Commun. Korean Math. Soc. **22** (2007), No. 4, pp. 527–534

INTEGRATED RATE SPACE $\int \ell_\pi$

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Abstract. This paper deals with the BK-AK property of the integrated rate space $\int \ell_\pi$. Importance of $\delta^{(k)}$ in this content is pointed out. We investigate a determining set for the integrated rate space $\int \ell_\pi$. The set of all infinite matrices transforming $\int \ell_\pi$ into BK-AK space Y is denoted $\left(\int \ell_\pi : Y\right)$. We characterize the classes $\left(\int \ell_\pi : Y\right)$. When $Y = \ell_\infty, c_0, c, \ell^p, bv, bv_0, bs, cs, \ell_\rho, \ell_\pi$. In summary we have the following table:

	ℓ_∞	c_0	c	ℓ^p	bv	bv_0	bs	cs	ℓ_ρ	ℓ_π
$\int \ell_\pi$	Necessary and sufficient conditions on the matrix are obtained									

Introduction

Let (π_k) be a sequence of positive numbers. Let ℓ_π be the space of all (complex) sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} \left|\frac{x_k}{\pi_k}\right| < \infty$ with the norm $\|x\| = \sum_{k=1}^{\infty} \left|\frac{x_k}{\pi_k}\right|$. Then with this norm, ℓ_π is a BK-space. (See [2])

ℓ_∞ := The BK-space of all bounded (complex) sequences.
c_0 := The BK-space of all null sequences.
c := The BK-space of all convergent sequences.

In respect of c_0, c, ℓ_∞, we have

$$\|x\| = \sup_{(k)} |x_k|,$$

Received January 22, 2007.
2000 *Mathematics Subject Classification.* 46A45, 40C05.
Key words and phrases. BK-AK spaces, absolutely convexhull, determining set.

where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$.

ℓ^p := The BK-space of all sequences $x = (x_k)$ such that (for $p \geq 1$) $\sum_{k=1}^{\infty} |x_k|^p$ converges with the norm $\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$.

bv := The BK-space of all sequences $x = (x_k)$ such that

$$\sum |x_k - x_{k+1}| < \infty$$

with the norm $\|x\| = \lim_{k \to \infty} |x_k| + \sum_{k=1}^{\infty} |x_k - x_{k+1}|$.

bv_0 := $bv \cap c_0$.

bs := The BK-space of all sequences $x = (x_k)$ such that the sequence $\left\{\sum_{k=1}^{m} x_k : m = 1, 2, 3, \ldots,\right\}$ is bounded with the norm

$$\|x\| = \sup_{(m)} \left|\sum_{k=1}^{m} x_k\right|.$$

cs := The BK-space of all sequences $x = (x_k)$ such that the sequence $\left\{\sum_{k=1}^{m} x_k : m = 1, 2, 3, \ldots,\right\}$ is convergent with the norm

$$\|x\| = \sup_{(m)} \left|\sum_{k=1}^{m} x_k\right|.$$

In place of $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$ take $\rho = (\rho_1, \rho_2, \rho_3, \ldots)$ to obtain ℓ_ρ.

Let ℓ_ρ be the BK-space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} \left|\frac{x_k}{\rho_k}\right| < \infty$ with the norm $\|x\| = \sum_{k=1}^{\infty} \left|\frac{x_k}{\rho_k}\right| < \infty$.

Definition 1. Let $\{\pi_k\}$ be a sequence of positive numbers with $\left|\frac{\pi_k}{k}\right| \leq M \; \forall k$ where $M \geq 1$. Let ℓ be the sequence space of all those sequence $x = (x_k)$ such that $\sum_{k=1}^{\infty} |x_k| < \infty$. We define $\int \ell_\pi = \left\{x = (x_k) : \sum_{k=1}^{\infty} \left|\frac{kx_k}{\pi_k}\right| < \infty\right\}$. The concept of integrated space was introduced in [1].

Definition 2. A BK-space is a Banach space of sequences having the property that the coordinate functionals are continuous. Given a sequence $x = \{x_k\}$ the coordinate functionals are given by $f_k(x) = x_k$ for $k = 1, 2, 3, \ldots$.

