Characterization of Function Rings Between $C^*(X)$ and $C(X)$

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Abstract. Let $X$ be a Tychonoff space and $\sum(X)$ the set of all the subrings of $C(X)$ that contain $C^*(X)$. For any $A(X)$ in $\sum(X)$ suppose $v_A X$ is the largest subspace of $\beta X$ containing $X$ to which each function in $A(X)$ can be extended continuously. Let us write $A(X) \sim B(X)$ if and only if $v_A X = v_B X$, thereby defining an equivalence relation on $\sum(X)$. We have shown that an $A(X)$ in $\sum(X)$ is isomorphic to $C(Y)$ for some space $Y$ if and only if $A(X)$ is the largest member of its equivalence class if and only if there exists a subspace $T$ of $\beta X$ with the property that $A(X) = \{f \in C(X) : f^*(p) \text{ is real for each } p \text{ in } T\}$, $f^*$ being the unique continuous extension of $f$ in $C(X)$ from $\beta X$ to $\mathbb{R}^*$, the one point compactification of $\mathbb{R}$. As a consequence it follows that if $X$ is a realcompact space in which every $C^*$-embedded subset is closed, then $C(X)$ is never isomorphic to any $A(X)$ in $\sum(X)$ without being equal to it.

1. Introduction

It is well known in the theory of rings of continuous functions that for a Tychonoff space $X$, $C^*(X)$ is isomorphic to $C(\beta X)$, where $\beta X$ is the Stone-Čech compactification of $X$; in other words every $C^*$ is a function ring in the sense that it is isomorphic to some $C$. Intimately connected with this fact is the result that the structure space of each of the rings $C(X)$ and $C^*(X)$ is $\beta X$. This result has been superseded to a great extent by D. Plank [6], who has proved that the structure space of any ring that lies between $C^*(X)$ and $C(X)$ is also $\beta X$. We have

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shown that in case \( C^*(X) \neq C(X) \), there exists at least \( 2^c \) many such rings, where \( c \) is the cardinality of the continuum. It is therefore quite natural to ask which are function rings amongst these. We answer this in terms of a number of conditions, each necessary as well as sufficient for the positive answer to this question. A bit of elaboration is needed to explain these things. Suppose \( \sum(X) \) is the set of all rings that lie between \( C^*(X) \) and \( C(X) \), and \( A(X) \) is in \( \sum(X) \). Since the structure space of \( A(X) \) is \( \beta X \), the set of all maximal ideals can be written as \( \{ M^p_A : p \in \beta X \} \). \( M^p_A \) is called real if and only if the residue class field \( A(X)/M^p_A \) is isomorphic to \( \mathbb{R} \), otherwise it is called hyper-real. The set of all those points \( p \) in \( \beta X \) for which \( M^p_A \) is real, is denoted by \( v_A X \) and is called the \( A \)-compactification of \( \beta X \). In this terminology \( v_{C^*} X = v_X \) and \( v_{C^*} X = \beta X \). \( \beta X \) is called \( A \)-compact if every real maximal ideal in it is fixed. Therefore \( X \) becomes \( A \)-compact if and only if \( X = v_A X \), and it is established in [2] that \( v_A X \) is the largest subspace of \( \beta X \) containing \( X \) to which each function in \( A(X) \) can be extended continuously. Furthermore the space \( v_A X = \{ p \in \beta X : f^*(p) \in \mathbb{R} \text{ for each } f \in A(X) \} \), where \( f^* : \beta X \mapsto \mathbb{R} \equiv \mathbb{R} \cup \{ \infty \} \) is the unique continuous extension of \( f \) in \( C(X) \) over \( \beta X \). It follows that every \( A \)-compact space is realcompact. (See [2] for a detailed discussion on all these topics.)

For any \( A(X) \), \( B(X) \in \sum(X) \) we write \( A(X) \sim B(X) \) if and only if \( v_A X = v_B X \). Then \( \sim \) defines an equivalence relation on \( \sum(X) \). It was established in [2] that each equivalence class has a largest member, which we record for our ready reference.

**Theorem 1.1.** The largest member of the equivalence class \([A(X)]\) containing \( A(X) \) is given by \( \{ g|_X : g \in C(v_A X) \} \).

We establish in section 2 that these largest members can indeed be achieved by considering suitable subsets of \( \beta X \) in the form of the following proposition:

**Theorem 1.2.** \( A(X) \in \sum(X) \) is the largest member of \([A(X)]\) if and only if there exists a subset \( T \) of \( \beta X \) with the property:

\[
A(X) = \{ f \in C(X) : f^*(p) \in \mathbb{R} \text{ for each } p \in T \}.
\]

It is clear from Theorem 1.1 that if \( A(X) \) is the largest member of its equivalence class then the canonical map: \( f \mapsto f^{v_A} \) establishes an isomorphism from the ring \( A(X) \) onto the ring \( C(v_A X) \), and in particular \( A(X) \) is identified as a function ring. (Here \( f^{v_A} \) stands for the unique real valued continuous extension of \( f \) from \( X \) to \( v_A X \).) It is interesting to note that any function ring in the family \( \sum(X) \) also shares this property. Indeed in section 2, we prove the following result:

**Theorem 1.3.** Let \( A(X) \in \sum(X) \) be a function ring. Then the map \( f \mapsto f^{v_A} \) defines an isomorphism from \( A(X) \) onto the ring \( C(v_A X) \).
This Theorem 1.3 has turned out to be crucial towards the following characterization of function rings, which we also establish in section 2.

