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R-fuzzy F-closed Spaces

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Abstract

In this paper, we introduce the concepts of r-fuzzy feebly open and r-fuzzy feebly closed sets in Šostak's fuzzy topological spaces and by using them, we explain the notions of r-fuzzy F-closed spaces. Also, we give some characterization of r-fuzzy F-closedness in terms of fuzzy filterbasis and r-fuzzy feebly-θ-cluster points.

Key Words : Fuzzy feebly open and fuzzy feebly closed sets, Sostak's fuzzy topology

1 Introduction

The concept of fuzzy topology was first defined in 1968 by Chang[?] and later redefined in somewhat different way by Lowen [?] and by Hutton[?]. According to Šostak, these definitions, a fuzzy topology is a crisp subfamily of family of fuzzy sets and fuzziness in the openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore, Šostak introduced a new definition of fuzzy topology in 1985[?], which we shall call "fuzzy topology in Šostake sense". In general topology, feebly open sets and relevant topological theory were introduced in [?] and [?]. Byung [?] introduced the concepts of fuzzy feebly open sets, fuzzy feebly closed sets in fuzzy topological spaces (Chang's sense) and by using them he explained the notions of fuzzy F-closed space and several fuzzy mappings. In this paper we introduce the concepts of fuzzy feebly open and fuzzy feebly closed sets in Šostak's fuzzy topology and by using them, we explain the notions of r-fuzzy F-closed spaces.

2 Preliminaries

Throughout this, paper, let X be a non-empty set $I = [0,1]$ and $I_0 = (0,1]$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. For fuzzy sets λ and μ in X, we write $\lambda q \mu$ to mean that λ is quasi-coincident (q-coincident, for short) with μ,i.e., there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \bar{q} \mu$.

DEFINITION 2.1. [?] A function $\tau\colon I^X \longrightarrow I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$.

(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.

(O3) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ for any $\{\lambda_i\}_{i\in\Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological spaces (fts, for short).

THEOREM 2.1. *[?]Let (X,τ) be a fts. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operator $C_\tau\colon I^X \times I_0 \longrightarrow I^X$ as follows:*

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu,\, \tau(\mu') \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator c_τ satisfies the following statements:

(C1) $C_\tau(\underline{0}, r) = \underline{0}$.

(C2) $\lambda \leq C_\tau(\lambda, r)$.

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(C3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.

(C4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ *if* $r \leq s$.

(C5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

THEOREM 2.2. *[?]Let* (X, τ) *be a fts. Then for each* $r \in I_0$, $\lambda \in I^X$ *we define an operator* $I_\tau\colon I^X \times I_0 \longrightarrow I^X$ *as follows:*

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \tau(\mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ *and* $r, s \in I_0$, *the operator* I_τ *satisfies the following statements:*

(I1) $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda, r)$.

(I2) $I_\tau(\underline{1}, r) = \underline{1}$.

(I3) $I_\tau(\lambda, r) \leq \lambda$.

(I4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.

(I5) $I_\tau(\lambda, s) \leq I_\tau(\lambda, r)$ *if* $r \leq s$.

(I6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

(I7) If $I_\tau(C_\tau(\lambda, r), r) = \lambda$, *then* $C_\tau(I_\tau(\underline{1} - \lambda, r), r) = \underline{1} - \lambda$.

DEFINITION 2.2. [?]Let (X, τ) be a fts, $\mu \in I^X$, $x_t \in \mathcal{P}(X)$ and $r \in I_0$ where $\mathcal{P}(X)$ is the family of all fuzzy points in X.

(a) μ is called a r-fuzzy open Q-neighborhood of x_t if $\tau(\mu) \geq r$ and $x_t q \mu$. We denote the set of all r fuzzy open Q-neighborhoods of x_t by $\mathcal{Q}_\tau(x_t, r)$.

(b) μ is called a r-fuzzy open R-neighborhood of x_t if $x_t q \mu$ and $\mu = I_\tau(C_\tau(\mu, r), r)$. We denote the set of all r-fuzzy open R-neighborhoods of x_t by $\mathcal{R}_\tau(x_t, r)$.

DEFINITION 2.3. [?]Let (X, τ) be a fts, $\lambda \in I^X$, $x_t \in \mathcal{P}(X)$ and $r \in I_0$.

(a) x_t is called r-fuzzy θ-cluster point of λ if for every $\mu \in \mathcal{Q}_\tau(x_t, r)$, we have $C_\tau(\mu, r) q \lambda$. We denote $\theta C_\tau(\lambda, r) = \bigvee\{x_t \in \mathcal{P}(X) \mid x_t \text{ is } r\text{-fuzzy } \theta\text{-cluster point of } \lambda\}$. Where $\theta C_\tau(\lambda, r)$ is called r-fuzzy θ-closure of λ.

(b) x_t is called a r-fuzzy δ-cluster point of λ if for every $\mu \in \mathcal{R}_\tau(x_t, r)$, we have $\mu q \lambda$. We denote $\delta C_\tau(\lambda, r) = \bigvee\{x_t \in \mathcal{P}(X) \mid x_t \ \ x_t \text{ is a } r\text{-fuzzy } \delta\text{-cluster point of } \lambda\}$. Where $\delta C_\tau(\lambda, r)$ is called a r-fuzzy δ-closure of λ.

THEOREM 2.3. *[?]Let* (X, τ) *a fts. For* λ, $\mu \in I^X$ *and* r, $s \in I_0$, *we have the following:*

(1) $\theta C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r),\ \tau(\mu') \geq r\}$

(2) x_t *is* r*-fuzzy* θ*-cluster point of* λ *iff* $x_t \in \theta C_\tau(\lambda, r)$.

(3) $C_\tau(\lambda, r) \leq \theta C_\tau(\lambda, r)$.

(4) If $\tau(\lambda) \geq r$, *then* $C_\tau(\lambda, r) = \theta C_\tau(\lambda, r)$.

(5) If λ *is* r*-fuzzy preopen and* $\lambda = C_\tau(I_\tau(\lambda, r), r)$, *then* $\theta C_\tau(\lambda, r) = \lambda$.

The complement of r-fuzzy θ-closed set is called r-fuzzy θ-open and the r-fuzzy θ-interior operator denoted by $\theta I_\tau(\lambda, r)$ is defined by $\theta I_\tau(\lambda, r) = \bigvee\{\nu \in I^X \mid C_\tau(\nu, r) \leq \lambda,\ \tau(\nu) \geq r\}$.

