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SOME ESTIMATES OF LITTLEWOOD-PALEY TYPE OPERATORS IN ARITHMETIC

YONG-CHEOL KIM

ABSTRACT. We prove that certain square functions of Littlewood-Paley type satisfy certain mapping properties on $L^q(\mathbb{Q}_p^d)$.

1. Introduction

For a prime number p, let $\mathbb{Q}_p$ denote the p-adic field. From the standard p-adic analysis [8], we see that any non-zero element $x \in \mathbb{Q}_p$ is uniquely represented in the canonical form $x = p^{\gamma} \sum_{j=0}^{\infty} x_j\, p^j$, $\gamma = \gamma(x) \in \mathbb{Z}$, where $x_j \in \{0, 1, \ldots, p-1\}$ and $x_0 \neq 0$. Here we call $\gamma = \gamma(x)$ the p-adic valuation of x and we write $\gamma = \text{ord}_p(x)$ with convention $\text{ord}_p(0) = \infty$. Then it is well-known [1, 8] that the nonnegative function $|\cdot|_p$ on $\mathbb{Q}_p$ given by $|x|_p = p^{-\text{ord}_p(x)}$ becomes a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ denotes the vector space over $\mathbb{Q}_p$ which consists of all points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, $x_1, x_2, \ldots, x_d \in \mathbb{Q}_p$. If we define $|\mathbf{x}|_p = \max_{1 \le j \le d} |x_j|_p$ for $\mathbf{x} \in \mathbb{Q}_p^d$, then it is easy to see that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$ and moreover $\mathbb{Q}_p^d$ is a locally compact Hausdorff and totally disconnected Banach space with respect to the norm $|\cdot|_p$. For $\gamma \in \mathbb{Z}$, we denote the ball $B_\gamma(\mathbf{a})$ with center $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ and its boundary $S_\gamma(\mathbf{a})$ by $B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p \le p^\gamma\}$ and $S_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\}$, respectively. Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Haar measure $d_H\mathbf{x}$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant (i.e., $d_H(\mathbf{x} + \mathbf{a}) = d_H\mathbf{x}$) and is normalized by

$$(1.1) \qquad \int_{B_0(\mathbf{0})} d_H\mathbf{x} \doteq |B_0(\mathbf{0})|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^d$. From this integration theory, it is easy to obtain $|B_\gamma(\mathbf{a})|_H = p^{\gamma d}$ and $|S_\gamma(\mathbf{a})|_H = p^{\gamma d}(1 - p^{-d})$ for any $\mathbf{a} \in \mathbb{Q}_p^d$.

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In what follows, we say that a (real-valued) measurable function f defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d), 1 \leq q \leq \infty$, if it satisfies

$$
\begin{aligned}
(1.2) \qquad & \|f\|_{L^q(\mathbb{Q}_p^d)} \fallingdotseq \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d_H\mathbf{x} \right)^{1/q} < \infty, \ 1 \leq q < \infty, \\
& \|f\|_{L^\infty(\mathbb{Q}_p^d)} \fallingdotseq \inf\{\alpha : |\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}|_H = 0\}\} < \infty.
\end{aligned}
$$

Here the integral in (1.2) is defined as

$$
\begin{aligned}
(1.3) \qquad \int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d_H\mathbf{x} &= \lim_{N \to \infty} \int_{B_N(\mathbf{0})} |f(\mathbf{x})|^q \, d_H\mathbf{x} \\
&= \lim_{N \to \infty} \sum_{-\infty < \gamma \leq N} \int_{S_\gamma(\mathbf{0})} |f(\mathbf{x})|^q \, d_H\mathbf{x},
\end{aligned}
$$

if the limit exists. We now mention some of the previous works on harmonic analysis on the p-adic field $\mathbb{Q}_p$ as follows; Haran [2, 3] obtained the explicit formula of Riesz potentials on $\mathbb{Q}_p$ and developed an analytical potential theory on the p-adic field $\mathbb{Q}_p$.

Let $f(\mathbf{x})$ be a complex-valued function on $\mathbb{Q}_p^d$. Then we say that f is *locally-constant* if for any $\mathbf{x} \in \mathbb{Q}_p^d$ there exists some integer $\ell(\mathbf{x}) \in \mathbb{Z}$ such that $f(\mathbf{x} + \mathbf{x}') = f(\mathbf{x})$ for $|\mathbf{x}'|_p \leq p^{\ell(\mathbf{x})}$. We denote by $\mathcal{E}(\mathbb{Q}_p^d)$ the class of all locally-constant functions on $\mathbb{Q}_p^d$ and we denote by $\mathcal{D}(\mathbb{Q}_p^d)$ the subclass of all functions in $\mathcal{E}(\mathbb{Q}_p^d)$ with compact support. We call a function in $\mathcal{D}(\mathbb{Q}_p^d)$ a *test function* on $\mathbb{Q}_p^d$. Any nonzero p-adic number $\eta \in \mathbb{Q}_p$ with $|\eta|_p = p^{-\gamma}$ may be written in the unique form $\eta = \sum_{j=\gamma}^{\infty} \eta_j p^j$, where $\eta_j \in \{0, 1, \ldots, p-1\}$ and $\eta_\gamma \neq 0$, as above. We define a function χ_p on $\mathbb{Q}_p$ by

$$
(1.4) \qquad \chi_p(\eta) = \begin{cases} \prod_{j=\gamma}^{-1} \exp(2\pi i \eta_j p^j), & \gamma < 0, \\ 1, & \gamma \geq 0 \text{ or } \eta = 0. \end{cases}
$$

