreted as the net rental revenue per unit time. λc_L implies the expected cost rate of lost opportunities of rental. The third part of (1) is the cost incurred when renting Q items from company 2 and holding them in stock during the infinite length of time. Hence **Assumption P** states that the expected cost rate of lost opportunities of rental cannot exceed the net rental revenue rate. It also says that the cost incurred when expanded capacity is never rented should not exceed the expected cost rate of lost opportunities of rental. If this is not the case, company 1 does not have to expand its capacity because the cost of losing customer is cheaper than that of capacity addition. In fact, **Assumption P** is a necessary condition which guarantees the existence of the threshold type of an optimal policy. From numerical investigation with a variety of examples, however, we find that the optimal policy still has a threshold property even when **Assumption P** is violated. For this reason, we conjecture that the optimal properties verified in this paper are valid regardless of **Assumption P.**

In order to establish the structural properties of the optimal capacity management policy, it is sufficient to show that certain properties of the functions defined on state space Γ are preserved under the operator T ([18]). Let F be the set of all functions defined on Γ such that if $f \in F$, then

$$-c_R \le f(x_1+Q, x_2, 1) - f(x_1, x_2, 0) \le c_S, \quad (2)$$

$$\triangle_{11} f(x_1+Q, x_2, 1) \le \triangle_{11} f(x_1, x_2, 0), \quad (3)$$

$$f(x_1+Q, x_2, 1) - f(x_1, x_2, 0) \ge -(c_h + c_k)Q/(1-\gamma), \quad (4)$$

$$f(x_1+1, x_2, \delta) \le f(x_1, x_2+1, \delta),\ \delta = 0, 1, \quad (5)$$

$$\triangle_{11} f(x_1, x_2, \delta) \le c_L \quad (6)$$

where

$$\triangle_{11} f(x_1, x_2, \delta) \equiv f(x_1+1, x_2, \delta) - f(x_1, x_2+1, \delta),\ \delta = 0, 1.$$

Operators $\triangle_{11}$ implies the value gained when holding one more item in stock and having one less item rented.

Lemma 2 is straightforward because we assume that the delivery time is negligible. It states that company 1 should not expand its capacity when the stock is not empty.

Lemma 2. If $f \in F$,

$$T_U f(x_1, x_2, 0) > T_R f(x_1, x_2, 0),\ x_1 > 0. \quad (6)$$

Proof : See the Appendix.

The following lemma states that it is always profitable for company 1 to rent Q items from company 2 when the stock is empty.

Lemma 3. If $f \in F$,

$$T_R f(0, N, 0) > T_U f(0, N, 0). \quad (7)$$

Proof : See the Appendix.

In the next lemma, we establish company 1's optimal decision when it is in the capacity expansion mode.

Lemma 4. If $f \in F$,

$$\triangle_{11} T_U f(x_1, x_2, 1) \le \triangle_{11} T_R f(x_1, x_2, 1),\ x_1 \ge Q. \quad (8)$$

Proof : See the Appendix.

When company 1 is in the capacity expansion mode, Equation (8) provides greater incentive to return expanded Q items as the stock increases. To see this, suppose that $T_U f(x_1, x_2+1, 1) - T_R f(x_1, x_2+1, 1) < 0$. Hence, it is optimal to return Q items in $(x_1, x_2, 1)$. By (8), $T_U f(x_1+1, x_2, 1) - T_R f(x_1+1, x_2, 1) \le T_U f(x_1, x_2+1, 1) - T_R f(x_1, x_2+1, 1) < 0$. It means if it is optimal to return Q items withx_1 items in stock, then it is also optimal to return Q items with x_1+1 items in stock, which establishes the optimality of a threshold function.

The following lemma guarantees that Equation (2)~(6) are preserved under$T$.

Lemma 5. If $f \in F$, $Tf \in F$.

Proof : See the Appendix.

Now we can identify the optimal return policy when company 1 is in the capacity expansion mode.

Theorem 1.

(i) The optimal value function J satisfies Equation (2)~(6), that is, $J \in F$.

(ii) Let

$$\Theta^* := \min\{x_1 \ge Q : T_R f(x_1, x_2, 1) > T_U f(x_1, x_2, 1)\}.$$

If company 1 is renting Q items from company 2, then, it is optimal to return Q items whenever $x_1 \ge \Theta^*$.