DEFINITION 2.4. Let (X, τ) be an fts, $\lambda \in I^X$, $r \in I_0$.

(a) λ is called a r fuzzy semiopen [] if $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$. The r-fuzzy semiclosure of λ, denoted by $sC_\tau(\lambda, r)$ is defined by $sC_\tau(\lambda, r) = \bigwedge\{\nu \in I^X \mid \nu \geq \lambda,\ \nu \text{ is } r\text{-fuzzy semiclosed}\}$

(b) λ is called a r-fuzzy preopen [?] if $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$. The r-fuzzy preclosure of λ, denoted by $pC_\tau(\lambda, r)$ is defined by $pC_\tau(\lambda, r) = \bigwedge\{\nu \in I^X \mid \nu \geq \lambda,\ \nu \text{ is } r\text{-fuzzy preclosed}\}$

THEOREM 2.4. *[?]If* λ *is rfuzzy preopen, then* $C_\tau(\lambda, r) = \delta C_\tau(\lambda, r) = \theta C_\tau(\lambda, r)$.

3 r-Fuzzy feebly open and r-Fuzzy feebly closed sets

DEFINITION 3.1. Let (X, τ) be a fts and λ, $r \in I_0$. Then λ is said to be

(a) r-fuzzy feebly open if there exists $\mu \in I^X$ such that $\tau(\mu) \geq r$ and $\mu \leq \lambda \leq sC_\tau(\mu, r)$.

(b) r-fuzzy feebly closed if its complement is r-fuzzy feebly open.

REMARK 3.1. It is obvious that every r-fuzzy open sets is r-fuzzy feebly open and every r-fuzzy feebly open set is r-fuzzy semiopen, but the converse may not be true.

EXAMPLE 3.1. *Let λ_1, λ_2, λ_3 be fuzzy sets of $X = \{a, b, c\}$ defined as follows:*

$$\lambda_1(a) = 0.5, \quad \lambda_1(b) = 0.4, \quad \lambda_1(c) = 0.4;$$
$$\lambda_2(a) = 0.6, \quad \lambda_2(b) = 0.7, \quad \lambda_2(c) = 0.8;$$
$$\lambda_3(a) = 0.5, \quad \lambda_3(b) = 0.4, \quad \lambda_3(c) = 0.6.$$

(1) *Consider τ_1 defined as*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0}, \underline{1}; \\ \frac{1}{2} & \text{if} \quad \lambda = \lambda_3; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda_3 \leq \lambda_2 \leq sC_{\tau_1}(\lambda_3, \frac{1}{2}) = \underline{1}$. Thus λ_2 is $\frac{1}{2}$-fuzzy feebly open, but λ_2 is not $\frac{1}{2}$-fuzzy open.

(2) *Consider τ_2 defined as*

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0}, \underline{1}; \\ \frac{1}{2} & \text{if} \quad \lambda = \lambda_1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda_1 \leq \lambda_3 \leq C_{\tau_2}(\lambda_3, \frac{1}{2})$. Thus λ_3 is $\frac{1}{2}$-fuzzy semiopen. Clearly $sC_{\tau_2}(\lambda_1, r) = \lambda_1$, then $\lambda_1 \leq \lambda_3 \not\leq sC_{\tau_2}(\lambda_1, r)$. Hence λ_3 is not $\frac{1}{2}$-fuzzy feebly open.

DEFINITION 3.2. Let (X, τ) be a fts and let $\lambda \in I^X$, $r \in I_0$. Then:

(a) r-fuzzy feebly closure of λ, denoted by $FC_\tau(\lambda, r)$ defined as follows:

$$FC_\tau(\lambda, r) = \bigwedge\{\nu \in I^X \mid \lambda \leq \nu, \; \nu \text{ is } r\text{-fuzzy feebly closed}\}$$

(b) r-fuzzy feebly interior of λ, denoted by $FI_\tau(\lambda, r)$ defined as follows:

$$FI_\tau(\lambda, r) = \bigvee\{\nu \in I^X \mid \nu \leq \lambda, \; \nu \text{ is } r\text{-fuzzy feebly open}\}$$

It is evident that $FC_\tau(\lambda, r) = \lambda$ iff λ is r-fuzzy feebly closed set and $FI_\tau(\lambda, r) = \lambda$ iff λ is r-fuzzy feebly open set.

DEFINITION 3.3. Let (X, τ) be a fts and $\lambda \in I^X$, $r \in I_0$. λ is called r-fuzzy feebly Q-nbd of a fuzzy point x_t if there exists $\mu \in I^X$ such that $\tau(\mu) \geq r$ and $x_t q \mu \leq \lambda$. We denote the set of all r-fuzzy feebly Q-nbd of x_t by $\mathcal{FQ}_\tau(x_t, r)$.

Lemma 3.1. *Let (X, τ) be a fts and $\lambda \in I^X$, $r \in I_0$. Then:*

(1) *$I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r) = I_\tau(C_\tau(\lambda, r), r)$ and $C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r) = C_\tau(I_\tau(\lambda, r), r)$;*

(2) *$I_\tau(C_\tau(\lambda, r), r)' = C_\tau(I_\tau(\lambda', r), r)$ and $C_\tau(I_\tau(\lambda, r), r)' = I_\tau(C_\tau(\lambda', r), r)$;*

(3) *$I_\tau(C_\tau(\lambda, r), r) \leq sC_\tau(\lambda, r)$*

(4) *$I_\tau(C_\tau(\lambda, r), r) = sC_\tau(\lambda, r)$, for each r-fuzzy preopen set $\lambda \in I^X$, $r \in I_0$.*

(5) *λ is r-fuzzy feebly open iff λ is rfuzzy α-open.*

Proof. The proof of (1), (2) and (3) are trivial. We only prove (4) and (5).

(4). By (3), it suffices to show that $sC_\tau(\lambda, r) \leq I_\tau(c_\tau(\lambda, r), r)$. Let $x_t \notin I_\tau(C_\tau(\lambda, r), r)$. Then $x_t q(I_\tau(C_\tau(\lambda, r), r)' = \mu$; μ is r-fuzzy semiopen. Since λ is r-fuzzy preopen, then $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$ and hence $\lambda \bar{q} \mu$. Hence there exist r-fuzzy semiopen set $\mu \in I^X$ with $x_t q \mu$ such that $\lambda \bar{q} \mu$. Then $x_t \notin sC_\tau(\lambda, r)$.