Definition 3. Let $x = \{x_k\}$ be given; the n^{th} section is the sequence $x^{[n]} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$, $\delta^{(k)} = (0, 0, \ldots, 1, 0, 0, \ldots)$, 1 in the k^{th} place and zero else where. A BK-space X is said to have AK (or sectional convergence) if $\left\|x^{[n]} - x\right\| \to 0$ as $n \to \infty$.

Definition 4. Let Φ stand for all finite sequences. Let $D = \{x \in \Phi : \|x\| \leq 1\}$ in BK-space X, that is, D is the intersection of closed unit sphere (disc) with Φ. A subset E of Φ is called a determining set for X if its absolutely convex hull E is identical with D.

Definition 5. Let X be an FK-space and E be a determining set for X. Let Y be an FK-space and A be a matrix. Suppose that either X has AK or A is row finite. Then $A \in (X : Y)$ if and only if

(1) The columns of A belong to Y and

(2) $A\,[E]$ is a bounded subset of Y.

Given a sequence $x = \{x_k\}$ and infinite matrix $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$, then transform A is the sequence $y = (y_n)$ where $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$ $(n, k = 1, 2, 3, \ldots)$ whenever $\sum a_{nk} x_k$ exists.

Theorem 1. *$\int \ell_\pi$ is a BK-space.*

Proof. Let $x = (x_k) \in \int \ell_\pi$. Define $\|x\| = \sum_{k=1}^{\infty} k \left|\frac{x_k}{\pi_k}\right|$ with this norm. Then $\int \ell_\pi$ is a Banach space. If we define $f_k(x) = x_k$ for all $k = 1, 2, 3, \ldots$ we have

$$\|x\| = 1 \left|\frac{x_1}{\pi_1}\right| + 2 \left|\frac{x_2}{\pi_2}\right| + \cdots + k \left|\frac{x_k}{\pi_k}\right| + \cdots .$$

Hence

$$\frac{k}{\pi_k} |x_k| \leq \|x\| .$$

So that

$$\begin{aligned} |x_k| \leq \mathrm{M}\,\|x\| \quad &\Rightarrow \quad |f_k(x)| \ \leq \mathrm{M}\,\|x\| \text{ for all } k \text{ , where } \mathrm{M} \geq 1 \\ &\Rightarrow \quad f_k \text{ is continuous linear functional for each fixed } k. \end{aligned}$$

Thus $\int \ell_\pi$ is a BK-space. □

Theorem 2. *The BK-space $\int \ell_\pi$ has AK-property.*

Proof. Let $x = (x_k) \in \int \ell_\pi$. Then $x^{[n]} = \{x_1, x_2, x_3, \ldots, x_n, 0, 0, \ldots\}$.

Hence

$$x - x^{[n]} = \{0, 0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots\}$$

$$\Rightarrow \left\| x - x^{[n]} \right\| = \|0, 0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots\| = \sum_{k \geq n+1} k \left| \frac{x_k}{\pi_k} \right| \to 0 \text{ as } n \to \infty,$$

$$\text{because } x = (x_k) \in \int \ell_\pi$$

$$\Rightarrow \lim_{n \to \infty} \left\| x - x^{[n]} \right\| = 0$$

$$\Rightarrow x^{[n]} \to x \text{ as } n \to \infty \text{in } \int \ell_\pi$$

$$\Rightarrow \int \ell_\pi \text{ has AK-property .}$$

This completes the proof. □

Theorem 3. *Let* $A = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \lambda_k^2 \left(\frac{x_k}{\pi_k} \right) = 0 \right\}$. *Then* $A \subset \int \ell_\pi$ *and* $\delta^1 \in \bar{A}$ *the closure of* A *in* $\int \ell_\pi$. *Here* $\delta^1 = (1, 0, 0, \ldots)$.

Proof. Let $x = (x_k) \in A$. Then $\sum \lambda_k^2 \left| \frac{x_k}{\pi_k} \right| = 0$.

Now $\sum_{k \geq n+1} \lambda_k^2 \left| \frac{x_k}{\pi_k} \right| \leq \sum_{k \geq 1} \lambda_k^2 \left| \frac{x_k}{\pi_k} \right| \to 0$ as $n \to \infty$. Hence $x = (x_k) \in \int \ell_\pi$.

