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Categories of two types of uniform spaces

Yong Chan Kim[1] and Seok Jong Lee[2]

[1] **Department of Mathematics, Kangnung National University, Gangneung, 210-702, Korea**
[2] **Department of Mathematics, Chungbuk National University, Cheongju, 361-763, Korea**

Abstract

In a strictly two-sided, commutative biquantale, we study the relationships between the categories of Hutton $(L,\otimes)$-uniform spaces and $(L,\odot)$-uniform spaces. We investigate the properties of them.

Key words : Hutton $(L,\otimes)$-uniform spaces, $(L,\odot)$-uniform spaces

1. Introduction

Recently, Gutiérrez García and his colleagues [1] introduced L-valued Hutton unifomity where a quadruple $(L,\leq,\otimes,*)$ is defined by a GL-monoid $(L,*)$ dominated by $\otimes$, a cl-quasi-monoid $(L,\leq,\otimes)$. Kubiak and his colleagues [10] studied the relationships between the categories of $I(L)$-uniform spaces and L-uniform spaces. Kim and his colleagues [7], as a somewhat different aspect in [1], introduced the notion of Hutton $(L,\otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [13] and $(L,\odot)$-uniformities in a sense Lowen [11] and Höhle [12] based on powersets of the form $L^{X\times X}$.

In this paper, we show that the category **HUnif** of all Hutton $(L,\otimes)$-uniform spaces and H-uniformly continuous maps and the category **Unif** of all $(L,\odot)$-uniform spaces and uniformly continuous maps are isomorphic. Moreover, we define the subspaces of them.

2. Preliminaries

Definition 2.1. [3,4,7,8,12] A triple $(L,\leq,\odot)$ is called a *strictly two-sided, commutative biquantale* (stsc-biquantale, for short) iff it satisfies the following properties:

(L1) $L=(L,\leq,\vee,\wedge,\top,\bot)$ is a completely distributive lattice where $\top$ is the universal upper bound and $\bot$ is the universal lower bound;

(L2) $(L,\odot)$ is a commutative semigroup;

(L3) $a=a\odot\top$, for each $a\in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i)\odot b=\bigvee_{i\in\Gamma}(a_i\odot b).$$

(L5) $\odot$ is distributive over arbitrary meets, i.e.

$$(\bigwedge_{i\in\Gamma} a_i)\odot b=\bigwedge_{i\in\Gamma}(a_i\odot b).$$

Remark 2.2. [3,4,7,8,12] Let $(L,\leq,\odot)$ be a stsc-biquantale. For each $x,y\in L$, we define

$$x\rightarrow y=\bigvee\{z\in L\mid x\odot z\leq y\}.$$

Then it satisfies Galois correspondence, that is, $(x\odot y)\leq z$ iff $x\leq(y\rightarrow z)$.

In this paper, we always assume that $(L,\leq,\odot,^*)$ is a stsc-biquantale with strong negation * where $a^*=a\rightarrow 0$ unless otherwise specified.

Let X be a nonempty set. All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x\in X$, $f,g\in L^X$, $\lambda\in L^X$ and $\alpha\in L$,

(1) $f\leq g$ iff $f(x)\leq g(x)$;
(2) $(f\odot g)(x)=f(x)\odot g(x)$;
(3) $1_X(x)=\top$, $\alpha\odot 1_X(x)=\alpha$ and $1_\emptyset(x)=\bot$;
(4) $(\alpha\rightarrow\lambda)(x)=\alpha\rightarrow\lambda(x)$ and $(\lambda\rightarrow\alpha)(x)=\lambda(x)\rightarrow\alpha$;
(5) $(\alpha\odot\lambda)(x)=\alpha\odot\lambda(x)$.

Definition 2.3. [7] Let $\Omega(X)$ be a subset of $(L^X)^{(L^X)}$ such that

(O1) $\lambda\leq\phi(\lambda)$, for every $\lambda\in L^X$,
(O2) $\phi(\bigvee_{i\in\Gamma}\lambda_i)=\bigvee_{i\in\Gamma}\phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma}\subset L^X$,
(O3) $\alpha\odot\phi(\lambda)=\phi(\alpha\odot\lambda)$, for $\lambda\in L^X$.

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Corresponding Author : Seok Jong Lee, sjl@chungbuk.ac.kr

Lemma 2.4. [7] For $\phi, \phi_1, \phi_2 \in \Omega(X)$, we define, for all $\lambda \in L^X$,

$$\phi^{-1}(\lambda) = \bigwedge\{\rho \in L^X \mid \phi(\rho^*) \leq \lambda^*\},$$

$$\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),$$

$$\phi_1 \otimes \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\}.$$

For $\phi_1, \phi_2, \phi_3 \in \Omega(X)$, the following properties hold:

(1) If $\phi(1_{\{x\}}) = \rho_x$ for all $x \in X$, then $\phi(\lambda) = \bigvee_{z\in X} \lambda(z) \odot \rho_z$.

(2) If $\phi_1(1_{\{x\}}) = \phi_2(1_{\{x\}})$ for all $x \in X$, then $\phi_1 = \phi_2$.

(3) $\phi^{-1} \in \Omega(X)$, $(\phi^{-1})^{-1} = \phi$ and $\phi_1 \circ \phi_2 \in \Omega(X)$.

(4) If $\phi_1 \leq \phi_2$, then $\phi_1^{-1} \leq \phi_2^{-1}$.

(5) $\phi_1 \otimes \phi_2 \in \Omega(X)$.

(6) $\phi_1 \otimes \phi_2 \leq \phi_1$ and $\phi_1 \otimes \phi_2 \leq \phi_2$.

(7) $(\phi_1 \otimes \phi_2) \otimes \phi_3 = \phi_1 \otimes (\phi_2 \otimes \phi_3)$.

(8) $(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2) \leq (\phi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)$.

(9) Define $\phi_\top \in \Omega(X)$ as $\phi_\top(1_{\{x\}}) = 1_X, \forall x \in X$. Then $\phi \leq \phi_\top$ for all $\phi \in \Omega(X)$.

Definition 2.5. [7] A nonempty subset $\mathbf{U}$ of $\Omega(X)$ is called a *Hutton* $(L, \otimes)$*-quasi-uniformity* on X if it satisfies the following conditions:

(QU1) If $\phi \leq \psi$ with $\phi \in \mathbf{U}$ and $\psi \in \Omega(X)$, then $\psi \in \mathbf{U}$.

(QU2) For each $\phi, \psi \in \mathbf{U}$, $\phi \otimes \psi \in \mathbf{U}$.

(QU3) For each $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ such that $\psi \circ \psi \leq \phi$.

The pair $(X, \mathbf{U})$ is said to be a *Hutton* $(L, \otimes)$*- quasi-uniform space*.

A Hutton $(L, \otimes)$- quasi-uniform space is said to be a *Hutton* $(L, \otimes)$*-uniform space* if it satisfies

(U) For each $\phi \in \mathbf{U}$, there exists $\phi^{-1} \in \mathbf{U}$.

Definition 2.6. [7] Let $E(X \times X) = \{u \in L^{X\times X} \mid u(x, x) = 1\}$ be a subset of $L^{X\times X}$. A nonempty subset $\mathbf{D}$ of $E(X \times X)$ is called an $(L, \odot)$*-quasi-uniformity* on X if it satisfies the following conditions:

(QD1) If $u \leq v$ with $u \in \mathbf{D}$ and $v \in E(X \times X)$, then $v \in \mathbf{D}$.