Proof :

(i) The result follows directly from Lemma 5.

(ii) Suppose that it is not optimal to return in state $(\Theta^*+1, x_2-1, 1)$, i.e., $T_U J(\Theta^*+1, x_2-1, 1) > T_R J(\Theta^*+1, x_2-1, 1)$. From the definition of Θ^*, we have $T_U J(\Theta^*, x_2, 1) < T_R J(\Theta^*, x_2, 1)$. By subtracting the second inequality from the first one, we have $\Delta_{11} T_U J(\Theta^*, x_2-1, 1) > \Delta_{11} T_R J(\Theta^*, x_2-1, 1)$. This is a contradiction by Lemma 4. □

4. Numerical Study

In this section, we discuss how to evaluate the optimal performance with respect to the problem parameters. Since it is intractable to analytically find the optimal return point Θ^* and the optimal size of capacity addition Q^*, we focus on numerically finding these optimal values, based on the value iteration (VI) method. For the clarity and easiness of the study, we put our attention to the average profit problem rather than the discounted profit one. VI enables us to find the optimal return point Θ^* as well as the optimal average profit, $\overline{J}^*$. Refer to Chapter 7 of Bertsekas [3] for the detail of VI.

Unfortunately, the computation of the optimal average profit violates **Assumption P**. Since $\beta=0$ for the average profit problem and thus $\gamma=1$, the third part of **Assumption P** goes to infinity. However, the numerical investigation indicates that the structure of the optimal policy exploited in section 3 can be valid for the average profit problem. We tested the optimality of the threshold property with a variety of test examples. Without any exceptional cases, we observed that the optimality of the threshold property is true for the average profit problem. Even though test examples in <Table 1> do not meet **Assumption P**, we believe that the parameter settings of these examples are reasonable.

In searching for Q^*, it is natural to investigate whether or not the optimal average profit is a convex function of Q. Our numerical investigations suggest that the optimal average profit is convex in Q even though we could not prove it. The following notations are introduced in the optimal solution procedure which jointly finds Q^* and Θ^* :

$\pi^k(x_1, x_2, \delta)$: the optimal action in state (x_1, x_2, δ) found at iteration k of VI given Q

$H^k(x_1, x_2, \delta)$: the value function in state (x_1, x_2, δ) evaluated at iteration k of VI given Q

ϵ : termination criterion of the procedure

$\overline{J}(Q)$: the optimal average profit given Q

$\pi_Q(x_1, x_2, \delta)$: the optimal action in state (x_1, x_2, δ) found at the termination of VI given Q

Optimal solution procedure :

1. Start with $Q>0$.
2. Implementation of Value Iteration
   (a) **Initialization** : Choose the reference state as $(0,0,0)$, set $k=0$, and for each state (x_1, x_2, δ), pick the value function $H^0(x_1, x_2, \delta)=0$.
   (b) **Value iteration step** : Implement a VI on the current value function estimate H^k :
   $$TH^k(x_1, x_2, \delta) = \max\{T_U H^k(x_1, x_2, \delta), T_R H^k(x_1, x_2, \delta)\}$$
   where
   $$T_U H^k(x_1, x_2, \delta) = g(x_1, x_2, \delta) + \lambda H^k(D(x_1, x_2, \delta)) + x_2 \mu H^k(I(x_1, x_2, \delta)) + (N+Q-x_2)\mu H^k(x_1, x_2, \delta),$$
   $$T_R H^k(x_1, x_2, 0) = -c_S + T_U H^k(x_1+Q, x_2, 1),$$
   $$T_R H^k(x_1, x_2, 1) = -c_R + T_U H^k(x_1-Q, x_2, 0).$$

When $\delta=0$, if $T_U H^k(x_1,x_2,\delta) < T_R H^k(x_1,x_2,\delta)$, let $\pi^k(x_1,x_2,\delta) = Rent$; otherwise, $\pi^k(x_1,x_2,\delta)$ = *Do not rent.*

When $\delta=1$, if $T_U H^k(x_1,x_2,\delta) < T_R H^k(x_1,x_2,\delta)$, let $\pi^k(x_1,x_2,\delta) = Return$; otherwise, $\pi^k(x_1,x_2,\delta)$ = *Do not return.*