Then it turns out (see [8]) that the function $\mathbf{x} \to \chi_p(\langle \boldsymbol{\xi}, \mathbf{x} \rangle)$ for each $\boldsymbol{\xi} \in \mathbb{Q}_p^d$ is an additive character of $\mathbb{Q}_p^d$ and the group $B_\gamma(\mathbf{0})$, where $\langle \boldsymbol{\xi}, \mathbf{x} \rangle$ is the inner product of $\boldsymbol{\xi}, \mathbf{x} \in \mathbb{Q}_p^d$. For $g \in \mathcal{D}(\mathbb{Q}_p^d)$, we define the Fourier transformation of g by $\mathfrak{F}[g](\boldsymbol{\xi}) = \widetilde{g}(\boldsymbol{\xi}) = \int_{\mathbb{Q}_p^d} \chi_p(\langle \boldsymbol{\xi}, \mathbf{x} \rangle) g(\mathbf{x}) \, d_H\mathbf{x}$ for $\boldsymbol{\xi} \in \mathbb{Q}_p^d$. Then $\mathfrak{F} : \mathcal{D}(\mathbb{Q}_p^d) \to \mathcal{D}(\mathbb{Q}_p^d)$ is a unitary isomorphism with the inversion formula $g(\mathbf{x}) = \int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi} \rangle) \widetilde{g}(\boldsymbol{\xi}) \, d_H\boldsymbol{\xi}$ and with the *Parseval-Steklov equalities*

$$
\begin{aligned}
\int_{\mathbb{Q}_p^d} g(\mathbf{x}) \overline{h(\mathbf{x})} \, d_H\mathbf{x} &= \int_{\mathbb{Q}_p^d} \widetilde{g}(\boldsymbol{\xi}) \overline{\widetilde{h}(\boldsymbol{\xi})} \, d_H\boldsymbol{\xi}, \ \int_{\mathbb{Q}_p^d} g(\mathbf{x}) \widetilde{h}(\mathbf{x}) \, d_H\mathbf{x} \\
&= \int_{\mathbb{Q}_p^d} \widetilde{g}(\boldsymbol{\xi}) h(\boldsymbol{\xi}) \, d_H\boldsymbol{\xi}, \ g, h \in \mathcal{D}(\mathbb{Q}_p^d).
\end{aligned}
$$

Moreover, $\mathfrak{F}$ is a unitary isomorphism from $L^2(\mathbb{Q}_p^d)$ to $L^2(\mathbb{Q}_p^d)$ with the inversion formula

$$g(\mathbf{x}) = \lim_{\gamma\to\infty} \int_{B_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\widetilde{g}(\boldsymbol{\xi})\, d_H\boldsymbol{\xi} \text{ in } L^2(\mathbb{Q}_p^d)\ ,\ g \in \mathcal{D}(\mathbb{Q}_p^d),$$

and with the Parseval-Steklov equalities on $L^2(\mathbb{Q}_p^d)$, because $\mathcal{D}(\mathbb{Q}_p^d)$ is dense in $L^2(\mathbb{Q}_p^d)$ (see [8]).

Let $\mathcal{M}(\mathbb{Q}_p^d)$ denote the set of all measurable functions on $\mathbb{Q}_p^d$. For $f, g \in \mathcal{M}(\mathbb{Q}_p^d)$, we define the convolution $f * g$ of f and g by

$$f * g(\mathbf{x}) = \int_{\mathbb{Q}_p^d} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\, d_H\mathbf{y},\ \mathbf{x} \in \mathbb{Q}_p^d.$$

For a function $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$, we define a square function $\mathcal{S}_\varphi(f)$ of Littlewood-Paley type by

$$\mathcal{S}_\varphi(f)(\mathbf{x}) = \left(\int_{\mathbb{Q}_p} |\varphi_t * f(\mathbf{x})|^2 \frac{d_H t}{|t|_p} \right)^{1/2},\ \mathbf{x} \in \mathbb{Q}_p^d,$$

where $\varphi_t(\mathbf{x}) = |t|_p^{-d}\varphi(\mathbf{x}/t)$ for $t \in \mathbb{Q}_p$. For $\varphi, \psi^1, \psi^2, \ldots, \psi^n \in \mathcal{M}(\mathbb{Q}_p^d)$ and $n \in \mathbb{N}$, we define another square function $\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)$ by

$$\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)(\mathbf{x}) = \left(\int_{\mathbb{Q}_p} |(\varphi_t * f)^2 * \psi_t^1 * \psi_t^2 * \cdots * \psi_t^n(\mathbf{x})| \frac{d_H t}{|t|_p} \right)^{1/2},\ \mathbf{x} \in \mathbb{Q}_p^d,$$

where $\psi_t^i(\mathbf{x}) = |t|_p^{-d}\psi^i(\mathbf{x}/t)$ for $t \in \mathbb{Q}_p$ and $i = 1, 2, \ldots, n$. Our purpose of this article is to obtain $L^q(\mathbb{Q}_p^d)$-mapping properties of those square functions $\mathcal{S}_\varphi(f)$ and $\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)$ under certain conditions on $\varphi, \psi^1, \psi^2, \ldots, \psi^n \in \mathcal{M}(\mathbb{Q}_p^d)$ to be given later.