(QD2) For each $u, v \in \mathbf{D}$, $u \odot v \in \mathbf{D}$.

(QD3) For each $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ such that $v \circ v \leq u$ where

$$v \circ v(x, y) = \bigvee_{z\in X} (v(x, z) \odot v(z, y)).$$

The pair $(X, \mathbf{D})$ is said to be an $(L, \odot)$*- quasi-uniform space*.

An $(L, \odot)$-quasi-uniform space is said to be an $(L, \odot)$*-uniform space* if it satisfies

(D) For each $u \in \mathbf{D}$, there exists $u^s \in \mathbf{U}$ where $u^s(x, y) = u(y, x)$.

Definition 2.7. [7] A function $u : X \times X \to L$ is called an $\odot$*-quasi-equivalence relation* iff it satisfies the following properties

(E1) $u(x, x) = 1$ for all $x \in X$.

(E2) $u(x, y) \odot u(y, z) \leq u(x, z)$.

An $\odot$-quasi-equivalence relation is called an $\odot$-equivalence relation on X if it satisfies

(E) $u(x, y) = u(y, x)$.

We denote $u^2 = u \odot u$ and $u^{n+1} = u^n \odot u$ for each $u \in L^{X\times X}$.

Theorem 2.8. [7] Let $u : X\times X \to L$ be an $\odot$-equivalence relation. We define a mapping $\mathbf{D}_u$ as follows:

$$\mathbf{D}_u = \{v \in E(X \times X) \mid \exists n \in N, u^n \leq v\}.$$

Then $\mathbf{D}_u$ is an $(L, \odot)$-uniformity on X.

Theorem 2.9. [7] We define a mapping $\Gamma : E(X \times X) \to \Omega(X)$ as follows:

$$\Gamma(u)(\lambda)(y) = \bigvee_{x\in X} \lambda(x) \odot u(x, y).$$

Then we have the following properties:

(1) For $u \in E(X \times X)$, $\Gamma(u) \in \Omega(X)$ and $\Gamma(u)$ has a right adjoint mapping $\Gamma(u)^{\leftarrow}$ defined by

$$\Gamma(u)^{\leftarrow}(\lambda) = \bigvee\{\rho \in L^X \mid \Gamma(u)(\rho) \leq \lambda\}.$$

(2) Γ is injective and join preserving.

(3) Γ has a right adjoint mapping $\Lambda : \Omega(X) \to E(X \times X)$ as follows:

$$\Lambda(\phi)(x, y) = \phi(1_{\{x\}})(y).$$

(4) $\Gamma \circ \Lambda = 1_{\Omega(X)}$ and $\Lambda \circ \Gamma = E(X \times X)$.

Theorem 2.10. [7] Let $u, u_1, u_2 \in E(X \times X)$. Then we have the following properties:

(1) If $u_1 \leq u_2$, $\Gamma(u_1) \leq \Gamma(u_2)$.

(2) $\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2)$.

(3) $\Gamma(1_\Delta) = 1_{L^X}$.

(4) $\Gamma(u)^{-1} = \Gamma(u^s)$.

(5) $\Gamma(u)^{-1}(\lambda \to \bot) = \Gamma(u)^{\leftarrow}(\lambda) \to \bot$, for all $\lambda \in L^X$.

(6) $\Gamma(u_1 \circ u_2) = \Gamma(u_2) \circ \Gamma(u_1)$.

(7) $\Gamma(\alpha \odot u) = \alpha \odot \Gamma(u)$.

(8) If u is an $\odot$-equivalence relation on X, then

$$(\Gamma(u))^{-1} = \Gamma(u^s) = \Gamma(u), \quad \Gamma(u) \circ \Gamma(u) = \Gamma(u).$$

Theorem 2.11. [7] Let $u : X \times X \to L$ be an $\odot$-equivalence relation. We define a mapping $\mathbf{U}_u$ as follows:

$$\mathbf{U}_u = \{\phi \in \Omega(X) \mid \exists n \in N,\ \Gamma(u^n) \leq \phi\}.$$

Then $\mathbf{U}_u$ is a Hutton $(L, \otimes)$-uniformity on X.

Theorem 2.12. [7] Let $\phi, \phi_1, \phi_2 \in \Omega(X)$. Then we have the following properties:

(1) If $\phi_1 \leq \phi_2$, then $\Lambda(\phi_1) \leq \Lambda(\phi_2)$.
(2) $\Lambda(\phi_1) \odot \Lambda(\phi_2) = \Lambda(\phi_1 \otimes \phi_2)$.
(3) $\Lambda(1_{L^X}) = 1_\Delta$.
(4) $\Lambda(\phi)^s = \Lambda(\phi^{-1})$.
(5) $\Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1)$.
(6) $\Lambda(\alpha \odot \phi) = \alpha \odot \Lambda(\phi)$.
(7) If $\phi \circ \phi = \phi$ and $\phi = \phi^{-1}$, then $\Lambda(\phi)$ is an $\odot$-equivalence relation.

Theorem 2.13. [7] Let $\mathbf{D}$ be an $(L, \odot)$-uniform space. We define a mapping $\mathbf{U}_{\mathbf{D}} \subset \Omega(X)$ as follows:

$$\mathbf{U}_{\mathbf{D}} = \{\phi \in \Omega(X) \mid \exists u \in \mathbf{D}, \Gamma(u) \leq \phi\}.$$

Then $\mathbf{U}_{\mathbf{D}}$ is a Hutton $(L, \otimes)$- uniformity on X.

Theorem 2.14. [7] Let $\mathbf{U}$ be a Hutton $(L, \otimes)$-uniformity on X. We define a mapping $\mathbf{D}_{\mathbf{U}} \subset E(X \times X)$ as follows:

$$\mathbf{D}_{\mathbf{U}} = \{u \in E(X \times X) \mid \exists \phi \in \mathbf{U}, \Lambda(\phi) \leq u\}.$$

Then:
(1) $\mathbf{D}_{\mathbf{U}}$ is an $(L, \odot)$-uniformity on X.
(2) $\mathbf{D}_{\mathbf{U}_{\mathbf{D}}} = \mathbf{D}$ and $\mathbf{U}_{\mathbf{D}_{\mathbf{U}}} = \mathbf{U}$.