(c) **Termination test** : Perform the following convergence test :

$\underline{b}_k \equiv \min_{(x_1,x_2,\delta)\in\Gamma}\{TH^k(x_1,x_2,\delta) - H^k(x_1,x_2,\delta)\}$

$\overline{b}_k \equiv \max_{(x_1,x_2,\delta)\in\Gamma}\{TH^k(x_1,x_2,\delta) - H^k(x_1,x_2,\delta)\}$

If $(\overline{b}_k - \underline{b}_k) \geq \epsilon$, for every state (x_1,x_2,δ), let

$H^{k+1}(x_1,x_2,\delta) = TH^k(x_1,x_2,\delta) - TH^k(0,0,0)$,

increase k by one, and go to **Value iteration step.** Otherwise, go to **Evaluation step.**

(d) **Evaluation step** : Set $\overline{J}(Q) = TH^k(0,0,0)$ and $\pi_Q(x_1,x_2,\delta) = \pi^k(x_1,x_2,\delta)$ for every state (x_1,x_2,δ).

3. Decrease Q by one and go to STEP 2 and compute $\overline{J}(Q-1)$.

If $\overline{J}(Q-1) < \overline{J}(Q)$, go to STEP 4.

Otherwise, continue this step until finding Q^l such that $\overline{J}(Q^l-1) < \overline{J}(Q^l)$.

Set $Q^* = Q^l$, $\overline{J^*} = \overline{J}(Q^l)$, and

$\Theta^* = \arg min_{x_1}\{(x_1,x_2,1) : \pi_{Q^l}(x_1,x_2,1) = Return\}$.

Stop the procedure.

4. Increase Q by one and go to STEP 2 and compute $\overline{J}(Q+1)$.

If $\overline{J}(Q+1) < \overline{J}(Q)$, set $Q^* = Q$, $\overline{J^*} = \overline{J}(Q)$, and

$\Theta^* = \arg min_{x_1}(x_1,x_2,1) : \pi_Q(x_1,x_2,1) = \mathrm{Return}$.

Otherwise, continue this step until finding Q^r such that $\overline{J}(Q^r+1) < \overline{J}(Q^r)$.

Set $Q^* = Q^r$, $\overline{J^*} = \overline{J}(Q^r)$, and

$\Theta^* = \arg min_{x_1}\{(x_1,x_2,1) : \pi_{Q^r}(x_1,x_2,1) = Return\}$.

Stop the procedure.

<Table 1> Numerical test results on the optimal performance

Ex.	N	p	c_h	c_L	c_k	c_S	c_R	λ	μ	$\overline{J^*}$	Q^*	Θ^*
1	30	50	1	100	0	10	0	2.5	0.1	1241.124	13	15
2					10					1222.628	8	9
3					20					1209.961	6	7
4					30					1200.534	5	6
5					40					1193.267	4	5
6					50					1187.129	3	4
7					60					1183.296	3	4
8	30	50	1	100	30	0		2.5	0.1	1201.927	5	6
9						10				1200.534	5	6
10						100				1190.442	5	7
11						200				1183.39	4	7
12						300				1178.032	4	7
13						400				1173.515	4	8
14	30	50	1	0	30	10		2.5	0.1	1203.53	4	5
15				100						1200.534	5	6
16				200						1197.754	6	7
17				300						1195.748	6	7
18				400						1193.742	6	7
19				500						1192.146	7	8
20	30	30	1	100	30	10		2.5	0.1	707.654	4	5
21		40								953.53	4	5
22		50								1200.534	5	6
23		60								1447.754	6	7
24		70								1695.748	6	7

In <Table 1>, we present numerical test results which show the optimal performance. Example 1 represents a base case, and in Example 2 through 19 we systematically changed one of the problem parameters, c_k, c_S, and c_L to test the optimal performance. When we compute the optimal average profits, termination criterion ϵ is set to 10^{-3}.

Based on the computational results with a variety of test examples including ones in <Table 1>, we observe monotonic properties of the optimal policy with respect to the cost parameters :

- Θ^* is decreasing in capacity addition variable cost c_k.
- Θ^* is increasing in capacity addition setup cost c_S.
- Θ^* is increasing in customer lost cost c_L.