In what follows, we shall use notations; given two quantities A and B, we write $A \lesssim B$ or $B \gtrsim A$ if there is a positive constant c (possibly depending on the dimension d and a prime number p to be given) such that $A \le cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We denote by $\mathbb{Q}_p^* \risingdotseq \mathbb{Q}_p \setminus \{0\}$, and by C_F the characteristic function of a measurable subset F of $\mathbb{Q}_p^d$.

Theorem 1.1. *Let $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying that*

$$\sup_{\boldsymbol{\xi}\in S_0(\mathbf{0})} \int_{\mathbb{Q}_p} |\widetilde{\varphi}(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \le A, \tag{1.5}$$

$$\sup_{\mathbf{y}\in\mathbb{Q}_p^*} \int_{\{\mathbf{x}\in\mathbb{Q}_p^d : |\mathbf{x}|_p \ge |\mathbf{y}|_p\}} \left(\int_{\mathbb{Q}_p} |\varphi_t(\mathbf{x}-\mathbf{y}) - \varphi_t(\mathbf{x})|^2 \frac{d_H t}{|t|_p} \right)^{1/2} d_H\mathbf{x} \le B. \tag{1.6}$$

Then $\mathcal{S}_\varphi$ is a bounded operator from $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$ for $1 < q < \infty$ and it is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$. Moreover, if $\mathcal{J}(\boldsymbol{\xi}) \risingdotseq \int_{\mathbb{Q}_p} |\widetilde{\varphi}(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} > 0$ is constant a.e. *on $S_0(\mathbf{0})$, then we have $\|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|\mathcal{S}_\varphi(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)}$.*

We observe that if $\varphi(\mathbf{x}) = \varphi(|\mathbf{x}|_p)$ for $\mathbf{x} \in \mathbb{Q}_p^d$ then the assumption (1.6) can be omitted because $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$. Thus we have the following corollary.

Corollary 1.2. *Let $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying* (1.5) *and $\varphi(\mathbf{x}) = \varphi(|\mathbf{x}|_p)$ for $\mathbf{x} \in \mathbb{Q}_p^d$. Then $\mathcal{S}_\varphi$ is a bounded operator from $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$ for $1 < q < \infty$ and it is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$. Moreover, if $\mathcal{J}(\boldsymbol{\xi}) \doteq \int_{\mathbb{Q}_p} |\widetilde{\varphi}(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} > 0$ is constant* a.e. *on $S_0(\mathbf{0})$, then we have $\|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|\mathcal{S}_\varphi(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)}$.*

Theorem 1.3. *Let $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying* (1.5) *and* (1.6) *as in Theorem 1.1, and let $\psi^1, \psi^2, \ldots, \psi^n \in \mathcal{M}(\mathbb{Q}_p^d)$ be real-valued functions satisfying*

$$\int_{\mathbb{Q}_p^d} \psi_*^i(\mathbf{x})\, d_H\mathbf{x} = \alpha_i < \infty, \tag{1.7}$$

where $\psi_*^i(\mathbf{x}) = \sup_{\{\mathbf{y}\in\mathbb{Q}_p^d : |\mathbf{y}|_p \geq |\mathbf{x}|_p\}} |\psi^i(\mathbf{y})|$ for $i = 1, 2, \ldots, n$. Then, for each $n \in \mathbb{N}$, $\mathcal{S}^n_{\varphi,\{\psi^i\}}$ is a bounded operator from $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$ for $2 \leq q < \infty$.

Remark. We could not obtain $L^q(\mathbb{Q}_p^d)$-mapping properties of $\mathcal{S}^n_{\varphi,\{\psi^i\}}$ for $1 \leq q < 2$. It would be interesting to ask whether the unsettled problem is true or not.

Corollary 1.4. *Let $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying* (1.5) *and $\varphi(\mathbf{x}) = \varphi(|\mathbf{x}|_p)$ for $\mathbf{x} \in \mathbb{Q}_p^d$, and let $\psi^1, \psi^2, \ldots, \psi^n \in \mathcal{M}(\mathbb{Q}_p^d)$ be real-valued functions satisfying* (1.7). *Then $\mathcal{S}_{\varphi,\{\psi^i\}}$ is a bounded operator from $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$ for $2 \leq q < \infty$.*

Corollary 1.5. *Let $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying* (1.5) *and* (1.6), *and let $\{\psi^i\}_{i\in\mathbb{N}} \subset \mathcal{M}(\mathbb{Q}_p^d)$ be a family of real-valued functions satisfying* (1.7). *If $\sum_{n=1}^\infty \prod_{k=1}^n \alpha_k < \infty$, then $\sum_{n=1}^\infty \mathcal{S}^n_{\varphi,\{\psi^i\}}$ is a bounded operator from $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$ for $2 \leq q < \infty$.*

2. Preliminary estimates and examples

In this section, we obtain several propositions which shall be useful in furnishing examples to exemplify the main theorems.