3. Properties of two types of uniform spaces

Let $f : X \to Y$ be a function. We define the image and preimage operators

$$f^{\Rightarrow} : (L^X)^{(L^X)} \to (L^Y)^{(L^Y)},$$

$$f^{\Leftarrow} : (L^Y)^{(L^Y)} \to (L^X)^{(L^X)}$$

such that for each $\phi \in (L^X)^{(L^X)}$ and $\psi \in (L^Y)^{(L^Y)}$ for all $\mu, \mu_1, \mu_2 \in L^X, \rho_1, \rho_2 \in L^Y$,

$$f^{\Rightarrow}(\phi)(\rho) = (f^{\to} \circ \phi \circ f^{\leftarrow})(\rho) = f^{\to}(\phi(f^{\leftarrow}(\rho))),$$

$$f^{\Leftarrow}(\psi)(\mu) = (f^{\leftarrow} \circ \psi \circ f^{\to})(\mu) = f^{\leftarrow}(\psi(f^{\to}(\mu))).$$

Lemma 3.1. For each $\psi, \psi_1, \psi_2 \in \Omega(Y)$ and $\phi_1, \phi_2 \in \Omega(X)$, we have the following properties:

(1) The pair $(f^{\Rightarrow}, f^{\Leftarrow})$ is a Galois connection; i.e., $f^{\Rightarrow} \dashv f^{\Leftarrow}$.
(2) $f^{\to}(\mu_1 \odot \mu_2) \leq f^{\to}(\mu_1) \odot f^{\to}(\mu_2)$ with equality if f is injective and $f^{\leftarrow}(\rho_1 \odot \rho_2) = f^{\leftarrow}(\rho_1) \odot f^{\leftarrow}(\rho_2)$.
(3) $f^{\Leftarrow}(\psi) \in \Omega_X$.
(4) If $\psi_1 \leq \psi_2$, then $f^{\Leftarrow}(\psi_1) \leq f^{\Leftarrow}(\psi_2)$.
(5) $f^{\Leftarrow}(\psi_1) \circ f^{\Leftarrow}(\psi_2) \leq f^{\Leftarrow}(\psi_1 \circ \psi_2)$ with equality if f is onto.
(6) $(f^{\Leftarrow}(\psi))^{-1} = f^{\Leftarrow}(\psi^{-1}) \in \Omega_X$.
(7) $f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2) = f^{\Leftarrow}(\psi_1 \odot \psi_2)$ and $f^{\Rightarrow}(\phi_1) \odot f^{\Rightarrow}(\phi_2) \geq f^{\Rightarrow}(\phi_1 \odot \phi_2)$.
(8) $f^{\to}((f^{\Leftarrow}(\psi))^{-1}(\mu)) \leq \psi^{-1}(f^{\to}(\mu))$, for all $\mu \in L^X$.

Proof. (1) We prove the following statements:

$$\begin{aligned} f^{\Leftarrow}\Big(f^{\Rightarrow}(\psi)\Big)(\mu) &= f^{\leftarrow}\Big(f^{\Rightarrow}(\psi)\Big)(f^{\leftarrow}(\mu)) \\ &= f^{\leftarrow}(f^{\to}(\psi(f^{\leftarrow}(f^{\to}(\mu))))) \\ &\geq \psi(\mu). \end{aligned}$$

Similarly, $f^{\Rightarrow}(f^{\Leftarrow}(\phi))(\rho) \leq \phi(\rho)$. Thus, $f^{\Rightarrow} \dashv f^{\Leftarrow}$.

(2-5) can be easily proved.

(6) Suppose there exists $\mu \in L^X$ such that

$$(f^{\Leftarrow}(\psi))^{-1}(\mu) \not\leq f^{\Leftarrow}(\psi^{-1})(\mu).$$

By the definition of $f^{\Leftarrow}(\psi^{-1})(\mu) = f^{\leftarrow}(\psi^{-1}(f^{\to}(\mu)))$, there exists $\rho \in L^Y$ with $\psi(\rho^*) \leq (f^{\to}(\mu))^*$ such that

$$(f^{\Leftarrow}(\psi))^{-1}(\mu) \not\leq f^{\leftarrow}(\rho).$$

On the other hand, since

$$\psi(f^{\to}(f^{\leftarrow}(\rho)^*)) = \psi(f^{\to}(f^{\leftarrow}(\rho^*))) \leq \psi(\rho^*)$$

$$f^{\leftarrow}(\psi(\rho^*)) \leq f^{\leftarrow}(f^{\to}(\mu)^*) = (f^{\leftarrow}(f^{\to}(\mu)))^* \leq \mu^*,$$

we have $f^{\leftarrow}(\psi(f^{\to}(f^{\leftarrow}(\rho)^*))) \leq \mu^*$. So, $(f^{\Leftarrow}(\psi))^{-1}(\mu) \not\leq f^{\leftarrow}(\rho)$. It is a contradiction. Thus, $(f^{\Leftarrow}(\psi))^{-1} \leq f^{\Leftarrow}(\psi^{-1})$. It implies $(f^{\Leftarrow}(\psi^{-1}))^{-1} \leq f^{\Leftarrow}(\psi)$. So, $f^{\Leftarrow}(\psi^{-1}) \leq (f^{\Leftarrow}(\psi))^{-1}$. From (3) and Lemma 2.4 (3,4), $(f^{\Leftarrow}(\psi))^{-1} = f^{\Leftarrow}(\psi^{-1}) \in \Omega(X)$.

(7) Suppose there exist $\mu \in L^X$ and $x \in X$ such that

$$\Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(\mu)(x) \not\leq f^{\Leftarrow}(\psi_1 \odot \psi_2)(\mu)(x)$$

$$= f^{\leftarrow}(\psi_1 \odot \psi_2)(f^{\to}(\mu))(\psi(x)).$$

Then there exist $\nu_i \in L^Y$ with $f^{\to}(\mu) = \nu_1 \odot \nu_2$ such that

$$\Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(\mu)(x) \not\leq \psi_1(\nu_1)(f(x)) \odot \psi_2(\nu_2)(f(x))$$

Since $\mu \leq f^{\leftarrow}(f^{\rightarrow}(\mu)) = f^{\leftarrow}(\nu_1) \odot f^{\leftarrow}(\nu_2)$ from (2), we have

$$\begin{aligned}&\Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(\mu)\\ &\leq \Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(f^{\leftarrow}(\nu_1) \odot f^{\leftarrow}(\nu_2))\\ &\leq f^{\Leftarrow}(\psi_1)(f^{\leftarrow}(\nu_1)) \odot f^{\Leftarrow}(\psi_2)(f^{\leftarrow}(\nu_2))\\ &= f^{\leftarrow}(\psi_1(f^{\rightarrow}(f^{\leftarrow}(\nu_1)))) \odot f^{\leftarrow}(\psi_2(f^{\rightarrow}(f^{\leftarrow}(\nu_2))))\\ &\leq f^{\leftarrow}(\psi_1(\nu_1)) \odot f^{\leftarrow}(\psi_2(\nu_2)).\end{aligned}$$

Thus, $\Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(\mu)(x) \leq \psi_1(\nu_1)(f(x)) \odot \psi_2(\nu_2)(f(x))$. It is a contradiction. Hence $f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2) \leq f^{\Leftarrow}(\psi_1 \odot \psi_2)$.

