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Misunderstandings and Difficulties in Learning Sequence and Series: A Case Study

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This paper analyzes the difficulties with the learning of sequence and series of the second-year students who participated in a year long whole class at the university level. The research was carried out at the end of students' third semester. These students were randomly selected. They were applied to one paper and pencil test containing eight task items on sequence and series. In this study, qualitative method (case study design) was used to explore students' *difficulties and misunderstandings in learning sequence and series*. Students' responses to the questions were divided into three categories: These were "correct", "partial correct" and "false or no responses". Students' responses to the paper and pencil test were evaluated. The results show that students had difficulties and misunderstandings in series and sequence.

Keywords: sequence, series, students' learning difficulties, misunderstandings
ZDM Classification: D74
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1. INTRODUCTION AND LITERATURE REVIEW

For most students in mathematics, science, and engineering, calculus is the entry-point to undergraduate mathematics. Because of mathematics' importance in such a wide range of disciplines, and its size of enrollment, there have been many studies in the student understanding of calculus (Redish 2005; Sabella & Redish 2005). There has long been

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concern about calculus in many parts of the world. Learning difficulties and misunderstandings in calculus have been discussed and reported by numerous researchers and mathematics educators (Schwarzenberger & Tall 1978; Orton 1983a; 1983b; Davis & Vinner 1986; Mamona-Downs 1990; Amit *et al.* 1990, Tall 1990; Bezuidenhout 1998; Aspinwall & Miller 2001). The studies demonstrate that students, who enroll university calculus class, have a very superficial and incomplete understanding of many of the basic concepts in calculus. There has been much concern in the failure to develop a conceptual understanding of calculus topics because of the rote, manipulative learning that takes place in an introductory course (Cipra 1988; Steen 1998; White & Mitchelmore 1996).

In 1992, at the Seventh International Congress on Mathematical Education (ICME-7) in Québec, a working group focused on students' difficulties in calculus. Problems in the teaching and learning of calculus have also been discussed in a series of publications by the Mathematical Association of America (*cf.* Steen 1988). The traditional introduction to calculus depends on students' understanding the idea of a limit. Yet this concept is inherently difficult and causes problems, no matter how it is taught, partly because many students' intuitive ideas are in conflict with the formal definition (Barnes 1995). Tall & Vinner (1981), Davis & Vinner (1986), Cornu (1982; 1983), and Tall (1992) discussed *many of these problems and difficulties.* Most of difficulties include conceptual difficulties related to infinite processes, and logic and manipulative on dealing with a complicated definition. Tall (1992) divided these difficulties into two groups. These are fundamental difficulties with limits and infinite processes, and other difficulties in the calculus.

Many students are not understanding about the relationship between the sequence and series. Students in calculus class often are confused about infinite sequences and infinite series (Davis 1982). Students have difficulties in distinguishing a sequence and a series. These difficulties are compounded by the need for a sophisticated level of algebra as well as a sound knowledge of limits when dealing with convergence of a sequence or a series (Brown 1996). The difficulties have been recognized by other authors including (Morrel 1992). Students demonstrated a poor understanding of the difference between a sequence and a series as well as determining whether a series was convergent or not. The students were not only able to graph and tabulate sequences, partial sums of series; they were also able to compare algebraically derived limits for both sequences and series (Brown 1996). Lee (1993) indicates that students have three misunderstandings. These were finite point of view of concept of limit, misinterpretation of the geometrical representation, and confusion between sequences and series. Lee (1993) reported that several students in his class had difficulty in differentiating between sequence and series. Two of the students claimed that they did not know what the term "partial sums" meant. Half of the students did not form a new sequence from the series.

Many of students have difficulties in the definition of convergence of a sequence which is a fundamental concept of calculus. The definition is given as follows:

"A sequence $\{a_n\}$ converges to the number L if for each $\varepsilon > 0$ there is a corresponding positive integer N such that $|a_n - L| < \varepsilon$, whenever $n > N$. The number L is called the limit of the sequence $\{a_n\}$. It is written as $\lim_{n\to\infty} a_n = L$."

Graphically this definition says that eventually (for $n > N$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$ as illustrated following.

