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A METHOD TO MAKE BCK-ALGEBRAS

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ABSTRACT. Using the notion of posets, a method to make BCK-algebras is considered. We show that if a poset has the least element, then the induced BCK-algebra is bounded.

1. Introduction and basic results on BCK-algebras

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [3], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets ($= MV$-algebras). Founding new BCK-algebras is important in studying BCK-algebras and related algebraic structures. A way to make a new BCK-algebra from old is established by Abujabal [1]. Jun et al. [7] gave a method to make a BCK-algebra from a poset and an upper set. Hao [2] gave a method for constructing a proper BCC-algebra by the extension of a BCK-algebra with a small atom. Iséki [6] gave a method to make a BCI-algebra by using a group and a BCK-algebra. In this paper, we give a method to make a BCK-algebra by using a poset. We show that if a poset has the least element, then the induced BCK-algebra is bounded.

We first display basic concepts on BCK-algebras. By a *BCK-algebra* we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)$ $(x * x = 0, 0 * x = 0)$,
(a4) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *bounded* if there exists the bound 1 such that $x \leq 1$ for all $x \in X$. A mapping $f : X \to Y$ of BCK-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

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In any BCK-algebra X, the following hold:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b4) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

2. Making BCK-algebras

In what follows let X denote a BCK-algebra unless otherwise specified. The following definition is well-known.

Definition 2.1. A subset A of X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A \Rightarrow y \in A)$.

Note that every ideal A of X satisfies:

$$(\forall x \in A)\,(\forall y \in X)\,(y \leq x \Rightarrow y \in A). \tag{2.1}$$

The set of all ideals of X is denoted by $Id(X)$. It is known that $Id(X)$ is an infinitely distributive lattice (see [10]). If A is a nonempty subset of X, then the ideal of X generated by A, in symbol $\langle A\rangle$, is the set (see [4, Theorem 3])

$$\langle A\rangle = \left\{ x \in X \;\middle|\; \begin{array}{l} (\cdots((x * a_0) * a_1) * \cdots) * a_n = 0 \\ \text{for some } a_0, a_1, \ldots, a_n \in A \end{array} \right\}. \tag{2.2}$$

Definition 2.2. An ideal A of X is said to be *irreducible* (see [5]) if it satisfies:

$$(\forall B, C \in Id(X))\,(A = B \cap C \Rightarrow A = B \text{ or } A = C). \tag{2.3}$$

Denote by $IId(X)$ the set of all irreducible ideals of X.

Lemma 2.3 ([5, Theorem 2]). *Let $A \in Id(X)$. If A is irreducible, then*

$$(\forall a, b \in X \setminus A)\,(\exists c \in X \setminus A)\,(c \leq a,\ c \leq b). \tag{2.4}$$

Lemma 2.4. *Let $A \in Id(X)$. Then the following are equivalent:*

(i) *A is irreducible.*
(ii) $(\forall a, b \in X \setminus A)\ (\exists c \in X \setminus A)\ (c * a,\ c * b \in A)$.

Proof. (i) $\Rightarrow$ (ii) This follows from Lemma 2.3.

(ii) $\Rightarrow$ (i) Let $B, C \in Id(X)$ be such that $A = B \cap C$. Assume that $A \neq B$ and $A \neq C$. Then there exist $b \in B \setminus A$ and $c \in C \setminus A$. It follows from (ii) that there exists $d \in X \setminus A$ such that $d * b \in A$ and $d * c \in A$. From $b \in B \setminus A \subseteq B$ and $d * b \in A = B \cap C \subseteq B$, we have $d \in B$ since $B \in Id(X)$. Similarly, $d \in C$, and so $d \in B \cap C = A$. This is a contradiction. Therefore A is an irreducible ideal of X. □

Definition 2.5. A subset I of X is called an *order system* of X if it satisfies:

(c3) I is an upper set, that is, I satisfies:

$$(\forall x \in X)\,(\forall y \in I)\,(y \leq x \;\Rightarrow\; x \in I),$$

(c4) $(\forall x, y \in I)\ (\exists z \in I)\ (z \leq x,\ z \leq y)$.

Denote by $Os(X)$ the set of all order systems of X.