Suppose there exist $\rho \in L^X$ and $x \in X$ such that

$$\Big(f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2)\Big)(\rho)(x) \not\geq f^{\Leftarrow}(\psi_1 \odot \psi_2)(\rho)(x)$$
$$= (\psi_1 \odot \psi_2)(f^{\rightarrow}(\rho))(f(x)).$$

Then there exist $\rho_i \in L^X$ with $\rho = \rho_1 \odot \rho_2$ such that

$$\begin{aligned}&\Big(f^{\Leftarrow}(\psi_1)(\rho_1) \odot f^{\Leftarrow}(\psi_2)(\rho_2)\Big)(x)\\ &= (\psi_1(f^{\rightarrow}(\rho_1))(f(x)) \odot (\psi_2(f^{\rightarrow}(\rho_2))(f(x))\\ &\not\geq f^{\Leftarrow}(\psi_1 \odot \psi_2)(\rho)(x)\end{aligned}$$

Since $f^{\rightarrow}(\rho) \leq f^{\rightarrow}(\rho_1) \odot f^{\rightarrow}(\rho_2)$ from (2),

$$\begin{aligned}&f^{\Leftarrow}(\psi_1 \odot \psi_2)(\rho)(x)\\ &= (\psi_1 \odot \psi_2)(f^{\rightarrow}(\rho))(f(x))\\ &\leq (\psi_1 \odot \psi_2)(f^{\rightarrow}(\rho_1) \odot f^{\rightarrow}(\rho_2))(f(x))\\ &= \Big(\psi_1(f^{\rightarrow}(\rho_1)) \odot \psi_2(f^{\rightarrow}(\rho_2))\Big)(f(x)).\end{aligned}$$

It is a contradiction. Thus

$$f^{\Leftarrow}(\psi_1) \odot f^{\Leftarrow}(\psi_2) \geq f^{\Leftarrow}(\psi_1 \odot \psi_2).$$

We will show $f^{\Rightarrow}(\phi_1) \odot f^{\Rightarrow}(\phi_2) \geq f^{\Rightarrow}(\phi_1 \odot \phi_2)$ from:

$$\begin{aligned}&f^{\Rightarrow}(\phi_1) \odot f^{\Rightarrow}(\phi_2)(\mu)\\ &= \bigwedge\{f^{\Rightarrow}(\phi_1)(\mu_1) \odot f^{\Rightarrow}(\phi_2)(\mu_2) \mid \mu = \mu_1 \odot \mu_2\}\\ &= \bigwedge\{f^{\rightarrow}(\phi_1(f^{\leftarrow}(\mu_1))) \odot f^{\rightarrow}(\phi_2(f^{\leftarrow}(\mu_2))) \mid\\ &\qquad \mu = \mu_1 \odot \mu_2\}\\ &\geq \bigwedge\{f^{\rightarrow}\Big(\phi_1(f^{\leftarrow}(\mu_1)) \odot \phi_2(f^{\leftarrow}(\mu_2))\Big) \mid\\ &\qquad f^{\leftarrow}(\mu) = f^{\leftarrow}(\mu_1) \odot f^{\leftarrow}(\mu_2)\}\\ &\geq f^{\rightarrow}\Big(\bigwedge\{\phi_1(f^{\leftarrow}(\mu_1)) \odot \phi_2(f^{\leftarrow}(\mu_2)) \mid\\ &\qquad f^{\leftarrow}(\mu) = f^{\leftarrow}(\mu_1) \odot f^{\leftarrow}(\mu_2)\}\Big)\\ &\geq f^{\rightarrow}(\phi_1 \odot \phi_2)(f^{\leftarrow}(\mu)).\end{aligned}$$

(8) From (6), we have for all $\mu \in L^X$,

$$\begin{aligned}&f^{\rightarrow}((f^{\Leftarrow}(\psi))^{-1}(\mu))\\ &= f^{\rightarrow}(f^{\Leftarrow}(\psi^{-1})(\mu)) \text{ (by (6))}\\ &= f^{\rightarrow}(f^{\leftarrow}(\psi^{-1}(f^{\rightarrow}(\mu))))\\ &\leq \psi^{-1}(f^{\rightarrow}(\mu)).\end{aligned}$$

□

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{x, y\}$ be sets and $L = [0, 1]$ an unit interval. Define a binary operation $\otimes$ (called Łukasiewicz conjection) on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}.$$

Then $([0, 1], \vee, \odot, 0, 1)$ is a stsc-biquantale (ref.[2-4]). Let $\mu, \nu \in [0, 1]^X$ as follows:

$$\mu(a) = 0.7, \mu(b) = 0.5, \mu(c) = 0.8,$$
$$\nu(a) = 0.6, \nu(b) = 0.9, \nu(c) = 0.7.$$

Then $(\mu \odot \nu)(a) = 0.3$, $(\mu \odot \nu)(b) = 0.4$, $(\mu \odot \nu)(c) = 0.5$.

Let $f : X \to Y$ be a function by $f(a) = f(b) = x, f(c) = y$. Then $f^{\rightarrow}(\mu)(x) = 0.7, f^{\rightarrow}(\mu)(y) = 0.8$ and $f^{\rightarrow}(\nu)(x) = 0.9, f^{\rightarrow}(\nu)(y) = 0.7$. Thus, $(f^{\rightarrow}(\mu) \odot f^{\rightarrow}(\nu))(x) = 0.6$ and $(f^{\rightarrow}(\mu) \odot f^{\rightarrow}(\nu))(y) = 0.5$. But $f^{\rightarrow}(\mu \odot \nu)(x) = 0.4, f^{\rightarrow}(\mu \odot \nu)(y) = 0.5$. Hence $f^{\rightarrow}(\mu \odot \nu) \neq f^{\rightarrow}(\mu) \odot f^{\rightarrow}(\nu)$ because f is not injective.

Lemma 3.3. Let $f : X \to Y$ be a function. For each $v, v_1, v_2 \in E(Y \times Y), \phi \in \Omega(Y)$ and $\lambda \in L^X$, we have:

(1) $f^{\Leftarrow}(\Gamma(v)) = f^{\leftarrow} \circ \Gamma(v) \circ f^{\rightarrow} = \Gamma((f \times f)^{\leftarrow}(v))$.

(2) $(f \times f)^{\leftarrow}(\Lambda(\phi)) = \Lambda(f^{\Leftarrow}(\phi))$.

(3) $\Gamma((f \times f)^{\leftarrow}(v^s)) = \Gamma(((f \times f)^{\leftarrow}(v))^s) = \Gamma((f \times f)^{\leftarrow}(v))^{-1}$.

(4) $(f \times f)^{\leftarrow}(v_1 \odot v_2) = (f \times f)^{\leftarrow}(v_1) \odot (f \times f)^{\leftarrow}(v_2)$.

(5) $(f \times f)^{\leftarrow}(v) \circ (f \times f)^{\leftarrow}(v) \leq (f \times f)^{\leftarrow}(v \circ v)$.

(6) If v is an $\odot$-equivalence relation on Y, then $(f \times f)^{\leftarrow}(v)$ is an $\odot$-equivalence relation on X.