Proposition 2.1. *If $\mathfrak{m}$ is a function on $\mathbb{R}_+$ satisfying $\sum_{\gamma=0}^\infty |\mathfrak{m}(p^{-\gamma})| p^{-\gamma d} < \infty$, then we have that for any $\mathbf{x} \in \mathbb{Q}_p^d \setminus \{\mathbf{0}\}$,*

$$\int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\mathfrak{m}(|\boldsymbol{\xi}|_p)\, d_H\boldsymbol{\xi} = \frac{1-p^{-d}}{|\mathbf{x}|_p^d} \sum_{\gamma=0}^\infty p^{-\gamma d}\mathfrak{m}(p^{-\gamma}|\mathbf{x}|_p^{-1}) - \frac{1}{|\mathbf{x}|_p^d}\mathfrak{m}(p\,|\mathbf{x}|_p^{-1}).$$

Proof. It follows from (1.4) that $\int_{B_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\, d_H\boldsymbol{\xi} = p^{\gamma d}\mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x})$ for any $\gamma \in \mathbb{Z}$. Thus for $\gamma \in \mathbb{Z}$ we have that

$$\begin{aligned}&\int_{S_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\, d_H\boldsymbol{\xi} \\ &= p^{\gamma d}\mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x}) - p^{(\gamma-1)d}\mathcal{C}_{B_{-\gamma+1}(\mathbf{0})}(\mathbf{x}) \\ &= p^{\gamma d}(1-p^{-d})\mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x}) - p^{(\gamma-1)d}\mathcal{C}_{S_{-\gamma+1}(\mathbf{0})}(\mathbf{x}).\end{aligned} \tag{2.1}$$

Hence by (2.1) and simple calculation we obtain that

$$\begin{aligned}\int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\mathfrak{m}(|\boldsymbol{\xi}|_p)\, d_H\boldsymbol{\xi} &= \lim_{N\to\infty} \sum_{\gamma=-\infty}^{N} \mathfrak{m}(p^\gamma) \int_{S_\gamma(\mathbf{0})} \chi_p(-\langle \boldsymbol{\xi}, \mathbf{x}\rangle)\, d_H\mathbf{x} \\ &= \frac{1-p^{-d}}{|\mathbf{x}|_p^d} \sum_{\gamma=0}^{\infty} p^{-\gamma d}\mathfrak{m}(p^{-\gamma}|\mathbf{x}|_p^{-1}) - \frac{1}{|\mathbf{x}|_p^d}\mathfrak{m}(p\,|\mathbf{x}|_p^{-1}).\end{aligned}$$

□

Proposition 2.2. *If* $\mathfrak{m}$ *is a function on* $\mathbb{R}_+$ *satisfying* $\sum_{\gamma=0}^{\infty} |\mathfrak{m}(p^{-\gamma})|p^{-\gamma d} < \infty$ *and* $\varphi(\mathbf{x}) = \mathfrak{F}^{-1}[\mathfrak{m}(|\boldsymbol{\xi}|_p)](\mathbf{x})$, *then*

$$\sup_{\mathbf{y}\in\mathbb{Q}_p^*} \int_{\{\mathbf{x}\in\mathbb{Q}_p^d : |\mathbf{x}|_p \ge |\mathbf{y}|_p\}} \left(\int_{\mathbb{Q}_p} |\varphi_t(\mathbf{x}-\mathbf{y}) - \varphi_t(\mathbf{x})|^2 \frac{d_H t}{|t|_p}\right)^{1/2} d_H\mathbf{x} = 0.$$

Proof. Since $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$, it easily follows from Proposition 2.1. □

Example. (a) We consider the kernel φ_a defined by $\widetilde{\varphi}_a(\boldsymbol{\xi}) = |\boldsymbol{\xi}|_p\,(1-|\boldsymbol{\xi}|_p)_+^n$, $\boldsymbol{\xi} \in \mathbb{Q}_p^d$, $n \in \mathbb{N}$. By Proposition 2.2, the kernel φ_a satisfies (1.6) of Theorem 1.1. Thus it suffices to show that

$$\mathcal{J}(\boldsymbol{\xi}) \doteq \int_{\mathbb{Q}_p} |\widetilde{\varphi}_a(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} > 0$$

is constant on $S_0(\mathbf{0})$. Indeed, by the binomial theorem and simple calculation, we get that

$$\begin{aligned}&\int_{\mathbb{Q}_p} |\widetilde{\varphi}_a(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \\ &= \int_{|t|_p \le |\boldsymbol{\xi}|_p^{-1}} |t|_p\,|\boldsymbol{\xi}|_p^2(1-|t|_p\,|\boldsymbol{\xi}|_p)^{2n}\, d_H t \\ &= \left(1-\frac{1}{p}\right)\sum_{s=0}^{2n}(-1)^s\binom{2n}{s}\frac{1}{1-p^{-(2+s)}} \\ &\le p^2\left(1-\frac{1}{p}\right)\sum_{s=0}^{2n}(-1)^s\binom{2n}{s}p^s = p(p-1)^{2n+1} < \infty.\end{aligned}$$