**Theorem 1.4.** \( A(X) \in \sum(X) \) is a function ring if and only if it is the largest member of its equivalence class.

Combining Theorems 1.2, 1.3, 1.4 we have the following comprehensive result almost immediately.

**Theorem 1.5.** For any ring \( A(X) \) lying between \( C^*(X) \) and \( C(X) \) the following statements are equivalent.

1. \( A(X) \) is a function ring.
2. \( A(X) \) is the largest member of its equivalence class.
3. \( A(X) \) is isomorphic to the ring \( C(\nu_A X) \) under the canonical mapping \( f \mapsto f^{\nu_A} \).
4. There exists a subset \( T \) of \( \beta X \) such that \( A(X) = \{ f \in C(X) : f^*(p) \in \mathbb{R} \forall p \in T \} \).

We conclude this introductory section with the statement of our final theorem, which we also prove in section 2. It is well known that \( C(X) \) is never isomorphic to \( C^*(X) \) without being equal to it [4]. For a large class of spaces \( X \) we have improved this result in the following form.

**Theorem 1.6.** Suppose \( X \) is a non-compact realcompact space in which every \( C^* \)-embedded subset is closed (in particular therefore \( X \) may be a metrizable space with non measurable cardinal). Then given any \( A(X) \in \sum(X) \) with \( A(X) \neq C(X) \), \( C(X) \) is never isomorphic to \( A(X) \).

It is not known to us whether this theorem remains still valid without the assumption of the closedness of the \( C^* \)-embedded subset of \( X \).

## 2. Function rings: their characterizations

In this section our principal aim is to give proofs of the Theorems 1.2, 1.3, 1.4, 1.6 stated in the introductory section. For any subset \( T \) of \( \beta X \) let us set

\[
C_T(X) \equiv C_T = \{ f \in C(X) : f^*(p) \in \mathbb{R} \forall p \in T \}.
\]

Then it is easy to see that \( C_T(X) \) is a subring of \( C(X) \) containing \( C^*(X) \).

**Proof of Theorem 1.2.** Let \( T \) be a subset of \( \beta X \) and let \( B(X) \) be a member of \( \sum(X) \) with \( \nu_B X = \nu_{C_T} X \). We choose \( f \) in \( A(X) \) and \( p \) in \( T \) arbitrarily. Since \( T \subset \nu_{C_T} X \) it follows that \( p \in \nu_{C_T} X \) and therefore \( p \in \nu_B X \). Accordingly \( f^*(p) \in \mathbb{R} \). Thus \( f \in C_T \). Hence \( A(X) \subset C_T(X) \), consequently \( C_T(X) \) is the largest member of its
equivalence class. Conversely let $A(X)$ be the largest member of its equivalence class $[A(X)]$. We shall show that $A(X) = C_{v_A X}(X)$. Since for each $f$ in $A(X)$ and $p$ in $v_A X$, $f^*(p)$ is real, it is trivial that $A(X) \subseteq C_{v_A X}(X)$. Conversely let $f \in C_{v_A X}(X)$ and $p \in v_A X$. Then $f^*(p)$ is real and therefore $g = f^*|_{v_A X} \in C(v_A X)$. Theorem 1.1 tells us that $g|_X \in A$, but since $g|_X = f$ it follows that $f \in A(X)$. Thus $C_{v_A X}(X) \subseteq A(X)$. Hence $A(X) = C_{v_A X}(X)$.

For any point $p$ in $\beta X - v X$, let us write $C_p$ instead of $C_{\{p\}}$. Now choosing any point $q$ in $\beta X - v X$ different from $p$ we can find an $l \in C^*(X)$ with $l^3(p) = 0$ and $l^3(q) = 1$ with $l$ not vanishing anywhere on $X$ ($l^3$ of course stands for the Stone extension of $l$ from $X$ to $\beta X$.) On the other hand there also exists an $h \in C^*(X)$, not vanishing anywhere on $X$ for which $h^3(q) = 0$. On putting $k = l^2 + h^2$, we have $k \in C^*(X)$, $k^3(q) = 0$ and $k^3(p) \geq 1$. If we take $g = \frac{1}{k}$, then $g^*(q) = \infty$ and $g^*(p) \in \mathbb{R}$. Consequently $g \in C_p$ and $q \notin v_{C_p} X$. Thus we have for each element $p$ in $\beta X - v X$, $v_{C_p} X = v X \cup \{p\}$. Since for a non-pseudocompact space $X$, $\beta X - v X$ contains at least $2^\omega$ many points [4], it follows that $\{C_p : p \in \beta X - v X\}$ is an infinite set containing at least $2^\omega$ many members with $C_p \neq C_q$ whenever $p \neq q$. Clearly then if $C(X) \neq C^*(X)$, there exists at least $2^\omega$ many different equivalence classes in the family $\sum(X)$.