Proof. (1) It is proved from:

$$\begin{aligned}&f^{\Leftarrow}(\Gamma(v))(\lambda)(x)\\ &= f^{\leftarrow} \circ \Gamma(v) \circ f^{\rightarrow}(\lambda)(x)\\ &= \Gamma(v)(f^{\rightarrow}(\lambda))(f(x))\\ &= \bigvee_{y \in Y}\{f^{\rightarrow}(\lambda)(y) \odot v(y, f(x))\}\\ &= \bigvee_{z \in X}\{f^{\rightarrow}(\lambda)(f(z)) \odot v(f(z), f(x))\}\\ &= \bigvee_{z \in X}\{\lambda(z) \odot (f \times f)^{\leftarrow}(v)(z, x)\}\\ &= \Gamma((f \times f)^{\leftarrow}(v))(\lambda)(x).\end{aligned}$$

(2)

$$\begin{aligned}\Lambda(f^{\Leftarrow}(\phi))(x, y) &= f^{\Leftarrow}(\phi)(1_{\{x\}})(y)\\ &= f^{\leftarrow}(\phi(f^{\rightarrow}(1_{\{x\}})))(y)\\ &= \phi(1_{\{f(x)\}})(f(y))\\ &= (f \times f)^{\leftarrow}(\Lambda(\phi))(x, y).\end{aligned}$$

(3)

$$\begin{aligned}&\Gamma((f \times f)^{\leftarrow}(v^s))(v^s)(\lambda)(x)\\ &= ((f \times f)^{\leftarrow}(v))^s(\lambda)\\ &= \bigvee_{y \in X}\{\lambda(y) \odot (f \times f)^{\leftarrow}(v^s)(y, x))\}\\ &= \bigvee_{y \in X}\{\lambda(y) \odot v^s(f(y), f(x))\}\\ &= \bigvee_{y \in X}\{\lambda(y) \odot (f \times f)^{\leftarrow}(v)(x, y)\}\\ &= \bigvee_{y \in X}\{\lambda(y) \odot ((f \times f)^{\leftarrow}(v))^s(y, x)\}\\ &= \Gamma(((f \times f)^{\leftarrow}(v))^s)(\lambda)(x)\end{aligned}$$

Furthermore, by Lemma 2.10(4), $\Gamma(((f\times f)^{\leftarrow}(v))^s) = \Gamma((f\times f)^{\leftarrow}(v))^{-1}$.

(4) It is easily proved.

(5)

$$\begin{aligned}&(f\times f)^{\leftarrow}(v)\circ(f\times f)^{\leftarrow}(v)(x_1,x_2)\\ &=\bigvee_{z\in X}(f\times f)^{\leftarrow}(v)(x_1,z)\odot(f\times f)^{\leftarrow}(v)(z,x_2)\\ &=\bigvee_{z\in X}v(f(x_1),f(z))\odot v(f(z),f(x_2))\\ &\le\bigvee_{y\in Y}v(f(x_1),y)\odot v(y,f(x_2))\\ &=v\circ v(f(x_1),f(x_2))\\ &=(f\times f)^{\leftarrow}(v\circ v)(x_1,x_2).\end{aligned}$$

(6) We have to check the axioms of Definition 2.7.

(E1) $(f\times f)^{\leftarrow}(v)(x,x)=v(f(x),f(x))=1$.

(E2) $(f\times f)^{\leftarrow}(v)\circ(f\times f)^{\leftarrow}(v)\le(f\times f)^{\leftarrow}(v\circ v)=(f\times f)^{\leftarrow}(v)$.

(E) $(f\times f)^{\leftarrow}(v^s)=((f\times f)^{\leftarrow}(v))^s$.

□

Example 3.4. Let $X=\{a,b,c,d\}, Y=\{x,y,z\}$ be sets and $([0,1],\odot)$ a biquantale defined by $x\odot y=\max\{0,x+y-1\}$. Define a function $f: X\to Y$ as follows:

$$f(a)=f(b)=x,\ f(b)=y,\ f(d)=z.$$

Let $v\in E(Y\times Y)$ be defined as

$$v(x,x)=v(y,y)=v(z,z)=v(x,y)=1,$$

$$v(y,x)=0.7,\quad v(y,z)=v(z,y)=0.6,$$

$$v(x,z)=v(z,x)=0.5.$$

Then

$$\Gamma(v)(1_{\{x\}})=\rho_x,\ \ \rho_x(x)=1,\rho_x(y)=1,\rho_x(z)=0.5,$$

$$\Gamma(u)(1_{\{y\}})=\rho_y,\ \ \rho_y(x)=0.7,\rho_y(y)=1,\rho_y(z)=0.6,$$

$$\Gamma(u)(1_{\{z\}})=\rho_z,\ \ \rho_z(x)=0.5,\rho_z(y)=0.6,\rho_z(z)=1.$$

Furthermore,

$$f^{\Leftarrow}(\Gamma(v))(1_{\{a\}})=f^{\Leftarrow}(\Gamma(v))(1_{\{b\}})=f^{\leftarrow}(\rho_x),$$

$$f^{\Leftarrow}(\Gamma(v))(1_{\{c\}})=f^{\leftarrow}(\rho_y),\ \ f^{\Leftarrow}(\Gamma(v))(1_{\{d\}})=f^{\leftarrow}(\rho_z).$$

Since

$$\begin{aligned}&\Gamma((f\times f)^{\leftarrow}(v))(1_{\{a\}})\\ &=\bigvee_{x\in X}1_{\{a\}}(x)\odot(f\times f)^{\leftarrow}(v)(x,-)\\ &=(f\times f)^{\leftarrow}(v)(a,-)=v(x,f(-))=f^{\leftarrow}(\rho_x),\end{aligned}$$

by a similar method, $\Gamma((f\times f)^{\leftarrow}(v))(1_{\{a\}}) = f^{\Leftarrow}(\Gamma(v))(1_{\{a\}})$ for all $a\in X$. By Lemma 2.4(2), $\Gamma((f\times f)^{\leftarrow}(v))=f^{\Leftarrow}(\Gamma(v))$.

Definition 3.5. (1) Let $(X,\mathbf{U}_1)$ and $(Y,\mathbf{U}_2)$ be Hutton $(L,\otimes)$-uniform spaces. A function $f:(X,\mathbf{U}_1)\to(Y,\mathbf{U}_2)$ is *H-uniformly continuous* if $f^{\Leftarrow}(\psi)\in\mathbf{U}_1$, for every $\psi\in\mathbf{U}_2$.

(2) Let $(X,\mathbf{D}_1)$ and $(Y,\mathbf{D}_2)$ be $(L,\odot)$-uniform spaces. A function $f:(X,\mathbf{D}_1)\to(Y,\mathbf{D}_2)$ is *uniformly continuous* if $(f\times f)^{\leftarrow}(v)\in\mathbf{D}_1$, for every $v\in\mathbf{D}_2$.

Theorem 3.6. (1) Let $(X,\mathbf{U}_1)$, $(Y,\mathbf{U}_2)$ and $(Z,\mathbf{U}_3)$ be Hutton $(L,\otimes)$-uniform spaces. If $f:(X,\mathbf{U}_1)\to(Y,\mathbf{U}_2)$ and $g:(Y,\mathbf{U}_2)\to(Z,\mathbf{U}_3)$ are H-uniformly continuous, then $g\circ f:(X,\mathbf{U}_1)\to(Y,\mathbf{U}_3)$ is H-uniformly continuous.

(2) Let $(X,\mathbf{D}_1)$, $(Y,\mathbf{D}_2)$ and $(Z,\mathbf{D}_3)$ be $(L,\odot)$-uniform spaces. If $f:(X,\mathbf{D}_1)\to(Y,\mathbf{D}_2)$ and $g:(Y,\mathbf{D}_2)\to(Z,\mathbf{D}_3)$ are uniformly continuous, then $g\circ f:(X,\mathbf{D}_1)\to(Y,\mathbf{D}_3)$ is uniformly continuous.

Proof. (1) Since $f^{\Leftarrow}(g^{\Leftarrow}(\psi))=(g\circ f)^{\Leftarrow}(\psi)$ for each $\psi\in\mathbf{U}_3$, it is easily proved.