As long as other parameters remain the same, the decrease in capacity addition variable cost, c_k, will motivate company 1 to delay returning expanded items. Hence, it is reasonable to expect that given $(x_1, x_2, 1)$, the optimal return point Θ^* will be greater than before. The intuition behind the second observation is as follows. As long as other parameters remain the same, one can expect that the increase in capacity addition setup cost, c_S, will force company 1 to not use capacity expansion frequently. Hence, given $(x_1, x_2, 1)$, one might expect that the optimal return point will higher in a larger setup cost than in a smaller one. Using a similar reasoning, we can explain the third observation. As long as other parameters remain the same, the increase in customer lost cost, c_L, will force company 1 to keep expanded items in stock for a longer period, since the possibility of stockout can be reduced. Hence, given state $(x_1, x_2, 1)$, the optimal return point will be higher than before. <Table 2> and [Figure 3] illustrate the effect of c_k on the optimal return point, Θ^*, given Q.

Numerical investigation also indicates that $\overline{J^*}$ is decreasing as each of the cost parameters increases assuming all other parameters are held constant. In addition, it shows

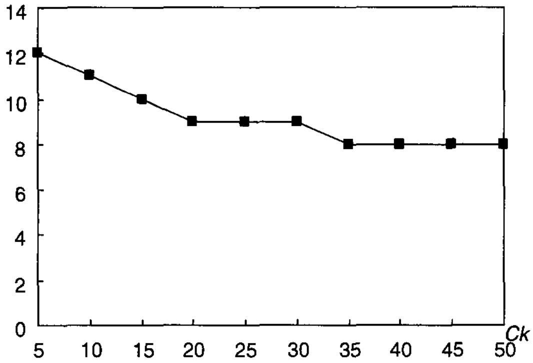
- Q^* is decreasing in capacity addition variable cost c_k,
- Q^* is increasing in capacity addition setup cost c_S,
- Q^* is increasing in customer lost cost c_L.

The first and second phenomenon can be explained using the reasoning of the economic order quantity (EOQ) model. In EOQ model, it can be easily seen that the optimal order quantity decreases in inventory holding cost and increases in order setup cost. The capacity addition variable cost and setup cost can be compared with inventory holding cost and order setup cost of EOQ model, respectively. The third observation confirms that the increase in the size of capacity expansion contributes to the reduction in the lost opportunities of rental.

〈Table 2〉 Effects of c_k on the optimal return point Θ^* given Q

Ex.	N	p	c_h	c_L	c_k	c_S	c_R	λ	μ	Q	$\overline{J^*}$	Θ^*
1	30	10	1	10	0	500	0	2.5	0.1	5	237.933	14
2					0.5						236.982	12
3					1						236.156	11
4					1.5						235.386	10
5					2						234.69	10
6					2.5						233.994	10
7					3						233.346	9
8					3.5						232.73	9
9					4						232.113	9
10					4.5						231.497	9

A thorough study of the optimal selection of Q is beyond the scope of this paper, which focuses on how best to use capacity addition/return. However, we believe that these monotonicity properties will be very useful in developing a heuristic formula which approximates Q^*.



[Figure 3] The change in the optimal return point as a function of c_k

5. Conclusions

In this paper, we proposed a new rental resource management model with capacity expansion and return. One aspect distinguishing the rental resource from traditional inventory resources is that it is repeatedly reused. This paper exploited the optimal properties of the capacity management policy and presented a numerical procedure which finds the optimal performance measures.

Major contributions of our work to the reverse logistics and capacity management literature are summarized as follows. First, we presented and analyzed a model which simultaneously considers the dependency of the return process on the demand process and the capacity addition/return. To the best of our knowledge, this is the first attempt to deal with capacity control issue in the area of reverse logistics. Second, we found a necessary condition that guarantees the existence of an optimal policy with threshold property, which is meaningful because general results found in infinite queueing models cannot be applied to our model.

Numerical investigation was performed with test examples. It exhibited that the optimal return point can be monotonically changed as a function of the cost parameters. We observed the monotonic properties of the optimal profit with respect to system parameters even though we could not prove it. Numerical investigation also suggested that the optimal profit should be convex with respect to the size of capacity addition. Hence, it is conjectured that there may exist an unique size of capacity addition which maximizes the company's profit.