(5).($\Rightarrow$) Let λ be r-fuzzy feebly open. Then there is $\mu \in I^X$, $r \in I_0$ such that $\tau(\mu) \geq r$ and $\mu \leq \lambda \leq sC_\tau(\mu, r)$. By (4) $\mu \leq \lambda \leq I_\tau(C_\tau(\mu, r), r)$.

($\Leftarrow$) Let $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$. By (3) $I_\tau(\lambda, r) \leq \lambda \leq sC_\tau(I_\tau(\lambda, r), r)$. Thus λ is r-feebly open. □

THEOREM 3.1. *Let (X, τ) be a fts and let λ, $r \in I_0$. Then:*

(1) *$I_\tau(\lambda, r) \leq FI_\tau(\lambda, r) \leq sI_\tau(\lambda, r) \leq \lambda \leq sC_\tau(\lambda, r) \leq FC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.*

(2) *$FC_\tau(\lambda', r) = FI_\tau(\lambda, r)'$ and $FI_\tau(\lambda', r) = FC_\tau(\lambda, r)'$.*

Proof. Straightforward. □

Lemma 3.2. *Let (X, τ) be a fts and $\lambda \in I^X$, $r \in I_0$. Then we have:*

(1) *λ is r-feebly closed iff λ is r-fuzzy α-closed.*

(2) *$C_\tau(FC_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.*

(3) *$C_\tau(I_\tau(C_\tau(\lambda, r), r), r) \leq FC_\tau(\lambda, r)$.*

(4) *$FC_\tau(\lambda, r) = C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$, for each $\lambda \in I^X$, $r \in I_0$; $\tau(\lambda) \geq r$.*

Proof. Straightforward. □

The r-fuzzy feebly closure operator of a fuzzy set can be characterized as follows:

THEOREM 3.2. *Let (X, τ) be a fts and let λ_1, $\lambda_2 \in I^X$, $r \in I_0$. Then the following are true:*

(FC1) $FC_\tau(\underline{0}, r) = \underline{0}$,

(FC2) $FC_\tau(\lambda, r) \leq FC_\tau(\mu, r)$, *for each* $\lambda \leq \mu$, $r \in I_0$,

(FC3) $FC_\tau(\lambda, r) \leq FC_\tau(\lambda, s)$, *for each* $r, s \in I_0; r \leq s$,

(FC4) $FC_\tau(\lambda \vee \mu, r) = FC_\tau(\lambda, r) \vee FC_\tau(\mu, r)$,

(FC5) $FC_\tau(FC_\tau(\lambda, r), r) = FC_\tau(\lambda, r)$,

(FC6) *If* λ *is r-fuzzy feebly open set, then* $\lambda q \mu$ *iff* $\lambda q FC_\tau(\mu, r)$.

Proof. Straightforward. □

The r-fuzzy feebly interior operator of a fuzzy set can be characterized as follows:

THEOREM 3.3. *Let (X, τ) be a fts and let λ_1, $\lambda_2 \in I^X$, $r \in I_0$. Then the following are true:*

(FI1) $FI_\tau(\underline{1}, r) = \underline{1}$,

(FI2) $FI_\tau(\lambda, r) \leq FI_\tau(\mu, r)$, *for each* $\lambda \leq \mu$, $r \in I_0$,

(FI3) $FI_\tau(\lambda, r) \geq FI_\tau(\lambda, s)$, *for each* $r, s \in I_0$; $r \leq s$,

(FI4) $FI_\tau(\lambda \wedge \mu, r) = FI_\tau(\lambda, r) \wedge FI_\tau(\mu, r)$,

(FI5) $FI_\tau(FI_\tau(\lambda, r), r) = FI_\tau(\lambda, r)$,

Proof. Straightforward. □

THEOREM 3.4. *$x_t \in FC_\tau(\lambda, r)$ iff for every r-fuzzy feebly open Q-nbd of x_t is quasi-coincident with λ.*

Proof. Let $x_t \in FC_\tau(\lambda, r)$ and suppose there is $\nu \in FQ_\tau(x_t, r)$ such that $\nu \bar{q} \lambda$. Then $\lambda \leq \nu'$. Since ν' is r-fuzzy feebly closed, $FC_\tau(\lambda, r) \leq \nu'$. But $x_t \notin \nu'$. Thus $x_t \notin FC_\tau(\lambda, r)$ which is contradiction.
Conversely, suppose $x_t \notin FC_\tau(\lambda, r)$. Then there r-fuzzy feebly closed set $\mu \geq \lambda$ with $x_t \notin \mu$ so μ' is r-fuzzy feebly open with $x_t q \mu'$ and $\lambda \bar{q} \mu'$. Hence $x_t \notin FC_\tau(\lambda, r)$. □

DEFINITION 3.4. A fuzzy point x_t in an fts (X, τ) is said to be r-fuzzy f-θ-cluster point of a fuzzy set $\lambda \in I^X$ if $FC_\tau(\nu, r) q \lambda$ for each $\nu \in FQ_\tau(x_t, r)$. The union of all r-fuzzy f-θ-cluster points of λ is called the r-fuzzy f-θ-closure of λ and is denoted by $\theta FC_\tau(\lambda, r)$. A fuzzy set λ is called r-fuzzy f-θ-closed iff $\theta FC_\tau(\lambda, r) = \lambda$.

THEOREM 3.5. *For r-fuzzy feebly open set in an fts (X, τ), then $FC_\tau(\lambda, r) = \theta FC_\tau(\lambda, r)$.*

Proof. It suffices to show that $\theta FC_\tau(\lambda, r) \leq FC_\tau(\lambda, r)$. Let $x_t \notin FC_\tau(\lambda, r)$, then there exists $\mu \in FQ_\tau(x_t, r)$ such that $\mu \bar{q} \lambda$. Hence $\mu \leq \lambda'$, which implies $FC_\tau(\mu, r) \leq FC_\tau(\lambda', r) = \lambda'$. Then $FC_\tau(\mu, r) \bar{q} \lambda$ and so $x_t \notin \theta FC_\tau(\lambda, r)$. Thus $\theta FC_\tau(\lambda, r) \leq FC_\tau(\lambda, r)$. □

4 r-Fuzzy F-Closed Spaces

DEFINITION 4.1. A collection $\{\mu_j \mid j \in J\}$ of r-fuzzy feebly open sets in a fts (X, τ) is said to be r-fuzzy feebly open cover of a fuzzy set λ if $(\bigvee_{i \in J} \mu_j)(x) = 1$ for each $x \in Sup(\lambda)$. r-fuzzy feebly open cover of a fuzzy set λ in an fts (X, τ) is said to have a finite r-fuzzy feebly open subcover iff there exists a finite subset J_0 of J such that $(\bigvee_{j \in J_o} \mu_j)(x) \geq \lambda(x)$ for each $x \in Sup(\lambda)$.