(2) For each $v\in\mathbf{D}_3$, $((g\circ f)\times(g\circ f))^{\leftarrow}(v)=(f\times f)^{\leftarrow}((g\times g)^{\leftarrow}(v))\in\mathbf{D}_1$.

□

Theorem 3.7. Let $(X,\mathbf{D}_1)$ and $(Y,\mathbf{D}_2)$ be $(L,\odot)$-uniform spaces. If $f:(X,\mathbf{D}_1)\to(Y,\mathbf{D}_2)$ is uniformly continuous, then $f:(X,\mathbf{U}_{\mathbf{D}_1})\to(Y,\mathbf{U}_{\mathbf{D}_2})$ is H-uniformly continuous.

Proof. For each $\psi\in\mathbf{U}_{\mathbf{D}_2}$, there exists $v\in\mathbf{D}_2$ with $\Gamma(v)\le\psi$. Since f is uniformly continuous, for $v\in\mathbf{D}_2$, $(f\times f)^{\leftarrow}(v)\in\mathbf{D}_1$. By Lemma 3.3(1), since

$$\Gamma((f\times f)^{\leftarrow}(v))=f^{\Leftarrow}(\Gamma(v))\le f^{\Leftarrow}(\phi)$$

we have $f^{\Leftarrow}(\phi)\in\mathbf{U}_{\mathbf{D}_1}$.

□

Theorem 3.8. Let $(X,\mathbf{U}_1)$ and $(Y,\mathbf{U}_2)$ be Hutton $(L,\otimes)$-uniform spaces.

(1) A function $f:(X,\mathbf{U}_1)\to(Y,\mathbf{U}_2)$ is H-uniformly continuous iff $f:(X,\mathbf{D}_{\mathbf{U}_1})\to(Y,\mathbf{D}_{\mathbf{U}_2})$ is uniformly continuous.

(2) In Theorem 3.7, $f:(X,\mathbf{D}_1)\to(Y,\mathbf{D}_2)$ is uniformly continuous iff $f:(X,\mathbf{U}_{\mathbf{D}_1})\to(Y,\mathbf{U}_{\mathbf{D}_2})$ is H-uniformly continuous.

Proof. (1) For each $v \in \mathbf{D}_{\mathbf{U}_2}$, there exists $\psi \in \mathbf{U}_2$ with $\Lambda(\psi) \leq v$. Since f is H-uniformly continuous, for $\psi \in \mathbf{U}_2$, $f^{\Leftarrow}(\psi) \in \mathbf{U}_1$. By Lemma 3.3(2), since

$$(f \times f)^{\leftarrow}(\Lambda(\phi)) = \Lambda(f^{\Leftarrow}(\phi)) \leq (f \times f)^{\leftarrow}(v)$$

we have $(f \times f)^{\leftarrow}(v) \in \mathbf{D}_{\mathbf{U}_1}$.

Conversely, since $(\mathbf{U}_i)_{\mathbf{D}_{\mathbf{U}_i}} = \mathbf{U}_i$ for $i = 1, 2$ from Theorem 2.14, it is easily proved.

(2) Since $(\mathbf{D}_i)_{\mathbf{U}_{\mathbf{D}_i}} = \mathbf{D}_i$ for $i = 1, 2$ from Theorem 2.14, it is easily proved.

□

The class of all Hutton $(L, \otimes)$-uniform spaces and H-uniformly continuous maps forms a category, which is denoted by **HUnif**.

Moreover, the class of all $(L, \odot)$-uniform spaces and uniformly continuous maps forms a category, which is denoted by **Unif**.

Theorem 3.9. Define maps F : **HUnif** $\to$ **Unif** and G : **Unif** $\to$ **HUnif** by $F(X, \mathbf{U}) = (X, \mathbf{D}_{\mathbf{U}}), F(f) = f$ and $G(X, \mathbf{D}) = (X, \mathbf{U}_{\mathbf{D}}), G(g) = g$, respectively. Then F and G are functors and **HUnif** and **Unif** are isomorphic.

Proof. By Theorems 3.6-8, F and G are functors. From Theorem 2.14, $F \circ G(X, \mathbf{D}) = (X, \mathbf{D})$ and $G \circ F(X, \mathbf{U}) = (X, \mathbf{U})$. So, **HUnif** and **Unif** are isomorphic.

□

Theorem 3.10. Let $(Y, \mathbf{U})$ be a Hutton $(L, \otimes)$-uniform space, X a set and $f : X \to Y$ a function. Define a subset $\mathbf{U}^f$ of $\Omega(X)$ as follows:

$$\mathbf{U}^f = \{\phi \in \Omega(X) \mid \exists \psi \in \mathbf{U},\ \ f^{\Leftarrow}(\psi) \leq \phi\}.$$

Then we have the following properties.

(1) The structure $\mathbf{U}^f$ is the coarsest Hutton $(L, \otimes)$-uniformity on X for which each f is H-uniformly continuous.

(2) A map $g : (Z, \mathbf{U}_1) \to (X, \mathbf{U}^f)$ is H-uniformly continuous iff $f \circ g : (Z, \mathbf{U}_1) \to (Y, \mathbf{U})$ is H-uniformly continuous.

Proof. (1) First, we will show that $\mathbf{U}^f$ is a Hutton $(L, \otimes)$-uniformity on X.

(QU1) Obvious. (QU2) If $\phi_1, \phi_2 \in \mathbf{U}^f$, there exists $\psi_i \in \mathbf{U}$ with $f^{\Leftarrow}(\psi_i) \leq \phi_i$ for $i = 1, 2$. Since $f^{\Leftarrow}(\psi_1) \otimes f^{\Leftarrow}(\psi_2) = f^{\Leftarrow}(\psi_1 \otimes \psi_2) \leq \phi_1 \otimes \phi_2$ from Lemma 3.1(7), we have $\phi_1 \otimes \phi_2 \in \mathbf{U}^f$.

(QU3) For each $\phi \in \mathbf{U}^f$, there exists $\psi \in \mathbf{U}$ with $f^{\Leftarrow}(\psi) \leq \phi_i$. For $\psi \in \mathbf{U}$, since $(Y, \mathbf{U})$ is a Hutton $(L, \otimes)$-uniform space, by (QU3), there exists $\gamma \in \mathbf{U}$ with $\gamma \circ \gamma \leq \psi$. By Lemma 3.1(5), since

$$f^{\Leftarrow}(\gamma) \circ f^{\Leftarrow}(\gamma) \leq f^{\Leftarrow}(\gamma \circ \gamma) \leq f^{\Leftarrow}(\psi) \leq \phi,$$

then $f^{\Leftarrow}(\gamma) \in \mathbf{U}^f$.

(U) For each $\phi \in \mathbf{U}^f$, there exists $\psi \in \mathbf{U}$ with $f^{\Leftarrow}(\psi) \leq \phi$. For $\psi \in \mathbf{U}$, since $(Y, \mathbf{U})$ is a Hutton $(L, \otimes)$-uniform space, by (U), there exists $\psi^{-1} \in \mathbf{U}$. By Lemma 3.1(6), we have

$$f^{\Leftarrow}(\psi^{-1}) = (f^{\Leftarrow}(\psi))^{-1} \leq \phi^{-1}.$$

Thus, $\phi^{-1} \in \mathbf{U}^f$

Second, by definition of $\mathbf{U}^f$, $f^{\Leftarrow}(\psi) \in \mathbf{U}^f$, for all $\psi \in \mathbf{U}$. Hence $f : (X, \mathbf{U}^f) \to (Y, \mathbf{U})$ is H-uniformly continuous.