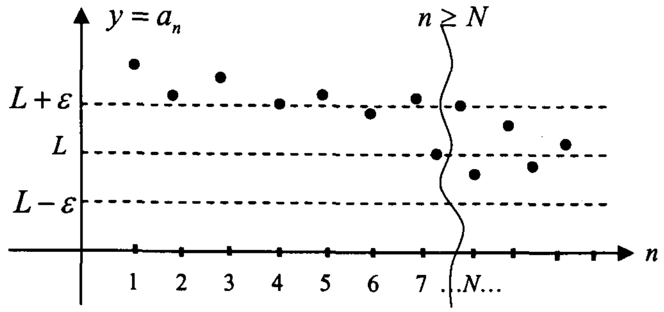

Figure 1. Geometrically figure of convergent sequences

In particular, the relationship between the mysterious thing called "epsilon" and some number N is usually vague and confuses students' minds. Teachers have no difficulty, because teachers have been there for a long time. But students have difficulty because it is the first time they have ever met these conceptions. Tall (1992) defines this situation as "difficulties in absorbing complex new ideas in a limited time". Students cannot contact the relationship between their intuitions and formal definitions. Because they have not necessary experiences to understand what formal definitions are trying to say.

When researching the literature, we were surprised by the lack of research dealing with the understanding of sequence and series. Sequence and function are the most two fundamental topics of calculus. Understanding of functions and sequence limits is very important in terms of understanding of further topics of calculus. Series is only one of these topics. Because the convergence or divergence of the given $\sum a_n$ series is dependent on the convergence or divergence of the sequence of partial sums of the given $\sum a_n$ series, understanding of series is dependent on understanding of sequence. The definition of convergence of infinite series is given as follows:

For the infinite series $\sum a_n$, the nth partial sum is given by

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n \cdots$$

If the sequence of partial sums $\{S_n\}$ converges to S. Then we say that the series $\sum a_n$ converges. We call S the sum of the series and write

$$S = a_1 + a_2 + a_3 + \cdots + a_n \cdots$$

If $\{S_n\}$ diverges, then we say that the series also diverges.

2.1 Purpose and research problems

The purpose of our study is to investigate students' difficulties in learning of sequence and series. For this purpose the problems were given as follows.

1. Can students distinguish the difference between sequence and series?
2. Can students use $(\varepsilon - \delta)$ method in finding of sequence limit?
3. Can students know which test, to use when determining the convergence or the divergence of series?

2.2 Participants

51 second-year students participated in the study. The students were randomly selected by researchers from Ataturk University undergraduate students in science teaching department. The research was carried out at the end of students' third semester. Calculus is a four-semester compulsory course for students in our sample. The course contains 224 hours of lectures. The course topics include functions, sequences, limits, continuity, derivatives and differentials, integrals, differential equations, series, etc. In our university, a calculus course is usually taught in a traditional way. Teaching is carried out in medium number-classes in which there are 40–80 students, teachers lecture to transfer knowledge, and students watch, listen, take notes and receive the information passively. At the each semester, students take two visas and one final examination. The total mark is 100, and its distribution is 40% for the final examination and 30% for each visa examination.

2.3 Data collection and analysis

In this study, qualitative method (case study design) was used to explore students' difficulties and misunderstandings in learning sequence and series. Data for the study was collected from one paper-and-pencil test containing eight task items about sequence and series. Students' responses to the questions were divided into three categories: These were "correct", "partial correct" and "false or no responses". Descriptive analysis was used in analyzing data, because it determines what students' responses mean or which consequences produce. Descriptive analysis presents data to reader depending on the original form of collected data and by directly quoting from the explanations of participants in research.

3. RESEARCH FINDINGS

The students' responses to one paper and pencil containing eight task items were evaluated. The results show that students had difficulties and misunderstandings in series and sequence. The majority of students cannot know the definition of sequence and series, cannot distinguish the difference between sequence and series. 20 of the students provided either no definition of sequence and series or gave wrong answers. While 23 of the students provided partial correct definition of sequence and series, 8 of students provided correct, whereas, the students must know the formal definitions of sequence and series. It is a deficiency that students do not know the formal definitions of sequence and series. This situation can make these conceptions hard to understand by the students. The concept images of sequence and series for many students are restricted, involving only formulas. Excerpts 1–2 from students' responses were given in Figures 2–3.

sequence	Sequence is the powers of series that increase and decrease
series	Series is the sum of sequence

Figure 2. Excerpt-1 related misunderstanding of students

Other students' responses were essentially the same. Another result was that students could not use $(\varepsilon - \delta)$ method in finding of sequence limit. The question: "use $(\varepsilon - \delta)$ to determine limit of sequence

$$\left(\frac{3n+1}{2n}\right)$$

with the given n-th term". 17 students (33%) of 51 students provided correct response by using $(\varepsilon - \delta)$ method to this question. While 10 students found sequence limit, nearly half of the students (24 of 51 students) provided no response or false response. Excerpts 3–5 from students' responses were given in Figures 4–6.