DEFINITION 4.2. An fts (X, τ) is said to be r-fuzzy F-closed space if for every r-fuzzy feebly open cover $\{\lambda_i \mid i \in J\}$. There is a finite subset J_0 of J such that $\bigvee\{FC_\tau(\lambda_i, r) \mid i \in J_0\} = \underline{1}$.

THEOREM 4.1. *An fts (X, τ) is r-fuzzy F-closed space iff for every family $\{\lambda_i \mid i \in J\}$ of r-fuzzy feebly closed sets with $\bigwedge\{\lambda_i \mid i \in J\} = \underline{0}$, there is finite subset J_0 of J such that $\bigwedge\{FI_\tau(\lambda_i, r) \mid i \in J_0\} = \underline{1}$.*

Proof. ($\Rightarrow$) Suppose (X, τ) is r-fuzzy F-closed space. Let $\{\lambda_i \mid i \in J\}$ be a family of r-fuzzy feebly closed sets such that $\bigwedge\{\lambda_i \mid i \in J\} = \underline{0}$. Then $\{\lambda'_i \mid i \in J\}$ is a family of r-fuzzy feebly open sets such that $\bigvee\{\lambda'_i \mid i \in J\} = \underline{1}$. Since (X, τ) is r-fuzzy F-closed space, there is a finite subset J_0 of J such that $\bigvee\{FC_\tau(\lambda'_i, r) \mid i \in J_0\} = \underline{1}$. Thus $\bigwedge\{FI_\tau(\lambda_i, r) \mid i \in J_0\} = \underline{0}$.

($\Leftarrow$) Let $\{\lambda_i \mid i \in J\}$ be r-fuzzy feebly open cover of X. Then $\{\lambda'_i \mid i \in J\}$ is a family of r-fuzzy feebly closed sets such that $\bigwedge\{\lambda'_i \mid i \in J\} = \underline{0}$. By hypothesis, there is a finite subset J_0 of J such that $\bigwedge\{FI_\tau(\lambda'_i, r) \mid i \in J_0\} = \underline{0}$, i.e., $\bigvee\{FC_\tau(\lambda_i, r) \mid i \in J_o\} = \underline{1}$. Thus (X, τ) is r-fuzzy F-closed space. □

DEFINITION 4.3. [?]An fts (X, τ) is said to be:

(a) r-fuzzy compact if for each r-fuzzy open cover $\{\lambda_i \mid i \in J\}$ of X, there is a finite subset J_0 of J such that $\bigvee\{\lambda_i \mid i \in J_0\} = \underline{1}$.

(b) r-fuzzy almost compact if for each r-fuzzy open cover $\{\lambda_i \mid i \in J\}$ of X, there is a finite subset J_0 of J such that $\bigvee\{C_\tau(\lambda_i, r) \mid i \in J_0\} = \underline{1}$.

(c) r-fuzzy nearly compact if for each r-fuzzy open cover $\{\lambda_i \mid i \in J\}$ of X, there is a finite subset J_0 of J such that $\bigvee\{I_\tau(C_\tau(\lambda_i, r), r) \mid i \in J_0\} = \underline{1}$.

THEOREM 4.2. *Every r-fuzzy compact space is r-fuzzy F-closed.*

Proof. Let $\{\lambda_i \mid i \in J\}$ be r-fuzzy feebly open cover of X. Then for each $i \in J$, $\lambda \leq I_\tau(C_\tau(I_\tau(\lambda_i, r), r), r)$ and hence $\{I_\tau(C_\tau(I_\tau(\lambda_i, r), r), r) \mid i \in J\}$ is r-fuzzy open cover of X. Since (X, τ) is r-fuzzy compact, there is a finite subset J_0 of J such that

$$\begin{aligned}\bigvee_{i\in J_0}\{FC_\tau(\lambda_i, r)\} &\geq \bigvee_{i\in J_0}\{FC_\tau(I_\tau(\lambda_i, r), r)\}\\ &\geq \bigvee_{i\in J_0}\{C_\tau(I_\tau(C_\tau(I_\tau(\lambda_i, r), r), r), r)\}\\ &\geq \bigvee_{i\in J_0}\{I_\tau(C_\tau(I_\tau(\lambda_i, r), r), r)\} = \underline{1}.\end{aligned}$$

Thus (X, τ) is r-fuzzy F-closed space. □

DEFINITION 4.4. An fts (X, τ) is said to be r-fuzzy feebly regular (r-fr, for short) if for each r-fuzzy feebly open set λ in X is a union of r-fuzzy feebly open $\mu_i \in I^X$ such that $FC_\tau(\mu_i, r) \leq \lambda$ for each $i \in J$.

THEOREM 4.3. *Let (X, τ) be r-fuzzy feebly regular and r-fuzzy F-closed space. Then (X, τ) is r-fuzzy compact.*

Proof. Let $\{\lambda_i \mid i \in J\}$ be r-fuzzy open cover of X. Then $\{\lambda_i \mid i \in J\}$ is r-fuzzy feebly open cover of X. Since X is r-fr, $\lambda_i = \bigvee_{j\in J}\mu_i^j$, where μ_i^j is r-fuzzy feebly open set such that $FC_\tau(\mu_i^j, r) \leq \lambda_i$ for each $j \in J$. It follows that $\underline{1} = \bigvee_{i\in J}\lambda_i = \bigvee_{j\in J}\mu_i^j$. By r-fuzzy F-closedness of X, there is a finite subset J_0 of J such that $\bigvee_{j\in J_0} FC_\tau(\mu_i^j, r) = \underline{1}$. But $\bigvee_{j\in J_0} FC_\tau(\mu_i^j, r) \leq \bigvee_{i\in J_0}\lambda_i$. Thus $\bigvee_{i\in J_0}\lambda_i = \underline{1}$ and hence X is r-fuzzy compact. □

THEOREM 4.4. *Every r-fuzzy nearly compact space is r-fuzzy F-closed space.*

Proof. Similar to the proof of Theorem ??. □

DEFINITION 4.5. [?]A function $f\colon (X, \tau_1) \longrightarrow (Y, \tau_2)$ is said to be r-fuzzy almost continuous (for short, r-f.a.c) iff for each fuzzy point $x_t \in \mathcal{P}(X)$ and each $\nu \in \mathcal{Q}_{\tau_2}(f(x_t), r)$, there exists $\lambda \in \mathcal{Q}_{\tau_1}(x_t, r)$ such that $f(\lambda) \leq I_\tau(C_\tau(\nu, r), r)$.