Finally, let $f : (X, \mathbf{U}_1) \to (Y, \mathbf{U})$ be H-uniformly continuous. For each $\phi \in \mathbf{U}^f$, there exists $\psi \in \mathbf{U}$ with $f^{\Leftarrow}(\psi) \leq \phi$. Since $f^{\Leftarrow}(\psi) \in \mathbf{U}_1$, then $\phi \in \mathbf{U}_1$. Hence $\mathbf{U}^f \subset \mathbf{U}_1$.

(2) Necessity of the composition condition is clear since the composition of H-uniformly continuous maps is H-uniformly continuous.

If $\phi \in \mathbf{U}^f$, there exists $\psi \in \mathbf{U}$ such that $f^{\Leftarrow}(\psi) \leq \phi$. Since $f \circ g$ is H-uniformly continuous, for $\psi \in \mathbf{U}$,

$$(f \circ g)^{\Leftarrow}(\psi) = g^{\Leftarrow} \circ f^{\Leftarrow}(\psi) \in \mathbf{U}_1.$$

Since $g^{\Leftarrow}(\phi) \geq g^{\Leftarrow} \circ f^{\Leftarrow}(\psi) \in \mathbf{U}_1$, we have $g^{\Leftarrow}(\phi) \in \mathbf{U}_1$. □

Theorem 3.11. Let $(Y, \mathbf{D})$ be an $(L, \odot)$-uniform space, X a set and $f : X \to Y$ a function. Define a subset $\mathbf{D}^f$ of $E(X \times X)$ as follows:

$$\mathbf{D}^f = \{u \in E(X \times X) \mid \exists v \in \mathbf{U},\ \ (f \times f)^{\leftarrow}(v) \leq u\}.$$

Then we have the following properties.

(1) The structure $\mathbf{D}^f$ is the coarsest $(L, \odot)$-uniformity on X for which each f is uniformly continuous.

(2) $g : (Z, \mathbf{D}_1) \to (X, \mathbf{D}^f)$ is uniformly continuous iff $f \circ g : (Z, \mathbf{D}_1) \to (Y, \mathbf{D})$ is uniformly continuous.

(3) $\mathbf{U}_{\mathbf{D}^f} = {\mathbf{U}_{\mathbf{D}}}^f$.

(4) If $(Y, \mathbf{U})$ be a Hutton $(L, \otimes)$-uniform space, then $\mathbf{D}_{\mathbf{U}^f} = {\mathbf{D}_{\mathbf{U}}}^f$.

Proof. (1) and (2) are similarly proved as in Theorem 3.10.

(3) If $\phi \in \mathbf{U}_{\mathbf{D}^f}$, there exists $u \in \mathbf{D}^f$ with $\Gamma(u) \leq \phi$. Since $u \in \mathbf{D}^f$, there exists $v \in \mathbf{D}$ such that $(f \times f)^{\leftarrow}(v) \leq u$. So, $v \in \mathbf{D}$ implies $\Gamma(v) \in \mathbf{U}_{\mathbf{D}}$. Since $f^{\Leftarrow}(\Gamma(v)) = \Gamma((f \times f)^{\leftarrow}(v)) \leq \Gamma(u) \leq \phi$ from Lemma 3.3(1), we have $\phi \in {\mathbf{U}_{\mathbf{D}}}^f$. Hence $\mathbf{U}_{\mathbf{D}^f} \subset {\mathbf{U}_{\mathbf{D}}}^f$.

If $\phi \in {\mathbf{U}_{\mathbf{D}}}^f$, there exists $\psi \in \mathbf{U}_{\mathbf{D}}$ with $f^{\leftarrow}(\psi) \leq \phi$. Since $\psi \in \mathbf{U}_{\mathbf{D}}$, there exists $v \in \mathbf{D}$ such that $\Gamma(v) \leq \psi$ and $(f \times f)^{\leftarrow}(v) \in \mathbf{D}^f$. Since

$$\Gamma((f \times f)^{\leftarrow}(v)) = f^{\Leftarrow}(\Gamma(v)) \leq f^{\Leftarrow}(\psi) \leq \phi,$$

we have $\phi \in \mathbf{U}_{\mathbf{D}^f}$. Hence $\mathbf{U}_{\mathbf{D}^f} \supset \mathbf{U_D}^f$.

(4) If $u \in \mathbf{D_U}^f$, there exists $v \in \mathbf{D_U}$ with $(f \times f)^{\leftarrow}(v) \leq u$. Since $v \in \mathbf{D_U}$, there exists $\psi \in \mathbf{U}$ such that $\Lambda(\psi) \leq v$. It follows $f^{\Leftarrow}(\psi) \in \mathbf{U}^f$. By Lemma 3.3(2), since $\Lambda(f^{\Leftarrow}(\psi)) = (f \times f)^{\leftarrow}(\Lambda(\psi) \leq (f \times f)^{\leftarrow}(v) \leq u$, we have $u \in \mathbf{D}_{\mathbf{U}^f}$. Hence $\mathbf{D}_{\mathbf{U}^f} \supset \mathbf{D_U}^f$.

If $u \in \mathbf{D}_{\mathbf{U}^f}$, there exists $\phi \in \mathbf{U}^f$ with $\Lambda(\phi) \leq u$. Since $\phi \in \mathbf{U}^f$, there exists $\psi \in \mathbf{U}$ such that $f^{\Leftarrow}(\psi) \leq \phi$. Since $\Lambda(\psi) \in \mathbf{D_U}$ and

$$(f \times f)^{\leftarrow}(\Lambda(\psi)) = \Lambda(f^{\Leftarrow}(\psi)) \leq \lambda(\phi) \leq u,$$

we have $u \in \mathbf{D_U}^f$. Hence $\mathbf{D}_{\mathbf{U}^f} \subset \mathbf{D_U}^f$.

□

Theorem 3.12. Let $w : Y \times Y \to L$ be an $\odot$-equivalence relation and $f : X \to Y$ a function. Then $\mathbf{U}_{\mathbf{D}^f_w} = \mathbf{U}^f_{\mathbf{D}_w}$ is defined as follows:

$$\mathbf{U}_{\mathbf{D}^f_w} = \{\phi \in \Omega(X) \mid \exists n \in N, f^{\Leftarrow}(\Gamma(w^n)) \leq \phi\}.$$

Proof. From Theorems 2.8 and 2.11, we obtain

$$\mathbf{D}_w = \{v \in E(Y \times Y) \mid \exists n \in N, w^n \leq v\},$$

$$\mathbf{U}_{\mathbf{D}_w} = \{\psi \in \Omega(Y) \mid \exists n \in N, \Gamma(w^n) \leq \phi\}.$$

Since $(f \times f)^{\leftarrow}(v)$ is an $\odot$-equivalence relation on X from Lemma 3.3(6), we obtain

$$\mathbf{D}^f_w = \{u \in E(X \times X) \mid \exists n \in N, (f \times f)^{\leftarrow}(v)^n \leq u\}.$$