Thus it follows from Theorem 1.1 that $\mathcal{S}_{\varphi_a}$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$ and

$$\|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|\mathcal{S}_{\varphi_a}(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)} \quad \text{for } 1 < q < \infty.$$

(b) Let φ_b be the kernel satisfying $\widetilde{\varphi}_b(\boldsymbol{\xi}) = \exp(-|\boldsymbol{\xi}|_p)|\boldsymbol{\xi}|_p^n$, $\boldsymbol{\xi} \in \mathbb{Q}_p^d$, $n \in \mathbb{N}$. Since it is easy to see that $\sum_{\gamma=0}^{\infty} |\widetilde{\varphi}_b(p^{-\gamma})|p^{-\gamma d} < \infty$, the kernel φ_b satisfies (1.6) of Theorem 1.1. Thus it is enough to show that

$$\sup_{\boldsymbol{\xi}\in S_\gamma(\mathbf{0})} \int_{\mathbb{Q}_p} |\widetilde{\varphi}_b(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \le A.$$

In order to show it, we observe that $\exp(x) \ge x^k/k!$, $x \in \mathbb{R}$, for all $k \in \mathbb{N}$. From simple calculation, we obtain that for $\boldsymbol{\xi} \in S_0(\mathbf{0})$

$$\begin{aligned}
&\int_{\mathbb{Q}_p} |\widetilde{\varphi}_b(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \\
&= |\boldsymbol{\xi}|_p^n \lim_{N\to\infty} \sum_{\gamma=-\infty}^{N} \exp(-|t|_p\,|\boldsymbol{\xi}|_p)|t|_p^{n-1} \int_{S_\gamma(\mathbf{0})} d_H t \\
&= \left(1-\frac{1}{p}\right)\left(\sum_{\gamma=0}^{\infty} \exp(-p^{-\gamma})p^{-\gamma n} + \sum_{\gamma=1}^{\infty} \exp(-p^{\gamma})p^{\gamma n}\right) \\
&\le (n-1)! + \frac{(n+1)!}{p} < \infty.
\end{aligned}$$

Thus by Theorem 1.1 we conclude that $\mathcal{S}_{\varphi_b}$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$ and

$$\|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|\mathcal{S}_{\varphi_b}(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)} \quad \text{for } 1 < q < \infty.$$

(c) We consider the kernel φ_c given by $\widetilde{\varphi}_c(\boldsymbol{\xi}) = |\boldsymbol{\xi}|_p\,(1-|\boldsymbol{\xi}|_p)_+^c$, $\boldsymbol{\xi} \in \mathbb{Q}_p^d$, $c > 0$. From Proposition 2.2, we see that the kernel φ_c satisfies (1.6) of Theorem 1.1. Thus it suffices to show that

$$\mathcal{J}(\boldsymbol{\xi}) \risingdotseq \int_{\mathbb{Q}_p} |\widetilde{\varphi}_c(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} > 0$$

is constant on $S_0(\mathbf{0})$. Indeed, by simple calculation, we obtain that for $\boldsymbol{\xi} \in S_0(\mathbf{0})$

$$\begin{aligned}
\int_{\mathbb{Q}_p} |\widetilde{\varphi}_c(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} &= \int_{|t|_p \le |\boldsymbol{\xi}|_p^{-1}} |t|_p\,|\boldsymbol{\xi}|_p^2(1-|t|_p\,|\boldsymbol{\xi}|_p)^{2c}\, d_H t \\
&= |\boldsymbol{\xi}|_p^2 \sum_{\gamma=-\infty}^{\log_p(|\boldsymbol{\xi}|_p^{-1})} p^\gamma(1-p^\gamma|\boldsymbol{\xi}|_p)^{2c} \int_{S_\gamma(\mathbf{0})} d_H t \le \frac{p}{p+1} < \infty.
\end{aligned}$$

Thus it follows from Theorem 1.1 that if $c > 0$ then $\mathcal{S}_{\varphi_c}$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$ and

$$\|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|\mathcal{S}_{\varphi_c}(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)} \quad \text{for } 1 < q < \infty.$$

3. The proof of Theorem 1.1

We consider the Hilbert space $\mathcal{H} = L^2(\mathbb{Q}_p, d_H t/|t|_p)$ with the inner product given by $\langle g_t(\mathbf{x}), h_t(\mathbf{x})\rangle_{\mathcal{H}} = \int_{\mathbb{Q}_p} g_t(\mathbf{x})h_t(\mathbf{x}) \frac{d_H t}{|t|_p}$, $g = (g_t)_{t\in\mathbb{Q}_p}, h = (h_t)_{t\in\mathbb{Q}_p} \in \mathcal{H}$. Then by applying Fubini's theorem we have the associative relation of convolution as follows; if $f \in \mathcal{M}(\mathbb{Q}_p^d)$ and $g = (g_t)_{t\in\mathbb{Q}_p} \in \mathcal{H}$, then

$$\text{(3.1)} \qquad \int_{\mathbb{Q}_p^d} \langle \varphi_t * f(\mathbf{x}), g_t(\mathbf{x})\rangle_{\mathcal{H}}\, d_H\mathbf{x} = \int_{\mathbb{Q}_p^d} \left(\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t(\mathbf{y}) \frac{d_H t}{|t|_p} \right) f(\mathbf{y})\, d_H\mathbf{y},$$

where $\bar{\varphi}(\mathbf{x}) = \varphi(-\mathbf{x})$. We write $|g(\mathbf{x})|_{\mathcal{H}} = \sqrt{\langle g_t(\mathbf{x}), g_t(\mathbf{x})\rangle_{\mathcal{H}}}$ for $g = (g_t)_{t\in\mathbb{Q}_p} \in \mathcal{H}$.