Many open problems remain to be explored. First, our formulation can be easily extends to general customer arrival and rental processes and we conjecture that our structural results continue to hold. Second, in practice, renting items from outside source requires setup times as well as setup costs. Hence, exploring how optimal capacity expansion and return policy are affected by the presence of these time parameters is a very important question.

Our model can be extended in many ways. One application area is the cash flow management in financial management decisions. In finance literature, there has been a tendency to isolate financing from investment decisions. For example, a firm's financing decisions are taken as given or as independent of the investment decisions, even though neither theory nor practice considerations support such a separation. As Mayers and Pogue [14] points out, financial management requires simultaneous consideration of the investment and financing options facing the firm.

At this moment, the capacity control mechanism presented in this paper can be applied to the cash flow management. Maintaining an optimal level of a firm's cash flow is a critical problem in corporate finance because cash flow shortfalls simultaneously affect both investment and financing decisions. Firms experiencing cash flow shortfalls should use external capital markets to cover shortfalls at the cost of accessing external capital or forgoing investment opportunities. In addition, because idle cash earns no interest, too much cash in firms generates opportunity costs and free cash flow costs. Thus, interaction of corporate financing and investment decisions should be considered in cash flow management. The financial management literature has spent much time examining the factors that cause cash flows to fluctuate (Minton and Schrand, [17]). Optimal cash management strategy is, however, only dealt with in the inventory cash management model of Miller and Orr [15]. The approach used in this paper might broaden the implications for corporate financial management.

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Appendix

Proof of Lemma 2.

$T_U f(x_1,x_2,0) - T_R f(x_1,x_2,0) = (c_h + c_k)Q + c_S + \lambda[f(x_1-1,x_2+1,0) - f(x_1-1+Q,x_2+1,1)]$

$+ x_2\mu[f(I(x_1,x_2),0) - f(I(x_1+Q,x_2),1)] + (N+Q-x_2)\mu[f(x_1,x_2,0) - f(x_1+Q,x_2,1)]$

$\geq (c_h + c_k)Q + c_S - \gamma c_S$ (by (2)) $\geqq 0$ (since $\gamma < 1$). □

Proof of Lemma 3.

$T_R f(0,N,0) - T_U f(0,N,0) = \lambda c_L - (c_h + c_k)Q - c_S + \lambda[f(Q-1,N+1,1) - f(0,N,0)]$

$+ N\mu[f(Q+1,N-1,1) - f(1,N-1,0)] + Q\mu[f(Q,N,1) - f(0,N,0)]$

$\geq \lambda c_L - (c_h + c_k)Q - c_S + \lambda[f(Q,N,1) - f(0,N,0)]$

$+ N\mu[f(Q+1,N-1,1) - f(1,N-1,0)] + Q\mu[f(Q,N,1) - f(0,N,0)]$ (by (5))

$\geq \lambda c_L - (c_h + c_k)Q - c_S - \gamma(c_h + c_k)Q/(1-\gamma)$ (by (4))

$= \lambda c_L - c_S - (c_h + c_k)Q/(1-\gamma) \geq 0$ (by (1)). □

Proof of Lemma 4. $\triangle_{11} T_U f(x_1,x_2,1) - \triangle_{11} T_R f(x_1,x_2,1)$

$= -\lambda c_L 1 x_1 = Q + \lambda[\triangle_{11} f(x_1-1,x_2+1,1) - \triangle_{11} f(x_1-1-Q,x_2+1,0)1\{x_1 > Q\}]$

$+ x_2\mu[\triangle_{11} f(x_1+1,x_2-1,1) - \triangle_{11} f(x_1-Q+1,x_2-1,0)]1\{x_2 > 1\}$

$+ (N+Q-x_2-1)\mu[\triangle_{11} f(x_1,x_2,1) - \triangle_{11} f(x_1-Q,x_2,0)] \leq 0.$

The inequality of λ term follows by (3) when $x_1 > Q$ and by (6) when $x_1 = Q$. The inequality corresponding to μ term follows by (3). □

Proof of Lemma 5. Denote by (U/R) the optimal action in (x_1,x_2,δ). U and R respectively represent *Do not rent* and *Rent* in $(x_1,x_2,0)$ and *Do not return* and *Return* in $(x_1,x_2,1)$.