Since $\Gamma((f \times f)^{\leftarrow}(w)^n) = \Gamma((f \times f)^{\leftarrow}(w^n)) = f^{\Leftarrow}(\Gamma(w^n))$ from Lemma 3.3(1), we have

$$\mathbf{U}_{\mathbf{D}^f_w} = \{\phi \in \Omega(X) \mid \exists n \in N, f^{\Leftarrow}(\Gamma(w^n)) \leq \phi\}.$$

□

Example 3.13. Let X, Y, f and $(L = [0,1], \odot)$ be defined as in Example 3.4. Let $w \in E(Y \times Y)$ be an $\odot$-equivalence relation on X as

$$w(x,x) = w(y,y) = w(z,z) = w(x,y) = w(y,x) = 1,$$

$$w(y,z) = w(z,y) = 0.6, w(x,z) = w(z,x) = 0.5.$$

Then

$$w^2(x,x) = w^3(y,y) = w^3(z,z)$$
$$= w^3(x,y) = w^3(y,x) = 1,$$
$$w^3(y,z) = w^3(z,y) = w^3(x,z) = w^3(z,x) = 0.$$

We obtain $\mathbf{D}_w, \mathbf{U}_{\mathbf{D}_w}, \mathbf{D}^f_w$ and $\mathbf{U}^f_{\mathbf{D}_w} = \mathbf{U}_{\mathbf{D}^f_w}$ as follows:

$$\begin{aligned}
\mathbf{D}_w &= \{v \in E(Y \times Y) \mid w^3 \leq v\} \\
\mathbf{U}_{\mathbf{D}_w} &= \{\psi \in \Omega(Y) \mid \Gamma(w^3) \leq \psi\} \\
\mathbf{D}^f_w &= \{u \in E(X \times X) \mid ((f \times f)^{\leftarrow}(w))^3 \leq u\} \\
\mathbf{U}^f_{\mathbf{D}_w} &= \mathbf{U}_{\mathbf{D}^f_w} = \{\phi \in \Omega(X) \mid f^{\Leftarrow}(\Gamma(w^3)) \leq \phi\}.
\end{aligned}$$

From Theorems 3.9 and 3.10, we can define subspaces in the obvious way.

Definition 3.14. Let A be a subset of X and $i : A \to X$ an inclusion function.

(1) Let $(X, \mathbf{U})$ be a Hutton $(L, \otimes)$-uniform space. The pair $(A, \mathbf{U}_A)$ where $\mathbf{U}_A = \{\phi \in \Omega(A) \mid \exists \psi \in \mathbf{U},\ i^{\Leftarrow}(\psi) \leq \phi\}$ is said to be a *subspace* of $(X, \mathbf{U})$.

(2) Let $(X, \mathbf{D})$ be an $(L, \odot)$-uniform space. The pair $(A, \mathbf{D}_A)$ where $\mathbf{D}_A = \{u \in E(A \times A) \mid \exists v \in \mathbf{D},\ (i \times i)^{\leftarrow}(v) \leq u\}$ is said to be a *subspace* of $(X, \mathbf{D})$.

Example 3.15. Let X, Y, f and $(L = [0,1], \odot)$ be defined as in Example 3.4.

(1) Define $\phi \in \Omega(Y)$ as follows:

$$\phi(1_{\{x\}}) = \phi(1_{\{y\}}) = 1_{\{x,y\}},\ \ \phi(1_{\{z\}}) = \phi(1_{\{z\}})$$

Since

$$\phi \otimes \phi(1_{\{x\}}) = \phi \otimes \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \otimes \phi(1_{\{z\}}) = 1_{\{z\}},$$

by Lemma 2.4(2), $\phi \otimes \phi = \phi$. We have $\phi \circ \phi = \phi$ because

$$\phi \circ \phi(1_{\{x\}}) = \phi \circ \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \circ \phi((1_{\{z\}})) = 1_{\{z\}}.$$

Since

$$\phi^{-1}(1_{\{x\}}) = \phi^{-1}(1_{\{y\}}) = 1_{\{x,y\}}, \phi^{-1}(1_{\{z\}}) = 1_{\{z\}},$$

Hence $\phi^{-1} = \phi$. Define $\mathbf{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$. Then $\mathbf{U}$ is a Hutton $(L, \otimes)$-uniformity on X. We obtain $\mathbf{D_U} = \{v \in E(X \times X) \mid \Lambda(\phi) \leq v\}$. Since $\phi \circ \phi = \phi$ and $\phi^{-1} = \phi$, by Theorem 2.12(7), $\Lambda(\phi)$ is an $\odot$-equivalence relation such that

$$\Lambda(\phi)(x,y) = \phi(1_{\{x\}})(y) = 1_{\{x,y\}}(y) = 1,$$
$$\Lambda(\phi)(x,x) = 1,\ \Lambda(\phi)(x,z) = 0$$
$$\Lambda(\phi)(y,x) = 1,\ \Lambda(\phi)(y,y) = 1,\ \Lambda(\phi)(y,z) = 0$$
$$\Lambda(\phi)(z,x) = 0,\ \Lambda(\phi)(z,y) = 0,\ \Lambda(\phi)(z,z) = 1$$

Furthermore, $\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi)$, $\Lambda(\phi^{-1}) = \Lambda(\phi)^s = \Lambda(\phi)$ and $\Lambda(\phi) \odot \Lambda(\phi) = \Lambda(\phi \otimes \phi) = \Lambda(\phi)$. Hence $\mathbf{D_U}$ is an $(L, \odot)$-uniformity on X and $\mathbf{U}_{\mathbf{D_U}} = \mathbf{U}$.

(2) We obtain $f^{\Leftarrow}(\phi) \in \Omega(X)$ as follows:

$$f^{\Leftarrow}(1_{\{a\}}) = f^{\Leftarrow}(1_{\{b\}}) = f^{\Leftarrow}(1_{\{c\}}) = 1_{\{a,b,c\}},$$
$$f^{\Leftarrow}(1_{\{d\}}) = 1_{\{d\}}$$

Then $\mathbf{U}^f = \{\psi \in \Omega(X) \mid f^{\Leftarrow}(\phi) \leq \psi\}$ is a Hutton $(L, \otimes)$-uniformity on X. We obtain $\mathbf{D}_{\mathbf{U}^f} = \{u \in E(X \times X) \mid \Lambda(f^{\Leftarrow}(\phi)) \leq u\}$ where

$$\Lambda(f^{\Leftarrow}(\phi))(x,y) = \begin{cases} 1 & x \in \{a,b,c\},\ y \in \{a,b,c\}, \\ 1 & x = d,\ y = d, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain $\mathbf{D_U}^f = \{u \in E(X \times X) \mid \exists v \in \mathbf{D_U}, (f \times f)^{\leftarrow}(v) \leq u\}$. Since $(f \times f)^{\leftarrow}(\Lambda(\phi)) = \Lambda(f^{\Leftarrow}(\phi))$, we have $\mathbf{D}_{\mathbf{U}^f} = \mathbf{D_U}^f$.

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Yong Chan Kim

Professor of Kangnung University.
Research Area: Fuzzy topology, Fuzzy logic
E-mail: yck@kangnung.ac.kr

Seok Jong Lee

Professor of Chungbuk National University
Research Area: Fuzzy mathematics, Fuzzy topology, General topology
E-mail : sjl@chungbuk.ac.kr