THEOREM 4.5. *Let f be a function from an fuzzy topological space (X, τ_1) into an fuzzy topological space (Y, τ_2), then the following statements are equivalent:*

(1) r-f.a.c.

(2) $f(C_{\tau_1}(\lambda, r)) \leq \delta C_{\tau_2}(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$.

(3) $C_{\tau_1}(f^{-1}(\nu), r) \leq f^{-1}(\delta C_{\tau_2}(\nu, r))$, for each $\nu \in I^Y$, $r \in I_0$.

(4) $f^{-1}(\nu)$ is r-fuzzy closed for each r-fuzzy δ-closed set $\nu \in I^Y$.

(5) $f^{-1}(\nu)$ is r-fuzzy open for each r-fuzzy δ-open set $\nu \in I^Y$.

Proof. (1) ⇒ (2) : Let $x_t \in C_{\tau_1}(\lambda, r)$ and let $\nu \in \mathcal{Q}(f(x_t), r)$. By (1) there exists $\mu \in \mathcal{Q}_{\tau_1}(x_t, r)$ such that $f(\mu) \leq I_{\tau_2}(C_{\tau_2}(\nu, r), r)$. Since $x_t \in C_{\tau_1}(\lambda, r)$ which implies $f(\mu)qf(\lambda)$ which implies $f(\lambda)qI_{\tau_2}(C_{\tau_2}(\nu, r), r)$ which implies $f(x_t) \in \delta C_{\tau_2}(f(\lambda), r)$ which implies $x_t \in f^{-1}(\delta C_{\tau_2}(f(\lambda), r))$. Thus $C_{\tau_1}(\lambda, r) \leq f^{-1}(\delta C_{\tau_2}(f(\lambda), r))$ so that $f(C_{\tau_1}(\lambda, r)) \leq \delta C_{\tau_2}(f(\lambda), r)$.

(2) ⇒ (3) : Let $\nu \in I^Y$, then $f^{-1}(\nu) \in I^X$. By (2)

$$f(C_{\tau_1}(f^{-1}(\nu), r)) \leq \delta C_{\tau_2}(ff^{-1}(\nu), r) \leq \delta C_{\tau_2}(\nu, r)$$

and hence $C_{\tau_1}(f^{-1}(\nu), r) \leq f^{-1}(\delta C_{\tau_2}(\nu, r)$.

(3) ⇒ (4) : Let ν be r-fuzzy δ-closed set in I^Y, then $\delta C_{\tau_2}(\nu, r) = \nu$. By (3)

$$C_{\tau_1}(f^{-1}(\nu), r) \leq f^{-1}(\delta C_{\tau_2}(\nu, r)) = f^{-1}(\nu)$$

which implies that $C_{\tau_1}(f^{-1}(\nu), r) = f^{-1}(\nu)$. Thus $f^{-1}(\nu)$ is r-fuzzy closed.

(4) ⇒ (5) : Straightforward.

(5) ⇒ (1) : Straightforward.

□

THEOREM 4.6. *Let f be an r-f.a.c. function from an fuzzy topological space (X, τ_1) into an fuzzy topological space (Y, τ_2). Then the following statements are true*

(1) $f^{-1}(\nu) \leq I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu, r), r), r)), r)$, for each r-fuzzy feebly open set $\nu \in I^Y$.

(2) $C_{\tau_1}(f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(\nu, r), r), r)), r) \leq f^{-1}(\nu)$, for each r-fuzzy feebly closed set $\nu \in I^X$.

Proof. (1). Let $\nu \in I^Y$ be r-fuzzy feebly open set, then $\nu \leq I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)$ which implies $f^{-1}(\nu) \leq f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r))$. Since $I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)$ is r-fuzzy regular, we have

$$f^{-1}(\nu) \leq I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)),r).$$

(2). Straightforward. □

THEOREM 4.7. *If $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ is r-f.a.c. and (X,τ_1) is almost compact, then (Y,τ_2) is r-fuzzy F-closed space.*

Proof. Let $\{\nu_i \mid i \in I\}$ be r-fuzzy feebly open cover of Y. By Theorem ?? $\{I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r)),r)) \mid i \in I\}$ is r-fuzzy open cover of X. Since (X,τ_1) is r-fuzzy almost compact, then $\{C_{\tau_1}(I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)),r),r) \mid i \in I_0$, I_0 is finite subset of $I\}$ is r-fuzzy cover of X.

$$\begin{aligned}\underline{1} &= \bigvee_{i\in I_0} \{C_{\tau_1}(I_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)),r),r)\}\\ &\leq \bigvee_{i\in I_0} \{C_{\tau_1}(f^{-1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r)),r)\})\\ &\leq \bigvee_{i\in I_0} \{f^{-1}(\delta C_{\tau_1}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r)),r))\}\\ &\leq \bigvee_{i\in I_0} \{f^{-1}(C_{\tau_2}(I_{\tau_2}(C_{\tau_2}(I_{\tau_2}(\nu,r),r),r),r))\}\\ &\leq \bigvee_{i\in I_0} \{f^{-1}(C_{\tau_2}(I_{\tau_2}(\nu,r),r))\}\end{aligned}$$

Then

$$\begin{aligned}\underline{1} = f(\underline{1}) &\leq \bigvee\{f(f^{-1}(C_{\tau_2}(I_{\tau_2}(\nu,r),r))) \mid i \in I_0\}\\ &\leq \bigvee\{C_{\tau_2}(I_{\tau_2}(\nu,r),r) \mid i \in I_0\}\\ &\leq \bigvee\{FC_{\tau_2}(\nu,r) \mid i \in I_0\}\end{aligned}$$

Then (Y,τ_2) is r-fuzzy F-closed space. □

DEFINITION 4.6. [?]An fts (X,τ) is said to be r-fuzzy extremally disconnected if for each $\lambda \in I^X$, $r \in I_0$; $\tau(\lambda) \geq r$ we have $\tau(C_\tau(\lambda,r)) \geq r$.