We take any $f \in L^1(\mathbb{Q}_p^d) \cap L^2(\mathbb{Q}_p^d)$ and $g = (g_t)_{t\in\mathbb{Q}_p} \in L^1(\mathbb{Q}_p^d; \mathcal{H}) \cap L^2(\mathbb{Q}_p^d; \mathcal{H})$. We may assume that $\mathcal{J}(\boldsymbol{\xi}) = \beta > 0$ a.e. on $S_0(\mathbf{0})$, because the first part can similarly be obtained from the second part. By the Parseval-Steklov equalities on $L^2(\mathbb{Q}_p^d)$ and (1.5), we easily obtain that

(3.2)

$$\begin{aligned} \|\varphi_t * f\|^2_{L^2(\mathbb{Q}_p^d;\mathcal{H})} = \|\mathcal{S}_\varphi(f)\|^2_{L^2(\mathbb{Q}_p^d)} &= \int_{\mathbb{Q}_p^d} \left(\int_{\mathbb{Q}_p} |\widetilde{\varphi}(t\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \right) |\widetilde{f}(\boldsymbol{\xi})|^2\, d_H\boldsymbol{\xi} \\ &= \int_{\mathbb{Q}_p^d} \left(\int_{\mathbb{Q}_p} |\widetilde{\varphi}(t|\boldsymbol{\xi}|_p\boldsymbol{\xi})|^2 \frac{d_H t}{|t|_p} \right) |\widetilde{f}(\boldsymbol{\xi})|^2\, d_H\boldsymbol{\xi} = \beta\, \|f\|^2_{L^2(\mathbb{Q}_p^d)}. \end{aligned}$$

Since $\langle \varphi_t * f(\mathbf{x}), g_t(\mathbf{x})\rangle_{\mathcal{H}} \le \mathcal{S}_\varphi(f)(\mathbf{x}) \cdot |g(\mathbf{x})|_{\mathcal{H}}$ by Schwarz's inequality, it follows from (3.1), (3.2), and the converse of Hölder's inequality that

$$\text{(3.3)} \qquad \left\| \int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t \frac{d_H t}{|t|_p} \right\|^2_{L^2(\mathbb{Q}_p^d)} \le \beta\, \|g\|^2_{L^2(\mathbb{Q}_p^d;\mathcal{H})}.$$

We now prove that $\mathcal{S}_\varphi$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$. For this, we employ the p-adic version [4] of the Calderón-Zygmund decomposition of f with aperture $\lambda > 0$ as follows;

$$f = \mathfrak{g} + \mathfrak{b} \eqqcolon \mathfrak{g} + \sum_{k=1}^{\infty} \mathfrak{b}_k,$$

where $\{B_k : k \in \mathbb{N}\}$ is a countable family of pairwise disjoint p-adic balls so that

(a) $|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathfrak{g}(\mathbf{x})| > p^d\lambda\}|_H = 0$,
(b) $\mathfrak{b}_k(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbb{Q}_p^d \setminus B_k$ and $\int_{\mathbb{Q}_p^d} \mathfrak{b}_k(\mathbf{x})\, d_H\mathbf{x} = 0$,
(c) $\sum_{k=1}^{\infty} |B_k|_H \le \frac{1}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}$,
(d) $\|\mathfrak{g}\|_{L^1(\mathbb{Q}_p^d)} + \sum_{k=1}^{\infty} \|\mathfrak{b}_k\|_{L^1(\mathbb{Q}_p^d)} \le 3\, \|f\|_{L^1(\mathbb{Q}_p^d)}$.

Then it follows from (3.2) and the p-adic version of Chebyshev's inequality that

$$\text{(3.4)} \qquad |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{S}_\varphi(\mathfrak{g})(\mathbf{x}) > \lambda/2\}|_H \le \frac{1}{\lambda^2} \|\mathfrak{g}\|^2_{L^2(\mathbb{Q}_p^d)} \le \frac{p^d}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}.$$

If we set $\Omega = \cup_{k\in\mathbb{N}} B_k$, then we have that

$$|\Omega|_H \leq \frac{1}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}. \tag{3.5}$$