**Proof of Theorem 1.3.** Since $A(X)$ is a function ring, there exists a realcompact space $Y$ with an isomorphism $t$ from $A(X)$ onto $C(Y)$. As the property of being a real maximal ideal is an algebraic invariant, for any $p \in v_A X$, $t(M^p_A)$ is a real maximal ideal in $C(Y)$ and therefore it is fixed due to the realcompactness of $Y$. Accordingly $\bigcap_{g \in t(M^p_A)} Z_Y(g)$ is a singleton set, where $Z_Y(g)$ stands for the zero set of the function $g$ in the space $Y$. We define a mapping $\Psi : v_A X \rightarrow Y$ by the rule $\Psi(p) = \bigcap_{g \in t(M^p_A)} Z_Y(g)$. Then $\Psi$ is clearly one-to-one. Again for any point $y$ in $Y$ if $M_y = \{h \in C(Y) : h(y) = 0\}$ is the corresponding fixed maximal ideal in $C(Y)$, then there is a unique point $p$ in $v_A X$ with $M_y = t(M^p_A)$ and so $\Psi(p) = y$. Now $\{S_A(f) : f \in A(X)\}$ being a base for closed subsets of $\beta X$, where $S_A(f) = \{p \in \beta X : f \in M^p_A\}$ [6], it is obvious that $\{S_A(f) \cap v_A X : f \in A(X)\}$ is a base for closed subsets of $v_A X$ and we observe that for any $f \in A(X)$, $\Psi(S_A(f) \cap v_A X) = Z_Y(f^*(f)).$ Hence $\Psi$ carries the basic closed sets in $v_A X$ onto $Y$. Suppose $s : C(Y) \rightarrow C(\beta X)$ is the isomorphism induced by $\Psi$, that is, for any $g$ in $C(Y)$, $s(g) = g \circ \Psi$. Since $t : A(X) \rightarrow C(Y)$ is already an isomorphism, we see that $s \circ t$ becomes an isomorphism from $A(X)$ onto $C(v_A X)$. Let us choose an $f \in A(X)$. To prove the theorem it is sufficient to prove that $t(f) \circ \Psi = f^\psi A$. Since $t(f) \circ \Psi$ is clearly a real-valued continuous function on $v_A X$ it is enough to show that it is an extension of $f$. Now if we choose $x \in X$ then for any $h$ in the fixed maximal ideal $M^x_A$ of $A(X)$, we have $t(h)(\Psi(x)) = 0$. Hence it follows that $t(f - f(x))(\Psi(x)) = 0$ (where $f(x)$ is the constant function on $X$ which takes the value $f(x)$ at all points of $X$), so that $t(f)(\Psi(x)) = f(x)$. □

**Proof of Theorem 1.4.** If $A(X)$ is the largest member of its equivalence class, then we have already observed in the introductory section that $A(X)$ is a function ring.
Conversely, suppose $A(X)$ is not the largest member of its equivalence class. Then from Theorem 1.1 there exists a $g \in C(v_A X)$ for which $g|_X$ is not in $A(X)$. Accordingly there cannot exist any $f \in A(X)$ with $f^v = g$ and therefore the canonical map $T : A(X) \to C(v_A X)$ defined by $T(f) = f^v$ is not an isomorphism on $A(X)$ onto $C(v_A X)$. Hence by Theorem 1.3, $A(X)$ is not a function ring. □

Proof of Theorem 1.6. If $A(X)$ is not the largest member of its equivalence class, then from Theorem 1.4 it is not a function ring and therefore it can not be isomorphic to $C(X)$. Next suppose that $A(X)$ is the largest member of its equivalence class contained properly in $C(X)$. Hence $A(X)$ does not belong to $[C(X)]$. Now since $X$ is realcompact, $X = vX$ and therefore $X \subseteq v_A X$. Again the result of Theorem 1.5 tells us that $A(X)$ is not isomorphic to $C(v_A X)$. Suppose now that $A(X)$ is isomorphic to $C(X)$. Since $X$ and $v_A X$ are both realcompact, it follows in view of Hewitt’s isomorphism theorem [4] that $X$ is homeomorphic to $v_A X$ under a mapping say, $\alpha : v_A X \to X$. As $X$ is dense in $v_A X$, it is plain that $\alpha(X)$ is also dense in $X$ and is also contained in $X$ properly; but $X$ is also $C^*$-embedded in $v_A X$ from which it follows that $\alpha(X)$ is $C^*$-embedded in $X$ and therefore closed in $X$. Altogether we get $\alpha(X) = X$, a contradiction. Hence $A(X)$ is not isomorphic to $C(X)$. □

References