Lemma 4.1. *Let (X,τ) be r-fuzzy extremally disconnected space and let $\lambda \in I^X$. Then λ is r-fuzzy feebly open iff λ is r-semiopen.*

Proof. Since every r-fuzzy feebly open set is r-fuzzy semiopen, it suffices to show that each r-fuzzy semiopen is r-fuzzy feebly open. Let λ be r-semiopen. Then $\lambda \leq C_\tau(I_\tau(\lambda,r),r)$. Since X is r-fuzzy extremally disconnected, $\lambda \leq C_\tau(I_\tau(\lambda,r),r) = I_\tau(C_\tau(I_\tau(\lambda,r),r),r)$. Thus λ is r-fuzzy feebly open. □

THEOREM 4.8. *Let λ be r-fuzzy feebly open set in r-fuzzy extremally disconnected space (X,τ). Then $FC_\tau(\lambda,r) = C_\tau(\lambda,r)$.*

Proof. It suffices to prove that $C_\tau(\lambda,r) \leq FC_\tau(\lambda,r)$. Let λ be r-fuzzy feebly open set and let $x_t \notin FC_\tau(\lambda,r)$. Then there is $\mu \in \mathfrak{FQ}_\tau(x_t,r)$ such that $\mu\bar{q}\lambda$, so $I_\tau(\mu,r)\bar{q}I_\tau(\lambda,r)$. But X is r-fuzzy extremally disconnected,

$$C_\tau(I_\tau(\mu,r),r)\bar{q}C_\tau(I_\tau(\lambda,r),r).$$

Hence $x_t \notin C_\tau(I_\tau(\lambda,r),r)$. But $\lambda \leq I_\tau(C_\tau(I_\tau(\lambda,r),r),r) = C_\tau(I_\tau(\lambda,r),r)$, so $C_\tau(\lambda,r) \leq C_\tau(I_\tau(\lambda,r),r)$. Thus $x_t \notin C_\tau(\lambda,r)$. This shows that $C_\tau(\lambda,r) \leq FC_\tau(\lambda,r)$ and this complete the proof. □

THEOREM 4.9. *Every r-fuzzy extremally disconnected, almost compact space is r-fuzzy F-closed space.*

Proof. Let $\{\lambda_i \mid i \in J\}$ be r-fuzzy feebly open cover of X. Then for each $i \in J$, $\lambda_i \leq I_\tau(C_\tau(I_\tau(\lambda_i,r),r),r)$ so $\{I_\tau(C_\tau(I_\tau(\lambda_i,r),r),r) \mid i \in J\}$ is r-fuzzy open cover of X. Since X is r-fuzzy almost compact, there is a finite subset J_0 of J such that $\bigvee\{C_\tau(I_\tau(C_\tau(I_\tau(\lambda_i,r),r),r),r) \mid i \in J_0\} = \underline{1}$. It follows that $\bigvee\{C_\tau(\lambda_i,r) \mid i \in J_0\} = \underline{1}$. By using Theorem ?? we have $\bigvee\{FC_\tau(\lambda_i,r) \mid i \in J_0\} = \underline{1}$, and this complete the proof. □

THEOREM 4.10. *Every r-fuzzy extremally disconnected , r-fuzzy F-closed space is r-fuzzy nearly compact.*

Proof. Let $\{\lambda_i \mid i \in J\}$ be r-fuzzy open cover of X. Then $\{\lambda_i \mid i \in J\}$ is r-fuzzy feebly open cover of X. Since X is r-fuzzy F-closed space, there is a finite subset J_0 of J such that $\bigvee\{FC_\tau(\lambda_i,r) \mid i \in J_0\} = \underline{1}$. Furthermore, since X is r-fuzzy extremally disconnected, then $FC_\tau(\lambda_i,r) = C_\tau(\lambda_i,r)$ and $C_\tau(\lambda_i,r) = I_\tau(C_\tau(\lambda_i,r),r)$ for each $i \in J_0$. Hence,

$$\begin{aligned}\bigvee\{I_\tau(C_\tau(\lambda_i,r),r) \mid i \in J_0\} &= \bigvee\{C_\tau(\lambda_i,r) \mid i \in J_0\}\\ &= \bigvee\{FC_\tau(\lambda_i,r) \mid i \in J_0\}\\ &= \underline{1}.\end{aligned}$$

Thus X is r-fuzzy nearly compact. □

DEFINITION 4.7. A mapping $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ is said to be:

(a) r-fuzzy feebly irresolute if $f^{-1}(\nu)$ is r-fuzzy feebly open for each r-fuzzy feebly open set $\nu \in I^Y$.

(b) r-fuzzy feebly continuous if $f^{-1}(\nu)$ is r-fuzzy feebly open for each $\nu \in I^Y$; $\tau_2(\nu) \geq r$.

THEOREM 4.11. *Let $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ be a mapping. Then the following are equivalent:*

(1) f is r-fuzzy feebly continuous,

(2) $f^{-1}(\nu)$ is r-fuzzy feebly closed for each r-fuzzy feebly closed set $\nu \in I^Y$,

(3) $f(FC_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$,

(4) $FC_{\tau_1}(f^{-1}(\nu), r) \leq f^{-1}(C_{\tau_2}(\nu, r))$, for each $\nu \in I^Y$, $r \in I_0$.

Proof. Straightforward. □

THEOREM 4.12. *Let $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ be a mapping. Then the following are equivalent:*

(1) f is r-fuzzy feebly irresolute,

(2) $f^{-1}(\nu)$ is r-fuzzy feebly closed for each r-fuzzy feebly closed set $\nu \in I^Y$,

(3) $f(FC_{\tau_1}(\lambda, r)) \leq FC_{\tau_2}(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$,

(4) $FC_{\tau_1}(f^{-1}(\nu), r) \leq f^{-1}(FC_{\tau_2}(\nu, r))$, for each $\nu \in I^Y$, $r \in I_0$.