Theorem 2.6. *Let $A \in Id(X)$ and $I \in Os(X)$. If A and I are disjoint, then there exists an irreducible ideal B of X such that $A \subseteq B$ and $B \cap I = \emptyset$.*

Proof. Let

$$\mathscr{X} := \{H \in Id(X) \mid A \subseteq H,\ H \cap I = \emptyset\}. \tag{2.5}$$

Then $\mathscr{X} \neq \emptyset$ since $A \in \mathscr{X}$. Obviously, the union of a chain of elements of $\mathscr{X}$ is contained in $\mathscr{X}$. Using Zorn's lemma, $\mathscr{X}$ has a maximal element, say B. Let $a, b \in X \setminus B$ and consider ideals $\langle B \cup \{a\}\rangle$ and $\langle B \cup \{b\}\rangle$ generated by $B \cup \{a\}$ and $B \cup \{b\}$, respectively. Clearly $B \subsetneq \langle B \cup \{a\}\rangle \cap \langle B \cup \{b\}\rangle$, and so $\langle B \cup \{a\}\rangle \cap I \neq \emptyset$ and $\langle B \cup \{b\}\rangle \cap I \neq \emptyset$. If not, then $\langle B \cup \{a\}\rangle \in \mathscr{X}$ or $\langle B \cup \{b\}\rangle \in \mathscr{X}$. This contradicts to the fact that B is a maximal element of $\mathscr{X}$. Hence there exist $x, y \in I$ such that $x \in \langle B \cup \{a\}\rangle$ and $y \in \langle B \cup \{b\}\rangle$. It follows that $x * a \in B$ and $y * b \in B$. Since I is an order system of X, there exists $z \in I$, and so $z \in X \setminus B$, such that $z \leq x$ and $z \leq y$. It follows from (b4) that $z * a \leq x * a$ and $z * b \leq y * b$ so from (2.1) that $z * a \in B$ and $z * b \in B$. We conclude from Lemma 2.4 that B is irreducible. □

Let $(P, \leq)$ be a poset. For any $a \in P$, we put

$$(a] := \{x \in P \mid x \leq a\}. \tag{2.6}$$

For any $X \subseteq P$, we put

$$(X] := \bigcup_{a \in X} (a]. \tag{2.7}$$

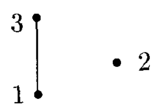
For any $X \subseteq P$, if $X = (X]$ then we say that X is a *decreasing subset* of P. Denote by $\mathscr{D}(P)$ the family of all decreasing subsets of P.

Theorem 2.7. *If $(P, \leq)$ is a poset, then $(\mathscr{D}(P), \odot, P)$ is a BCK-algebra where the operation $\odot$ is defined as follows:*

$$(\forall X, Y \in \mathscr{D}(P))\,(X \odot Y = \{a \in P \mid (a] \cap Y \subseteq X\}). \tag{2.8}$$

Proof. The proof is routine. □

Example 2.8. Let $P = \{1, 2, 3\}$ be a poset with the following Hasse diagram:



Then $(1] = \{1\}$, $(2] = \{2\}$ and $(3] = \{1,3\}$. It is routine to verify that

$$\mathscr{D}(P) = \{P, \{1,2\}, \{1,3\}, \{1\}, \{2\}\},$$

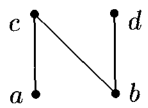
and it is a BCK-algebra with the following Cayley table:

$\odot$	P	$\{1,2\}$	$\{1,3\}$	$\{1\}$	$\{2\}$
P	P	P	P	P	P
$\{1,2\}$	$\{1,2\}$	P	$\{1,2\}$	P	P
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	P	P	$\{1,3\}$
$\{1\}$	$\{1\}$	$\{1,3\}$	$\{1,2\}$	P	$\{1,3\}$
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	P

Example 2.9. Consider the letter **N** poset (see [9]):

$$P := \{a, b, c, d\} \text{ and } \leq := \{(a,a), (b,b), (c,c), (d,d), (a,c), (b,c), (b,d)\}.$$

The Hasse diagram for this poset is as follows:



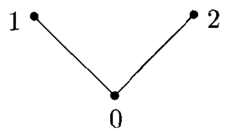
Then $(a] = \{a\}$, $(b] = \{b\}$, $(c] = \{a,b,c\}$ and $(d] = \{b,d\}$. Decreasing subsets of P are $\{a\}$, $\{b\}$, $\{a,b\}$, $\{b,d\}$, $\{a,b,c\}$, $\{a,b,d\}$ and P, that is,

$$\mathscr{D}(P) = \{P, \{a\}, \{b\}, \{a,b\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}\}.$$

Using (2.8), we have the following Cayley table:

$\odot$	P	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,c\}$	$\{a,b,d\}$
P	P	P	P	P	P	P	P
$\{a\}$	$\{a\}$	P	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\{b\}$	$\{b,d\}$	P	$\{b,d\}$	$\{a,b,c\}$	$\{b,d\}$	$\{b\}$
$\{a,b\}$	$\{a,b\}$	P	P	P	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,b,c\}$
$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	P	$\{b,d\}$	P	$\{b,d\}$	$\{b,d\}$
$\{a,b,c\}$	$\{a,b,c\}$	P	P	P	P	P	$\{a,b,c\}$
$\{a,b,d\}$	$\{a,b,d\}$	P	P	P	P	$\{a,b,d\}$	P