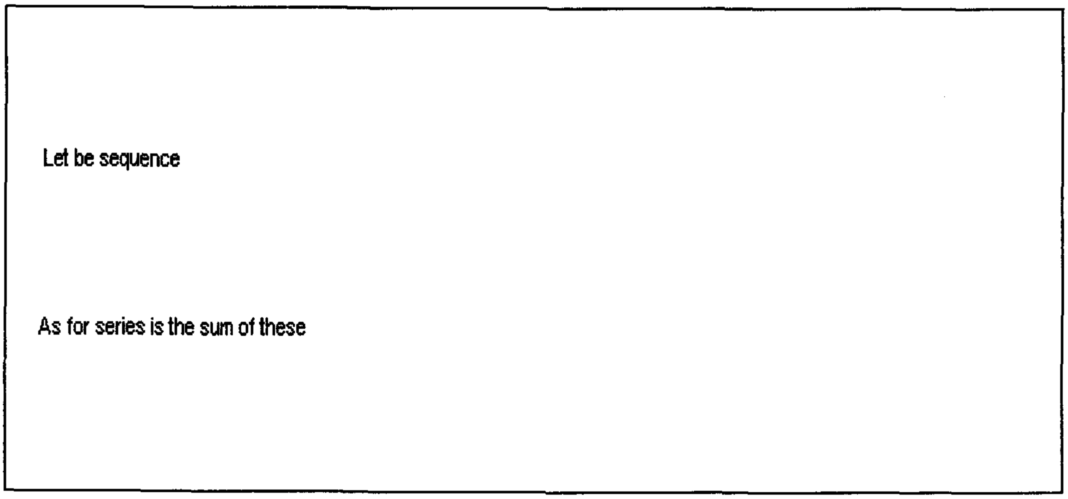

Figure 3. Excerpt-2 related misunderstanding of students

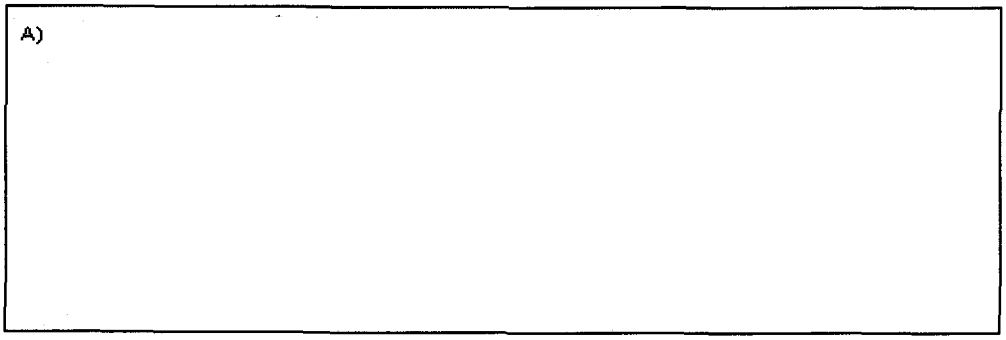

Figure 4. Excerpt-3 related misunderstanding of students

To us, the most striking thing about this result is the fact that 34 students (67%) made the same mistake in this question. Presumably, if asked

$$\lim_{n\to\infty}\frac{3n+1}{2n},$$

all or nearly all of these students could find limit of it. Because the students could not conceptually understand the $(\varepsilon-\delta)$ method, they provided no responses to this question. When determining the convergence or the divergence of the given sequence, the students preferred finding the limit of a sequence $\{a_n\}$ rather than using the theorem of "if a sequence $\{a_n\}$ is bounded and monotonic, then it converges". It also shows that students preferred procedural knowledge rather than conceptual knowledge.

Another result of this study is that students could not know which to use test while researching the convergence or the divergence of the given series. As stated above, because the majority of the students did not know definition and tips of series, they did not understand which test to use in finding of the convergence and the divergence of series.

For example; in excerpts (*e. g.* Figure 5) from students' responses, it showed that the student used the root test instead of the ratio test in finding of convergence and divergence of series