Proof. Straightforward. □

THEOREM 4.13. *Let $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ be r-fuzzy feebly irresolute surjection mapping from r-fuzzy F-closed space (X,τ_1) to an fts (Y,τ_2). Then (Y,τ_2) is also r-fuzzy F-closed space.*

Proof. Let f be r-fuzzy irresolute mapping and let $\{\nu_i \mid i \in J\}$ be r-fuzzy feebly open cover of Y. Then $\{f^{-1}(\nu_i) \mid i \in J\}$ is r-fuzzy feebly open cover of X. Since (X,τ_1) is r-fuzzy F-closed, there is a finite subset J_0 of J such that

$$\underline{1} = \bigvee_{i \in J_0} \{FC_\tau(f^{-1}(\nu_i), r)\} = FC_\tau(\bigvee_{i \in J_0} f^{-1}(\nu_i), r)$$

Thus,

$$\begin{aligned}\underline{1} = f(\underline{1}) &= f(\bigvee_{i \in J_0} FC_{\tau_1}(f^{-1}(\nu_i), r)) \\ &= \bigvee_{i \in J_0} (f(FC_{\tau_1}(f^{-1}(\nu_i), r))) \\ &\leq \bigvee_{i \in J_0} FC_{\tau_2}(f(f^{-1}(\nu_i)), r) \\ &\leq \bigvee_{i \in J_0} FC_{\tau_2}(\nu_i, r).\end{aligned}$$

This shows that (Y,τ_2) is r-fuzzy F-closed space. □

THEOREM 4.14. *If $f\colon (X,\tau_1) \longrightarrow (Y,\tau_2)$ is r-fuzzy feebly continuous surjection mapping from r-fuzzy F-closed space (X,τ_1) to an fts (Y,τ_2). Then (Y,τ_2) is r-fuzzy almost compact.*

Proof. Similar to the proof of Theorem ??. □

DEFINITION 4.8. A collection $\mathcal{B}$ of fuzzy sets in an fts (X,τ) is said to form a fuzzy filterbase in X if for every finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$; $\bigwedge \mathcal{B}_0 \neq \underline{0}$.

DEFINITION 4.9. A fuzzy point x_t in an fts (X,τ) is said to be r-fuzzy fc-accumulation point of a fuzzy filterbase $\mathcal{B}$ if for each $\mu \in \mathcal{FQ}_\tau(x_t, r)$ and for each $\lambda \in \mathcal{B}$, $\lambda q FC_\tau(\mu, r)$.

THEOREM 4.15. *An fts (X,τ) is r-fuzzy F-closed iff each filterbase in X has r-fuzzy fc-accumulation point in X.*

Proof. Let (X,τ) be r-fuzzy F-closed space. Suppose that a fuzzy filterbase $\mathcal{B} = \{\mu_i \mid i \in J\}$ has no r-fuzzy fc-accumulation point in X. Then for each $x \in X$ and for each $t \in (0,1]$, x_t is not r-fuzzy fc-accumulation point of $\mathcal{B}$, so there are $\nu_x \in FQ_\tau(x_t, r)$ and $\mu_{i_{(x)}} \in \mathcal{B}$ such that $\mu_{i_{(x)}} \bar{q} FC_\tau(\nu_x, r)$. Now $\nu_x(x) > 1 - t$ for each $t \in (0,1]$, and hence $\{\nu_x \mid x \in X\}$ is r-fuzzy feebly open cover of X. Since (X,τ) is r-fuzzy F-closed space, there is a finite subfamily $\{\nu_{x_1}, \nu_{x_2}, \nu_{x_3}, .., \nu_{x_n}\}$ such that $\bigvee_{i=1}^n FC_\tau(\nu_{x_i}, r) = \underline{1}$. Then there is a finite subfamily $\{\mu_{j_{(x_1)}}, \mu_{j_{(x_2)}}, \mu_{j_{(x_3)}}, .., \mu_{j_{(x_n)}}\}$ of $\mathcal{B}$ such that $\bigwedge_{i=1}^n \mu_{j_{(x_i)}} \bar{q} \underline{1}$. This implies $\bigwedge_{i=1}^n \mu_{j_{(x_i)}} = \underline{0}$ which is contradiction.
Let $\{\lambda_i \mid i \in I\}$ be a family of r-fuzzy feebly closed sets such that $\bigwedge\{\lambda_i \mid \in I\} = \underline{0}$. Suppose that for each finite subset I_0 of I, $\bigwedge\{FI_\tau(\lambda_i, r) \mid i \in I_0\} \neq \underline{0}$. Then the family $\mathcal{B} = \{FI_\tau(\lambda_i, r) \mid i \in I_0\}$ forms a fuzzy filterbase in X, and hence by hypothesis, $\mathcal{B}$ has r-fuzzy fc-accumulation point x_t in X. Since $x_t \notin \bigwedge\{\lambda_i \mid i \in I\}$, there is $i_0 \in I$ with $x_t \notin \lambda_{i_0}$, so $x_t q \lambda'_{i_0}$, Thus $\lambda_{i_0} \in FQ_\tau(x_t, r)$ such that $FI_\tau(\lambda_{i_0}, r) \bar{q} (FI_\tau(\lambda_{i_0}, r)' = FC_\tau(\lambda'_{i_0}, r)$. It follows that x_t is not r-fuzzy fc-accumulation point, which is contradiction. □

THEOREM 4.16. *An fts (X,τ) is r-fuzzy F-closed space iff for each r-fuzzy feebly open filterbasis $\mathcal{B}$ in X, $\bigwedge_{\lambda \in \mathcal{B}} FC_\tau(\lambda, r) \neq \underline{0}$.*

Proof. Let $\mathcal{A}$ be r-fuzzy feebly open cover of X and let for each finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$, $(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r))(x) < 1$ for some $x \in X$. Then $(\bigwedge_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r))(x) > 0$ for some $x \in X$. Thus $\{FC_\tau(\mu,r)' \mid \mu \in \mathcal{A}_0\} = \mathcal{B}$ forms r-fuzzy feebly open filterbasis in X. Since $\mathcal{A}$ is r-fuzzy feebly open cover of X, then $\bigwedge_{\mu\in\mathcal{A}} \mu' = \underline{0}$ which implies $\bigwedge_{\mu\in\mathcal{A}} FC_\tau(FC_\tau(\mu,r)',r) = \underline{0}$, which is a contradiction. Then each r-fuzzy feebly open cover $\mathcal{A}$ in X has a finite subfamily $\mathcal{A}_0$ such that $(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r))(x) = 1$ for each $x \in X$. Hence X is r-fuzzy F-closed space.

