Commun. Korean Math. Soc. **22** (2007), No. 4, pp. 509–512

A NOTE ON GAUSS'S SECOND SUMMATION THEOREM FOR THE SERIES ${}_2F_1(1/2)$

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ABSTRACT. We aim at deriving Gauss's second summation theorem for the series ${}_2F_1(1/2)$ by using Euler's integral representation for ${}_2F_1$. It seems that this method of proof has not been tried.

We begin by recalling the well-known *Beta function* $B(\alpha,\beta)$ defined by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha)>0;\ \Re(\beta)>0) \tag{1}$$

or, equivalently,

$$B(\alpha,\beta) = 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}\,d\theta \quad (\Re(\alpha)>0;\ \Re(\beta)>0). \tag{2}$$

It has been pointed out in almost all books on hypergeometric series that the well-known summation theorems for the series ${}_2F_1$ such as of Gauss, Bailey, and Kummer can be derived by evaluating the integral representation for the series ${}_2F_1$. But we carefully conclude that anyone has not ever succeeded in proving the following Gauss's second summation theorem [1]:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) \end{matrix}\,\middle|\,\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}, \tag{3}$$

by using its related integral representation for the series ${}_2F_1$.

However, the purpose of this note is show that how we can arrive at deriving Gauss's second summation theorem (3) by making use of the integral representation for the series ${}_2F_1$. Before we come to the derivation of (3), we will just browse the method of proof of Gauss's summation theorem for ${}_2F_1(1)$ and the Bailey's formula for ${}_2F_1(-1)$ by means of Euler's integral representation for ${}_2F_1$.

Received July 29, 2005.

2000 *Mathematics Subject Classification.* Primary 33C20, 33C60; Secondary 33C70, 33C65.

Key words and phrases. generalized hypergeometric series ${}_pF_q$, Gauss's second summation theorem for ${}_2F_1(1/2)$, Beta function.

We start with Euler's integral representation for ${}_2F_1$ (see [2, p. 46]):

$$(4)\qquad {}_2F_1(a,\, b;\, c;\, z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt.$$

If we take $z = 1$ in (4), then the resulting integral on the right-hand side of (4) can be evaluated by the Beta function (1) and we get the following Gauss's summation theorem [1]:

$$(5)\qquad {}_2F_1(a,\, b;\, c;\, 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b) > 0).$$

Again, in (4), if we take $z = -1$ and $c = 1-a+b$, then resulting the integral on the right-hand side of (4) can also be evaluated through Beta function (1) and we get the following Kummer's summation theorem [1]:

$$(5)\qquad {}_2F_1(a,\, b;\, 1-a+b;\, -1) = \frac{\Gamma\left(1-a+b\right)\,\Gamma\left(1+\frac{1}{2}b\right)}{\Gamma\left(1+\frac{1}{2}b-a\right)\,\Gamma\left(1+b\right)}.$$

And finally, if we set $z = 1/2$ and $b = 1-a$ in (4), then the resulting integral on the right-hand side of (4) can again be evaluated (by letting $1-t = u$) with the help of Beta function (1) and we arrive at the following Bailey's formula [1]:

$$(6)\qquad {}_2F_1\left(a,\, 1-a;\, c;\, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}c\right)\,\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}.$$

On the other hand, if we take $z = \frac{1}{2}$ and $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$ in (4), we have

$$(7)\qquad \begin{aligned} &{}_2F_1\left(a,\, b;\, \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2};\, z\right) \\ &= \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(b)\,\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})} \int_0^1 t^{b-1}\,(1-t)^{\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}} \left(1-\frac{t}{2}\right)^{-a} dt. \end{aligned}$$

Since the integral on the right-hand side of (7) does not seem to be evaluated easily as above, anybody has not ever succeeded in deriving Gauss's second summation theorem (3) through an integral representation for ${}_2F_1$.

However we show that Gauss's second summation theorem (3) can be derived by using Euler's integral representation for ${}_2F_1$ (4) as follows: First we shall prove the following result which is presumably new:

$$(8)\qquad \begin{aligned} &{}_2F_1\left(a,\, b;\, c;\, \frac{1}{2}\right) \\ &= \frac{2^a\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^{\pi/2} (\cos\theta)^{b-1} \left(\sin\frac{\theta}{2}\right)^{2c-2b-1} \left(\cos\frac{\theta}{2}\right)^{2a-2c+1} d\theta. \end{aligned}$$

Indeed, by taking $z = 1/2$ in (4), we get

$$
\begin{aligned}
&{}_2F_1\left(a,\, b\,;\, c\,;\, \frac{1}{2}\right) \\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,\left(1-\frac{t}{2}\right)^{-a}\,dt,
\end{aligned}
\tag{9}
$$

which can be rewritten as in the following very slightly modified form:

$$
\begin{aligned}
&{}_2F_1\left(a,\, b\,;\, c\,;\, \frac{1}{2}\right) \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,[1+(1-t)]^{-a}\,dt.
\end{aligned}
\tag{10}
$$

Now substituting $t = \tan^2\theta/2$, after a little simplification, we get (8).

Finally, upon taking $c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$ in (8), after a little simplification, we obtain

$$
\begin{aligned}
&{}_2F_1\left(a,\, b\,;\, \frac{1}{2}(a+b+1)\,;\, \frac{1}{2}\right) \\
&= \frac{2^a\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(b)\,\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})} \int_0^{\pi/2} (\sin\theta)^{a-b}\,(\cos\theta)^{b-1}\,d\theta,
\end{aligned}
\tag{11}
$$

which, upon using the Beta function (2) and Legendre's duplication formula for the Gamma function (see [2, p. 7, Eq.(49)]), yields Gauss's second summation theorem (3).

References

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[2] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.

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