(i) $Tf(x_1+Q,x_2,1) - Tf(x_1,x_2,0) \leq c_S$: We focus on the combination of actions admissible in $(x_1,x_2,0)$ and $(x_1+Q,x_2,1)$. Case(R,R) is excluded by Lemma 1. If (R,U), $x_1 = 0$ by Lemma 2 and $T_R f(0,x_2,0) - T_U f(Q,x_2,1) = -c_S$

If (U,U), $T_U f(x_1,x_2,0) - T_U f(x_1+Q,x_2,1) \geq T_R f(x_1,x_2,0) - T_U f(x_1+Q,x_2,1) = -c_S$

If (U,R), $T_U f(x_1,x_2,0) - T_R f(x_1+Q,x_2,1) = c_R \geq -c_S$

$-c_R \leq Tf(x_1+Q,x_2,1) - Tf(x_1,x_2,0)$: If (U,R), $T_R f(x_1+Q,x_2,1) - T_U f(x_1,x_2,0) = -c_R$.

If (U,U), $T_U f(x_1+Q,x_2,1) - T_U f(x_1,x_2,0) \geq T_R f(x_1+Q,x_2,1) - T_U f(x_1,x_2,0) = -c_R$.

If (R,U), then $x_1 = 0$ by Lemma 2 and $T_U f(Q,x_2,1) - T_R f(0,x_2,0) = c_S \geq -c_R$.

(ii) Suppose $x_1 > 0$. The optimal action in $(x_1+1, x_2-1, 0)$ and $(x_1, x_2+1, 0)$ becomes *Do not rent* by Lemma 2. We focus on the combination of actions admissible in $(x_1+Q+1, x_2, 1)$ and $(x_1+Q, x_2+1, 1)$. Case (U,R) is excluded by Lemma 4. For(R,R), by the definition of value function, $\triangle_{11} T_R f(x_1+Q, x_2, 1) - \triangle_{11} T_U f(x_1, x_2, 0) = 0$. For (U,U), we have

$\triangle_{11} T_U f(x_1+Q, x_2, 1) \le \triangle_{11} T_R f(x_1+Q, x_2, 1)$ (by Lemma 4) $= \triangle_{11} T_U f(x_1, x_2, 0)$. For (R,U),

$T_R f(x_1+Q+1, x_2, 1) - T_U f(x_1+Q, x_2+1, 1) \le \triangle_{11} T_R f(x_1+Q, x_2, 1) = \triangle T_U f(x_1, x_2, 0)$.

Suppose $x_1 = 0$. Using Lemma 1 and 2, the optimal action in $(Q, N, 1)$, $(1, N-1, 0)$, and $(0, N, 0)$ is *Do not return, Do not rent*, and *Rent*, respectively. If the optimal action in $(Q+1, N-1, 1)$ is *Do not return*,

$\triangle_{11} Tf(Q, N-1, 1) - \triangle_{11} Tf(0, N-1, 0)$

$= T_U f(Q+1, N-1, 1) - T_U f(Q, N, 1) - (T_U f(1, N-1, 0) - T_R f(0, N, 0))$

$= T_U f(Q+1, N-1, 1) - T_U f(1, N-1, 0) - c_S \le 0$ (by (2)).

If the optimal action in$(Q+1, N-1, 1)$ is *Return*, $\triangle_{11} Tf(Q, N-1, 1) - \triangle_{11} Tf(0, N-1, 0)$

$= T_R f(Q+1, N-1, 1) - T_U f(Q, N, 1) - (T_U f(1, N-1, 0) - T_R f(0, N, 0)) = -c_R - c_S \le 0$.

(iii) We focus on the combinations of actions admissible in states $(x_1+Q, x_2, 1)$ and $(x_1, x_2, 0)$. Case (R,R) is excluded by Lemma 1. For (U,U), $x_1 > 0$ by Lemma 2 and

$T_U f(x_1+Q, x_2, 1) - T_U f(x_1, x_2, 0) = -(c_h + c_k)Q + \lambda[f(x_1+Q-1, x_2+1, 1) - f(x_1-1, x_2+1, 0)]$

$+ x_2\mu[f(I(x_1+Q, x_2), 1) - f(I(x_1, x_2), 0)] + (N+Q-x_2)\mu[f(x_1+Q, x_2, 1) - f(x_1, x_2, 0)]$

$\ge -(c_h + c_k)Q - \gamma(c_h + c_k)Q/(1-\gamma)$ (by (4)) $= -(c_h + c_k)Q/(1-\gamma)$. For (R,U), by (U,U),

$T_R f(x_1+Q, x_2, 1) - T_U f(x_1, x_2, 0) \ge T_U f(x_1+Q, x_2, 1) - T_U f(x_1, x_2, 0) \ge -(c_h + c_k)Q/(1-\gamma)$.