It also follows from (1.6), the cancellation property of $\mathfrak{b}_k$, and the integral Minkowski's inequality that

$$\begin{aligned}
&|\{\mathbf{x} \in \mathbb{Q}_p^d \setminus \Omega : \mathcal{S}_\varphi(\mathfrak{b})(\mathbf{x}) > \lambda/2\}|_H \\
&\leq \frac{2}{\lambda} \int_{\mathbb{Q}_p^d\setminus\Omega} \mathcal{S}_\varphi(\mathfrak{b})(\mathbf{x})\, d_H\mathbf{x} \leq \frac{2}{\lambda} \sum_{k=1}^{\infty} \int_{\mathbb{Q}_p^d\setminus B_k} \mathcal{S}_\varphi(\mathfrak{b}_k)(\mathbf{x})\, d_H\mathbf{x} \\
&\leq \frac{2}{\lambda} \sum_{k=1}^{\infty} \int_{\mathbf{y}\in B_k} \int_{\mathbf{x}\in\mathbb{Q}_p^d\setminus B_k} \left(\int_{\mathbb{Q}_p} |\varphi_t(\mathbf{x}-\mathbf{y}) - \varphi_t(\mathbf{x})|^2 \frac{d_H t}{|t|_p} \right)^{1/2} d_H\mathbf{x}\, |\mathfrak{b}_k(\mathbf{y})|\, d_H\mathbf{y} \\
&\leq \frac{2B}{\lambda} \sum_{k=1}^{\infty} \|\mathfrak{b}_k\|_{L^1(\mathbb{Q}_p^d)} \leq \frac{2B}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}.
\end{aligned}$$

Combining this with (3.4) and (3.5), then we obtain that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : \mathcal{S}_\varphi(f)(\mathbf{x}) > \lambda\}|_H \leq \frac{p^d + 1 + 2B}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}. \tag{3.6}$$

We employ the p-adic version [4] of the Calderón-Zygmund decomposition of $g = (g_t)_{t\in\mathbb{Q}_p} \in L^1(\mathbb{Q}_p^d; \mathcal{H})$ with aperture $\lambda > 0$ as follows;

$$g = h + b \doteqdot h + \sum_{i=1}^{\infty} b^i = (h_t)_{t\in\mathbb{Q}_p} + \sum_{i=1}^{\infty} (b_t^i)_{t\in\mathbb{Q}_p},$$

where $\{B_i : i \in \mathbb{N}\}$ is a countable family of pairwise disjoint p-adic balls so that

(a) $|\{\mathbf{x} \in \mathbb{Q}_p^d : |h(\mathbf{x})|_{\mathcal{H}} > p^d\lambda\}|_H = 0$,
(b) $b_t^i(\mathbf{x}) = 0$ on $\mathbb{Q}_p^d \setminus B_i$ and $\int_{\mathbb{Q}_p^d} b_t^i(\mathbf{x})\, d_H\mathbf{x} = 0$ for any $t \in \mathbb{Q}_p$,
(c) $\sum_{i=1}^{\infty} |B_i|_H \leq \frac{1}{\lambda} \|g\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})}$,
(d) $\|h\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})} + \sum_{i=1}^{\infty} \|b^i\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})} \leq 3\, \|g\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})}$.

Very similarly to (3.6), we also obtain that

$$\left|\left\{\mathbf{x} \in \mathbb{Q}_p^d : \left| \int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t(\mathbf{x}) \frac{d_H t}{|t|_p} \right| > \lambda \right\}\right|_H \leq \frac{p^d + 1 + 2B}{\lambda} \|g\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})}. \tag{3.7}$$

Applying the general p-adic version [5] of the Marcinkiewicz interpolation theorem with (3.2) and (3.6), and with (3.3) and (3.7), for $1 < q \leq 2$ we have that

$$\|\varphi_t * f\|_{L^q(\mathbb{Q}_p^d;\mathcal{H})} = \|\mathcal{S}_\varphi(f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)}, \tag{3.8}$$

$$\left\| \int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t \frac{d_H t}{|t|_p} \right\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|g\|_{L^q(\mathbb{Q}_p^d;\mathcal{H})}. \tag{3.9}$$

Next we treat the case $q \geq 2$. Take any $f \in L^q(\mathbb{Q}_p^d)$. Then it follows from (3.1), (3.9), the converse of Hölder's inequality, and the p-adic version of Hölder's inequality that

$$\begin{aligned} &\|\mathcal{S}_\varphi(f)\|_{L^q(\mathbb{Q}_p^d)} \\ &= \|\varphi_t * f\|_{L^q(\mathbb{Q}_p^d;\mathcal{H})} \\ &= \sup_{\|g\|_{L^{q'}(\mathbb{Q}_p^d;\mathcal{H})}\leq 1} \int_{\mathbb{Q}_p^d}\Big(\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t(\mathbf{y})\,\frac{d_Ht}{|t|_p}\Big) f(\mathbf{y})\,d_H\mathbf{y} \qquad (3.10)\\ &\leq \sup_{\|g\|_{L^{q'}(\mathbb{Q}_p^d;\mathcal{H})}\leq 1} \Big\|\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t\,\frac{d_Ht}{|t|_p}\Big\|_{L^{q'}(\mathbb{Q}_p^d)} \|f\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|f\|_{L^q(\mathbb{Q}_p^d)}, \end{aligned}$$

where q' is the dual exponent of q. From (3.1), (3.8), the converse of Hölder's inequality, and the p-adic version of Hölder's inequality that, we also obtain that

$$\Big\|\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t\,\frac{d_Ht}{|t|_p}\Big\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|g\|_{L^q(\mathbb{Q}_p^d;\mathcal{H})}. \qquad (3.11)$$