Take

$$x^1 = (1, 0, 0, \ldots)$$

$$x^2 = \left(1, -\frac{\pi_2}{\lambda_2^2}, 0, 0, \ldots \right)$$

$$x^3 = \left(0, 0, -\frac{\pi_3}{\lambda_3^2}, 0, 0, \ldots \right)$$

$$\vdots$$

$$x^k = \left(1, 0, \ldots, -\frac{\pi_k}{\lambda_k^2}, 0, \ldots \right)$$

and so on. Then $x^k \in A$ for every k. Also

$$\|\delta^1 - x^k\| = \left\| \left(0, 0, \ldots, \frac{\pi_k}{\lambda_k^2}, 0, 0, \ldots \right) \right\| = \frac{\pi_k}{\lambda_k^2} \to 0 \text{ as } k \to \infty.$$

Thus $\delta^1 \in \bar{A}$. This completes the proof . □

Notation :

Let

$$s_1 = \frac{\pi_1}{1} \delta^{(1)} = \left(\frac{\pi_1}{1}, 0, \ldots \right)$$

$$s_2 = \frac{\pi_2}{2} \delta^{(2)} = \left(0, \frac{\pi_2}{2}, 0, \ldots \right)$$

$$\vdots$$

$$s_k = \frac{\pi_k}{k}\delta^{(k)} = \left(0, 0, \ldots, \frac{\pi_k}{k}, 0, \ldots\right)$$

and so on. Here the vectors $\delta^1 = (1, 0, 0, \ldots), \delta^2 = (0, 1, 0, \ldots), \ldots$.

Theorem 4. *Let $E = \{s_1, s_2, \ldots\}$. Then E is a determining set for $\int \ell_\pi$.*

Proof. **Step 1.** Let A be the absolutely convex hull of E and let $D = \Phi \cap$ (the closed unit ball in $\int \ell_\pi$).

Let $x \in A$. Then

$$\begin{aligned} & x = \sum_{k=1}^{m} t_k s_k \text{ with } \sum_{k=1}^{m} |t_k| \leq 1 \\ \Rightarrow\ & x = \left(t_1\frac{\pi_1}{1}, t_2\frac{\pi_2}{2}, \ldots, t_m\frac{\pi_m}{m}, 0, 0\right) \\ \Rightarrow\ & x \in \Phi. \end{aligned} \tag{4.1}$$

From $x = \sum_{k=1}^{m} t_k \frac{\pi_k}{k}\delta^k$, taking norm on both sides, we have

$$\|x\| = \sum_{k=1}^{m} |t_k| \left|\frac{\pi_k}{k}\right| \left\|\delta^{(k)}\right\|.$$

Note $\left\|\delta^{(k)}\right\| = \sum_{k=1}^{\infty} k \left|\frac{1}{\pi_k}\right| = \left|\frac{k}{\pi_k}\right|$ and so

$$\|x\| \leq \sum_{k=1}^{m} |t_k| \leq 1 \qquad \text{by using (4.1).} \tag{4.2}$$

Thus $\|x\| \leq 1$, so

$$x \in \text{ the closed unit ball in } \int \ell_\pi.$$

Furthermore

$$x = \sum_{k=1}^{m} t_k \frac{\delta^{(k)}\pi_k}{k} = \left\{\frac{1.\pi_1}{1}, \frac{2.\pi_2}{2}, \ldots, \frac{m.\pi_m}{m}, 0, 0, \ldots\right\}. \tag{4.3}$$

From (4.2) and (4.3), it follows that $x \in D$.

Thus

$$A \subset D. \tag{4.4}$$

Step 2. On the other hand, let $x \in D$. Then

$$\begin{aligned} & x \in \Phi \text{ and } \|x\| \leq 1 \\ \Rightarrow\ & x = \{x_1, x_2, x_3, \ldots, x_m, 0, \ldots\} \\ & = \left(\frac{1}{\pi_1}x_1\right) s_1 + \left(\frac{2}{\pi_2}x_2\right) s_2 + \cdots + \left(\frac{m}{\pi_m}x_m\right) s_m. \end{aligned} \tag{4.5}$$

But then $\|x\| \leq \sum \frac{k}{\pi_k} x_k$ and $\sum_{k=1}^{m} \left| \frac{k}{\pi_k} x_k \right| \leq \|x\| \leq 1$. Therefore $x \in A$.