$$\sum_{n=1}^{\infty} \frac{n \bullet 7^n}{n!}.$$

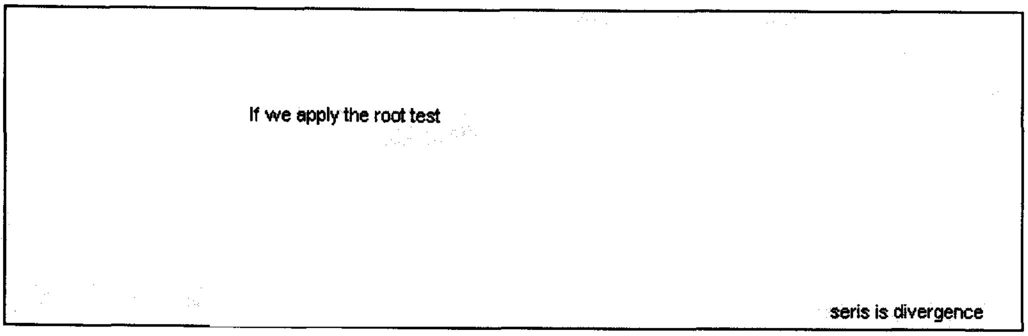

Figure 5. Excerpt-4 related misunderstanding of students

Another student used the root test (*see* Figure 6) instead of using the direct comparison test with geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

in finding of the convergence and divergence of series

$$\sum_{n=1}^{\infty} \frac{\text{Cos } n}{2^n}.$$

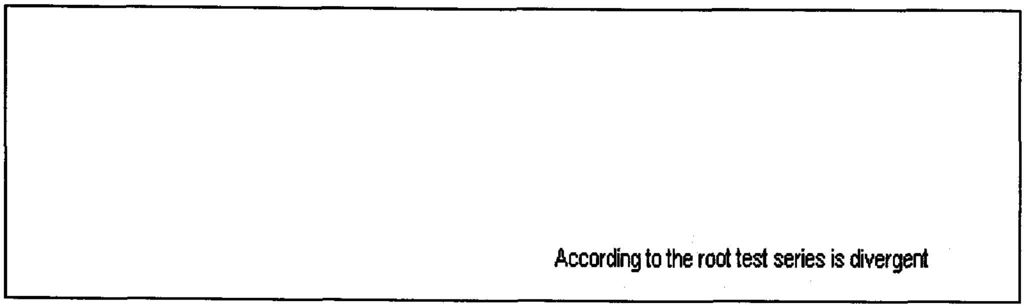

Figure 6. Excerpt-5 related misunderstanding of students

Most of the students made similar mistake.

Another mistake in determining the convergence and divergence of given series is students' misinterpretation of the direct comparison test.

Here, the student compared

$$\text{series } \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} \text{ with series } \sum_{n=1}^{\infty} \frac{1}{n}.$$

For instance, she or he wrote

$$\sum_{n=1}^{\infty} \frac{3}{n^{\frac{3}{2}}} < \sum_{n=1}^{\infty} \frac{1}{n} .$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

series is divergent, she or he said that:

"If the larger series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, the smaller series

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$$

must also diverge."

But, if the larger series diverges, the smaller series may diverge or converge. Here the interesting examples are given.

As many students could not understand the difference between sequence and series, when determining the convergence or divergence of sequence and series they perceived sequence instead of series or series instead of sequence. As stated below:

"The question is to determine the divergence or convergence of sequence

$$a_n = \frac{3n+1}{2n}$$

with the given n-th term."

The students used ratio test for the convergence and divergence of sequence.

The ratio test can not be applied

Figure 7. Excerpt-6 related misunderstanding of students

Or when finding the convergence and divergence of series students investigated monotonic and bounded of series. And they fell into misunderstanding:

"if a series

$$\sum a_n$$

is bounded and monotonic, then it converges".

4. CONCLUSION

The results of this study show that students have difficulties and misunderstanding in the sequence and series. Students have difficulty distinguishing between a sequence and a series (Excerpt-6: See Figure 7). These difficulties are compounded by the need for a sophisticated level of algebra as well as a sound knowledge of limits when dealing with convergence of a sequence or a series. These results were similar with results of Brown (1996), Davis (1982) & Lee (1993). Another difficulty is the definition of $(\varepsilon-\delta)$. In the study, students could not use $(\varepsilon-\delta)$ method in finding of sequence limit. The question:

"Use $(\varepsilon-\delta)$ to determine limit of sequence

$$\left(\frac{3n+1}{2n}\right)$$

with the given n-th term".

The students calculated the limit of sequence instead of $(\varepsilon-\delta)$ rule (Excerpt-3: See

Figure 4). This is a good example that Tall (1992) stated as *fundamental difficulties*.

Difficulties in selecting and using appropriate representations are known to be widespread. Another result of this study is that students cannot know which test to use while researching the convergence or the divergence of the given series (Excerpts 4–5: See Figure 5–6). Difficulties in understanding complex new concepts in a restricted time occur throughout university mathematics. Our students firstly face with sequences and series when they are sophomore. This situation can make these conceptions hard to understand by the students. Tall (1992) defines it as *difficulties in absorbing complex new ideas in a limited time*.

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