Finally, the polarization of (3.2) implies that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : f(\mathbf{x}) \neq \beta \int_{\mathbb{Q}_p} \bar{\varphi}_t * (\varphi_t * f)(\mathbf{x})\,\frac{d_Ht}{|t|_p}\}|_H = 0. \qquad (3.12)$$

Hence we complete the remaining part by applying (3.8), (3.9), (3.10), (3.11), and (3.12). □

Corollary 3.1. *Suppose that $\varphi \in \mathcal{M}(\mathbb{Q}_p^d)$ be a real-valued function satisfying* (1.5) *and* (1.6). *If $1 < q < \infty$, then we have*

$$\Big\|\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t\,\frac{d_Ht}{|t|_p}\Big\|_{L^q(\mathbb{Q}_p^d)} \lesssim \|g\|_{L^q(\mathbb{Q}_p^d;\mathcal{H})}.$$

Moreover, we have $\left|\{\mathbf{x} \in \mathbb{Q}_p^d : \left|\int_{\mathbb{Q}_p} \bar{\varphi}_t * g_t(\mathbf{x})\,\frac{d_Ht}{|t|_p}\right| > \lambda\}\right|_H \leq \frac{1}{\lambda}\|g\|_{L^1(\mathbb{Q}_p^d;\mathcal{H})}.$

4. The proof of Theorem 1.3

For a function $f \in L^1_{loc}(\mathbb{Q}_p^d)$, we define the Hardy-Littlewood maximal function of f on $\mathbb{Q}_p^d$ by

$$\mathcal{M}_p f(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|\,d_H\mathbf{y}.$$

Then we see [5] that $\mathcal{M}_p$ is a bounded operator of $L^r(\mathbb{Q}_p^d)$ into $L^r(\mathbb{Q}_p^d)$ for $1 < r < \infty$ and is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$. Moreover, it is well known [5] that for $i = 1, 2, \ldots, n$,

$$\sup_{t\in\mathbb{Q}_p} |\bar{\psi}_t^i * g(\mathbf{x})| \leq \|\psi_*^i\|_{L^1(\mathbb{Q}_p^d)}\,\mathcal{M}_p\, g(\mathbf{x}). \qquad (3.1)$$

Let $2 \le q < \infty$ be given and let $r \in (1,\infty)$ satisfy $1/r + 2/q = 1$. We take any $g \in L^r(\mathbb{Q}_p^d)$. Then by Fubini's theorem and (3.1) we have that

$$\begin{aligned}
&\int_{\mathbb{Q}_p^d} [\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)(\mathbf{x})]^2\, g(\mathbf{x})\, d_H\mathbf{x} \\
(3.2)\qquad &= \int_{\mathbb{Q}_p^d}\int_{\mathbb{Q}_p} [\varphi_t * f(\mathbf{x})]^2\, [\bar{\psi}_t^1 * \bar{\psi}_t^2 \cdots * \bar{\psi}_t^n * g(\mathbf{x})]\, \frac{d_H t}{|t|_p}\, d_H\mathbf{x} \\
&\le \prod_{i=1}^n \|\psi_*^i\|_{L^1(\mathbb{Q}_p^d)} \int_{\mathbb{Q}_p^d} [\mathcal{S}(f)(\mathbf{x})]^2\, \mathcal{M}_p^n\, g(\mathbf{x})\, d_H\mathbf{x},
\end{aligned}$$

where $\mathcal{M}_p^n$ denotes the n-times iterated composition of $\mathcal{M}_p$. Then it follows from the converse of Hölder's inequality, (3.2), the p-adic version of Hölder's inequality, and Theorem 1.1 that

$$\begin{aligned}
\|\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)\|^2_{L^q(\mathbb{Q}_p^d)} &= \sup_{\|g\|_{L^r(\mathbb{Q}_p^d)}\le 1} \int_{\mathbb{Q}_p^d} [\mathcal{S}^n_{\varphi,\{\psi^i\}}(f)(\mathbf{x})]^2\, g(\mathbf{x})\, d_H\mathbf{x} \\
&\lesssim \sup_{\|g\|_{L^r(\mathbb{Q}_p^d)}\le 1} \int_{\mathbb{Q}_p^d} [\mathcal{S}(f)(\mathbf{x})]^2\, \mathcal{M}_p^n\, g(\mathbf{x})\, d_H\mathbf{x} \\
&\le \sup_{\|g\|_{L^r(\mathbb{Q}_p^d)}\le 1} \|\mathcal{S}_\varphi(f)\|^2_{L^q(\mathbb{Q}_p^d)} \cdot \|\mathcal{M}_p^n g\|_{L^r(\mathbb{Q}_p^d)} \lesssim \|f\|^2_{L^q(\mathbb{Q}_p^d)}.
\end{aligned}$$

Hence this completes the proof. □

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Department of Mathematics Education
Korea University
Seoul 136-701, Korea
E-mail address: ychkim@korea.ac.kr