Consequently

$$D \subset A. \tag{4.6}$$

From (4.4) and (4.6) it follows that $A = D$.
This completes the proof . □

Theorem 5. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$, be an infinite matrix . Then*

$$A \in \left(\int \ell_\pi : \ell_\infty \right) \iff \sup_{(n,k)} \left| \frac{\pi_k a_{nk}}{k} \right| < \infty \tag{5.1}$$

where $\ell_\infty = \{$all bounded sequences$\}$.

Proof. Note that E is a determining set for $\int \ell_\pi$ and ℓ_∞ is a BK-space. Also $As^{(k)} = \left(\frac{\pi_1}{k} a_{1k}, \frac{\pi_k}{k} a_{2k}, \ldots, \frac{\pi_k}{k} a_{nk}, \ldots \right)$. We know that

$$\begin{aligned} A \in \left(\int \ell_\pi : \ell_\infty \right) &\iff A[E] \text{ is bounded in } \ell_\infty \\ &\iff \sup_{(n,k)} \left| \frac{\pi_k a_{nk}}{k} \right| < \infty. \end{aligned}$$

This completes the proof. □

The following proofs are similar. Hence we omit the proofs.

Theorem 6. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix . Then*

$$\begin{aligned} & A \in \left(\int \ell_\pi : c_0 \right) \\ \iff & \sup_{(n,k)} \left| \frac{\pi_k a_{nk}}{k} \right| < \infty, \\ & \lim_{n \to \infty} a_{nk} = 0 \text{ for } k = 1, 2, 3, \ldots. \end{aligned}$$

Theorem 7. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : c \right) \iff \begin{aligned} & \sup_{(n,k)} \left| \frac{\pi_k a_{nk}}{k} \right| < \infty, \\ & \lim_{n \to \infty} a_{nk} = \alpha \text{ for } k = 1, 2, 3, \ldots. \end{aligned}$$

Theorem 8. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : \ell^p \right), \; p \geq 1 \iff \sup_{(n,k)} \sum_{n=1}^{\infty} \left| \frac{\pi_k a_{nk}}{k} \right|^p < \infty.$$

Theorem 9. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : bv\right) \iff \sup_{(k)} \sum_{n=1}^{\infty} \frac{\pi_k}{k} |a_{nk} - a_{n-1,k}| < \infty$$

with $a_{0k} = 0$ by convention.

Theorem 10. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : bv_0\right) \iff \sup_{(k)} \sum_{n=1}^{\infty} \frac{\pi_k}{k} |a_{nk} - a_{n-1,k}| < \infty$$

with $a_{0k} = 0$ by convention,

$$\lim_{n \to \infty} a_{nk} = 0 \text{ for } k = 1, 2, 3, \ldots.$$

Theorem 11. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : bs\right) \iff \sup_{(m,k)} \sum_{n=1}^{m} \left|\frac{\pi_k a_{nk}}{k}\right| < \infty.$$

Theorem 12. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : cs\right) \iff \sup_{(m,k)} \sum_{n=1}^{k} \left|\frac{\pi_k a_{nk}}{k}\right| < \infty.$$

Theorem 13. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : \ell_\rho\right) \iff \sup_{(n,k)} \sum_{n=1}^{\infty} \left|\frac{\pi_k a_{nk}}{k\ \rho_k}\right| < \infty.$$

Theorem 14. *Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$ be an infinite matrix. Then*

$$A \in \left(\int \ell_\pi : \ell_\pi\right) \iff \sup_{(n,k)} \sum_{n=1}^{\infty} \left|\frac{\pi_k a_{nk}}{k}\right| < \infty.$$

References

[1] G. Goes and S. Goes, *Sequences of bounded variation and sequences of Fourier coefficients. I*, Math. Z. **118** (1970), 93–102.

[2] E. Jürimäe, *Properties of domains of matrix mappings on rate-spaces and spaces with speed*, Tartu Ül. Toimetised No. **970** (1994), 53–64.

[3] A. Wilansky, *Functional analysis*, Blaisdell Publishing Co. Ginn and Co., New York-Toronto-London, 1964.

[4] ———, *Summability through functional analysis*, North-Holland Mathematics Studies, 85. Notas de Matematica [Mathematical Notes], 91. North-Holland Publishing Co., Amsterdam, 1984.

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