Example 2.10. Let $P = \{0,1,2\}$ be a poset with the following Hasse diagram:



Then $(0] = \{0\}$, $(1] = \{0,1\}$ and $(2] = \{0,2\}$. We know that

$$\mathscr{D}(P) = \{P, \{0,1\}, \{0,2\}, \{0\}\},$$

and it is a bounded BCK-algebra with the following Cayley table:

$\odot$	P	$\{0,1\}$	$\{0,2\}$	$\{0\}$
P	P	P	P	P
$\{0,1\}$	$\{0,1\}$	P	$\{0,1\}$	P
$\{0,2\}$	$\{0,2\}$	$\{0,2\}$	P	P
$\{0\}$	$\{0\}$	$\{0,2\}$	$\{0,1\}$	P

Lemma 2.11. *If $(P,\leq)$ is a poset with the least element, then every decreasing subset of P contains the least element.*

Proof. Let X be a decreasing subset of P and let a be the least element of $(P,\leq)$. Then $a \in (x]$ for all $x \in P$, and so

$$a \in \bigcup_{w \in X} (w] = (X] = X. \tag{2.9}$$

This completes the proof. □

Theorem 2.12. *If $(P,\leq)$ is a poset with the least element a, then the induced BCK-algebra $(\mathscr{D}(P),\odot,P)$ is bounded with the bound $(a]$.*

Proof. Since a is the least element of $(P,\leq)$, we have $(a]=\{a\}$ and $a\in(w]$ for all $w \in P$. Hence $(w]\cap(a]=\{a\}$ for all $w\in P$. Now if $X\in\mathscr{D}(P)$, then $a \in X$ by Lemma 2.11. Therefore

$$X \odot (a] = \{y \in P \mid (y]\cap(a]=\{a\}\subseteq X\} = P,$$

that is, $(a]$ is the bound of the induced BCK-algebra $(\mathscr{D}(P),\odot,P)$. □

Let $\leq$ be an order relation on $IId(X)$ defined by

$$(\forall A,B \in IId(X))\,(A \leq B \Leftrightarrow B \subseteq A). \tag{2.10}$$

Then $(IId(X),\leq)$ is a poset, and $(\mathscr{D}(IId(X)),\odot,IId(X))$ is a BCK-algebra. Let $f : X \to \mathscr{D}(IId(X))$ be a mapping defined by

$$(\forall x \in X)\,(f(x)=\{A \in IId(X) \mid x \in A\}). \tag{2.11}$$

Lemma 2.13. *Let $A \in IId(X)$ be such that $x*y \in A$ for every $x,y \in A$. If $B \in IId(X)$ satisfies $B \in (A]\cap f(y)$, then $B \in f(x)$.*

Proof. Let $B \in IId(X)$ be such that $B \in (A]\cap f(y)$. Then $B\in(A]$ and $B \in f(y)$, and so $B \leq A$, i.e., $A \subseteq B$ and $y \in B$. Since $x*y\in A\subseteq B$, it follows from (c2) that $x \in B$ so that $B \in f(x)$. □

Theorem 2.14. *The mapping $f : X \to \mathscr{D}(IId(X))$ which is established in (2.11) is a homomorphism.*

Proof. Let $A \in f(x*y)$ for all $x,y\in X$. Then $A \in IId(X)$ and $x*y\in A$. Thus if $B \in IId(X)$ and $B \in (A]\cap f(y)$, then $B \in f(x)$ by Lemma 2.13, and so $A \in f(x)\odot f(y)$. Conversely, assume that $A \in IId(X)$ and $(A]\cap f(y)\subseteq f(x)$ for all $x,y \in X$. If $x*y \notin A$, then consider an ideal $A_y := \langle A \cup \{y\}\rangle$. Since

$x \notin A_y$, there exists $B \in IId(X)$ such that $A \subseteq B$, $y \in B$ and $x \notin B$. Hence $A \notin f(x) \odot f(y)$, which is a contradiction. Therefore $x * y \in A$ and $A \in f(x * y)$. Consequently, f is a homomorphism. □

Theorem 2.15. *Let $f : X \to \mathscr{D}(IId(X))$ be the mapping which is established in* (2.11). *Then*

$$f(X) := \{f(x) \mid x \in X\} \tag{2.12}$$

is a subalgebra of $\mathscr{D}(IId(X))$, which is isomorphic to X.

Proof. Clearly, $f(X)$ is a subalgebra of $\mathscr{D}(IId(X))$. It is also clear that the mapping $f : X \to \mathscr{D}(IId(X))$ is injective. Since f is a homomorphism, it follows that X and $f(X)$ are isomorphic. □

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