For (U,R), $x_1 = 0$ by Lemma 3 and $T_U f(Q, x_2, 1) - T_R f(0, x_2, 0) = c_S \ge -(c_h + c_k)Q/(1-\gamma)$.

(iv) Suppose $\delta = 1$. If $x_1 \ge Q$, case (U,R) is excluded by Lemma 4. For (R,R), $T_R f(x_1+1, x_2, 1) - T_R f(x_1, x_2+1, 1)$

$\le T_U f(x_1-Q+1, x_2, 0) - T_U f(x_1-Q, x_2+1, 0) \le 0$ (by (U,U) with $\delta = 0$).

For (U,U), $T_U f(x_1+1, x_2, 1) - T_U f(x_1, x_2+1, 1)$

$\le T_R f(x_1+1, x_2, 1) - T_R f(x_1, x_2+1, 1)$ (by Lemma 4) ≤ 0 (by(R,R)).

For (R,U), $T_R f(x_1+1, x_2, 1) - T_U f(x_1, x_2+1, 1) \le T_R f(x_1+1, x_2, 1) - T_R f(x_1, x_2+1, 1) \le 0$.

If $x_1 < Q$, *Do not return* is optimal in $(x_1+1, x_2, 1)$ and $(x_1, x_2+1, 1)$. Then,

$T_U f(x_1+1, x_2, 1) - T_U f(x_1, x_2+1, 1)$

$= -c_h - p + \lambda c_L 1x_1 = 0 + \lambda[f(x_1, x_2+1, 1) - f(D(x_1, x_2+1, 0))]1\{x_1 > 0\}$

$+ x_2\mu[f(I(x_1+1, x_2), 0) - f(x_1+1, x_2, 0)] + (N+Q-x_2-1)\mu[f(x_1+1, x_2, 0) - f(x_1, x_2+1, 0)]$

$\le -c_h - p + \lambda c_L 1x_1 = 0$ (by (5)) ≤ 0 (by (1)).

Suppose $\delta=0$. Then, the optimal action in $(x_1+1,x_2,0)$ is *Do not rent* by Lemma 2. If the optimal action in $(x_1,x_2+1,0)$ is *Do not rent,* $x_1>0$ by Lemma 2 and Lemma 3.
$T_U f(x_1+1,x_2,0)-T_U f(x_1,x_2+1,0)\le 0$ can be shown by the same argument in (U,U) with $\delta=1$. If *Rent* is optimal in $(x_1,x_2+1,0)$ is, $x_1=0$ by Lemma 3. And
$T_U f(1,N-1,0)-T_R f(0,N,0)\le T_U f(1,N-1,0)-T_U f(0,N,0)\le 0$ (by case (U,U)).

(v) Consider cases with $\delta=0$. Suppose that $x_1=0$. Then, the optimal action in $(1,x_2,0)$ and $(0,x_2+1,0)$ is *Do not rent* by Lemma 2 and *Rent,* respectively.
$T_U f(1,x_2,0)-T_R f(0,x_2,0)\ \le T_U f(1,x_2,0)-T_U f(0,x_2,0)$
$=\lambda c_L-p-c_h+x_2\mu\Delta_{11}f(1,x_2-1,0)-(N+Q-x_2-1)\mu\Delta_{11}f(0,x_2,0)$
$\le\lambda c_L-p-c_h+(N+Q-1)\mu c_L\le c_L-p-c_h\le c_L.$
Suppose that $x_1>0$. *Do not rent* is optimal in $(x_1+1,x_2,0)$ and $(x_1,x_2+1,0)$ by Lemma 2.
$T_U f(x_1+1,x_2,0)-T_U f(x_1,x_2,0)\le c_L$ can be shown using an argument similar to (U,U) with $x_1=0$. Cases with $\delta=1$ can be also shown by the same arguments in cases with $\delta=0$. □