Conversely, suppose there exists r-fuzzy feebly open filterbasis $\mathcal{B}$ in X such that $\bigwedge_{\mu\in\mathcal{B}} FC_\tau(\mu,r) = \underline{0}$, so that $(\bigvee_{\mu\in\mathcal{B}} FC_\tau(\mu,r)')(x) = 1$ for each $x \in X$ and hence $\mathcal{A} = \{FC_\tau(\mu,r)' \mid \mu \in \mathcal{B}\}$ is r-fuzzy feebly open cover of X. Since X is r-fuzzy F-closed space, then $\mathcal{A}$ has a finite subfamily $\mathcal{A}_0$ such that

$$(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(FC_\tau(\mu,r)',r))(x) = 1$$

for each $x \in X$, and hence

$$\bigwedge_{\mu\in\mathcal{A}_0} (FC_\tau(FC_\tau(\mu,r)',r)') = \underline{0}.$$

Thus $\bigwedge_{\mu\in\mathcal{A}_0} \mu = \underline{0}$ which is a contradiction. □

DEFINITION 4.10. A fuzzy set λ in an fts (X,τ) is said to be r-fuzzy F-closed relative to X iff each r-fuzzy feebly open cover $\mathcal{A}$ of λ, there exists a finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$ such that $(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r))(x) \geq \lambda(x)$ for each $x \in Sup(\lambda)$.

THEOREM 4.17. *A fuzzy set λ in an fts (X,τ) is r-fuzzy F-closed relative to X iff each r-fuzzy feebly open filterbasis $\mathcal{B}$ in X; $(\bigwedge_{\mu\in\mathcal{B}}(FC_\tau(\mu,r))) \wedge \lambda = \underline{0}$, there exists a finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $(\bigwedge_{\mu\in\mathcal{B}_0} \mu)\bar{q}\lambda$.*

Proof. Let λ be r-fuzzy F-closed set relative to X. Suppose $\mathcal{B}$ is r-fuzzy feebly open filterbasis in X such that for each finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$, $(\bigwedge_{\mu\in\mathcal{B}_0} \mu)q\lambda$, but $(\bigwedge_{\mu\in\mathcal{B}} FC_\tau(\mu,r)) \wedge \lambda = \underline{0}$. Then for each $x \in Sup(\lambda)$, $(\bigwedge_{\mu\in\mathcal{B}} FC_\tau(\mu,r))(x) = 0$ and hence $(\bigvee_{\mu\in\mathcal{B}}(FC_\tau(\mu,r)'))(x) = 1$ for each $x \in Sup(\lambda)$. Then $\mathcal{A} = \{FC_\tau(\mu,r)' \mid \mu \in \mathcal{B}\}$ is r-fuzzy feebly open cover of λ and hence there exists a finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $\bigvee_{\mu\in\mathcal{B}_0}(FC_\tau(FC_\tau(\mu,r)',r)) \geq \lambda$, so that

$$\bigwedge_{\mu\in\mathcal{B}_0} FC_\tau(FC_\tau(\mu,r)',r)' = \bigwedge_{\mu\in\mathcal{B}_0} FI_\tau(FC_\tau(\mu,r),r) \leq \lambda'$$

and hence $\bigwedge_{\mu\in\mathcal{B}_0} \mu \leq \lambda'$. Then $(\bigwedge_{\mu\in\mathcal{B}_0} \mu)\bar{q}\lambda$ which is a contradiction.

Conversely, let λ not be r-fuzzy F-closed set relative to X, then there exists r-fuzzy feebly open cover $\mathcal{A}$ of λ such that each finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$, $(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r))(x) < \lambda(x)$ for some $x \in Sup(\lambda)$ and hence

$$(\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r)')(x) > \lambda'(x) > 0$$

for some $x \in Sup(\lambda)$. Thus $\mathcal{B} = \{FC_\tau(\mu,r)' \mid \mu \in \mathcal{A}\}$ forms r-fuzzy feebly open filterbasis in X. Let there exists a finite subfamily $\{FC_\tau(\mu,r)' \mid \mu \in \mathcal{A}_0\}$ such that $(\bigwedge_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r)')\bar{q}\lambda$. Then $\lambda \leq \bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r)$. So there exists a finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$ such that $\bigvee_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r) \geq \lambda$ which is contradiction. Then for each finite subfamily $\mathcal{A}_0 = \{FC_\tau(\mu,r) \mid \mu \in \mathcal{A}_0\}$ of $\mathcal{A}$, we have $\bigwedge_{\mu\in\mathcal{A}_0} FC_\tau(\mu,r)'q\lambda$. Hence by the given condition $(\bigwedge_{\mu\in\mathcal{A}}(FC_\tau(FC_\tau(\mu,r)',r))) \wedge \lambda \neq \underline{0}$, so there exists $x \in Sup(\lambda)$ such that $(\bigwedge_{\mu\in\mathcal{A}}(FC_\tau(FC_\tau(\mu,r)',r)))(x) > 0$. Then $(\bigvee_{\mu\in\mathcal{A}}(FC_\tau(FC_\tau(\mu,r)',r)')(x) = \bigvee_{\mu\in\mathcal{A}} FI_\tau(FC_\tau(\mu,r),r))(x) < 1$ and hence $(\bigvee_{\mu\in\mathcal{A}} \mu)(x) < 1$ which contradicts the fact that $\mathcal{A}$ is r-fuzzy feebly open cover of λ. Therefore λ is r-fuzzy F-closed. □

THEOREM 4.18. *If each fuzzy filterbasis $\mathcal{B}$ in an fts (X,τ) such that for each finite subfamily $\mu_1, \mu_2, ..., \mu_n$ from $\mathcal{B}$ and for any r-fuzzy feebly closed set $\nu \geq \lambda$ has $(\mu_1 \wedge \mu_2 \wedge ...... \wedge \mu_n)q\nu$, $\mathcal{B}$ has r-fuzzy f-θ-cluster point in λ, then λ is r-fuzzy F-closed relative to X.*

Proof. Let $\mathcal{B}$ be r-fuzzy feebly open filterbasis in X not having r-f-cluster point in λ. By Theorem ?? $\mathcal{B}$ has no r-fuzzy f-θ-cluster point in λ. Then by hypothesis, there exists r-fuzzy feebly closed set $\nu \geq \lambda$ such that for some finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$; $(\bigwedge_{\mu\in\mathcal{B}_0} \mu)\bar{q}\nu$. Then $(\bigwedge_{\mu\in\mathcal{B}_0} \mu)\bar{q}\lambda$. Hence by Theorem ??, λ is r-fuzzy F-closed set relative to X. □

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