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DIMENSIONS OF DISTRIBUTION SETS IN THE UNIT INTERVAL

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ABSTRACT. The unit interval is not homeomorphic to a self-similar Cantor set in which we studied the dimensions of distribution subsets. However we show that similar results regarding dimensions of the distribution subsets also hold for the unit interval since the distribution subsets have similar structures with those in a self-similar Cantor set.

1. Introduction

We ([1]) studied the Hausdorff and packing dimensions of some distribution subsets in a self-similar Cantor set. Instead of computing the spherical density of the point in the distribution subset, we calculated its cylinder density since their values of densities are equivalent to each other. They have same values since the gaps between fundamental intervals are uniformly bounded away from zero. Similarly we also apply the arguments used in [1] to the computation of dimensions of distribution subsets in the unit interval. In this case, we find that the cylinder density and the spherical density of the point in the unit interval do not coincide. Still we can apply its cylinder density to find the dimensions of distribution subsets in the unit interval. We note that the cylinder density gives the information of Hausdorff and packing dimensions since the family of cylinders are a bounded Vital covering ([4, 7]) of the unit interval.

2. Preliminaries

We recall *a generalized expansion* of a number in $[0,1]$ ([4, 7]). For each $n = 1, 2, \ldots$ let $k_n \geq 2$ be an integer and choose values $0 < \alpha_{n,1} < \cdots < \alpha_{n,k_n-1} < 1$, setting $\alpha_{n,0} = 0$ and $\alpha_{n,k_n} = 1$. The initial proportions $\alpha_{1,1}, \ldots, \alpha_{1,k_1-1}$ determine a division of $[0,1]$ into the disjoint intervals $[\alpha_{1,i}, \alpha_{1,i+1})$, $i = 0, 1, \ldots, k_1 - 2$, and $[\alpha_{1,k_1-1}, 1]$. We will indicate that a point x in $[0,1]$ falls into the i^{th} interval ($i = 0, 1, \ldots, k_1 - 1$) by the notation $I_1(x) = i$. $I_1(x)$ will be the first term in the expansion of x (with respect to the choices

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$\alpha_{n,i}$). At the second stage each interval $\{x : I_1(x) = i\}$ is divided into k_2 disjoint subintervals determined by the given proportions $a_{2,1}, \ldots, a_{2,k_2-1}$. This splits $[0,1]$ into k_1k_2 disjoint intervals which are most conveniently expressed in the form $\{x : I_1(x) = i, I_2(x) = j\}$ for some choice of $i = 0, 1, \ldots, k_1 - 1$ and $j = 0, 1, \ldots, k_2 - 1$. Letting $d_{n,i} = \alpha_{n,i+1} - \alpha_{n,i}$ for each n and i, we can alternately write $\{x : I_1(x) = i, I_2(x) = j\} = \{x : \alpha_{1,i} + \alpha_{2,j}d_{1,i} \leq x < \alpha_{1,i} + \alpha_{2,j+1}d_{1,i}\}$ (but including the right hand endpoint if $i = k_1 - 1$ and $j = k_2 - 1$). $I_2(x)$ will be the second term in the expansion of x. Each interval $\{x : I_1(x) = i, I_2(x) = j\}$ is then divided according to the proportions $\alpha_{3,1}, \ldots, \alpha_{3,k_3-1}$. Continuing this subdivision process, the n^{th} stage produces a splitting of $[0,1]$ into $k_1k_2\cdots k_n$ disjoint intervals $\{x : I_1(x) = i_1, I_2(x) = i_2, \ldots, I_n(x) = i_n\}$ $= \{x : \alpha_{1,i_1} + \alpha_{2,i_2}d_{1,i_1} + \alpha_{3,i_3}d_{2,i_2}d_{1,i_1} + \cdots + \alpha_{n,i_n}d_{n-1,i_{n-1}} + \cdots + d_{1,i_1} \leq x < \alpha_{1,i_1} + \alpha_{2,i_2}d_{1,i_1} + \cdots + \alpha_{n,i_n+1}d_{n-1,i_{n-1}} + \cdots + d_{1,i_1}\}$. The sequence $I_1(x), I_2(x), \ldots$ is the *generalized expansion* of x, taking values in the countable set $S = \{0, 1, \ldots, k_n - 1 : n = 1, 2, \ldots\}$. If $r \geq 2$ is a positive integer and $k_n = r$, $\alpha_{n,i} = \frac{1}{r}$ for each n, then the result is the usual r-adic expansion of x. (If x has more than one r-adic expansion, this method produces the terminating one.)

Let $\mathbb{N}$ be the set of natural numbers. In this paper, we restrict $k_n = 2$ for all $n \in \mathbb{N}$ and $\alpha_{n,1} = a$ for each n with $0 < a < 1$. In this case, the generalized expansion of a number in $[0,1]$ will be called *a generalized dyadic expansion* of the number. We denote a fundamental interval or a cylinder $\{x : I_1(x) = i_1, I_2(x) = i_2, \ldots, I_k(x) = i_k\}$ by $I_{i_1,\ldots,i_k}$ where $i_j \in \{0,1\}$ and $1 \leq j \leq k$. Sometimes we use the notation F_a in which the point in $[0,1]$ has a generalized dyadic expansion with $\alpha_{n,1} = a$ for each n to distinguish it from another one in which the point in $[0,1]$ has different $\alpha_{n,1}$ from a for each n. We note that the point in $F_{\frac{1}{2}}$ has the dyadic expansion.

We note that if $x \in F_a$, then there is a generalized dyadic expansion $\sigma \in \{0,1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \ldots, i_k$ where $\sigma = i_1, i_2, \ldots, i_k, i_{k+1}, \ldots$). Without confusion, we identify $x \in F_a$ with $\sigma \in \{0,1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$. If $x \in [0,1]$ and $x \in I_\tau$ where $\tau \in \{0,1\}^k$, a cylinder $c_k(x)$ denotes the fundamental interval I_τ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \ldots$.

From now on $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E([6]). We note that $\dim(E) \leq \mathrm{Dim}(E)$ for every set E([6]). We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of $x = \sigma$(cf. [1]).

In F_a, for $r \in [0,1]$, we define the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 0 in proportion r by

$$\underline{F}(r) = \{x \in [0,1] : \liminf_{k\to\infty} \frac{n_0(x|k)}{k} = r\},$$

$$\overline{F}(r) = \{x \in [0,1] : \limsup_{k\to\infty} \frac{n_0(x|k)}{k} = r\}.$$

We write $\underline{F}(r) \cap \overline{F}(r) = F(r)$ and call it the distribution set containing the digit 0 in proportion r. Let $p \in (0,1)$ and denote γ_p a self-similar Borel probability measure on $[0,1]$ satisfying $\gamma_p(I_0) = p$ (cf. [1, 6]). We write $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$) for the set of points at which the lower(upper) local cylinder density of γ_p on $[0,1]$ is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \{x \in [0,1] : \liminf_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \alpha\},$$

$$\overline{E}_\alpha^{(p)} = \{x \in [0,1] : \limsup_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \alpha\}.$$

We write $\underline{E}_\alpha^{(p)} \cap \overline{E}_\alpha^{(p)} = E_\alpha^{(p)}$ and call it the local dimension set having local cylinder density α by a self-similar measure γ_p. In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. Main results

Without any additional condition, we assume that the distribution sets and local dimension sets are in F_a.

Lemma 1. *Let $p \in (0,1)$ and consider a self-similar measure γ_p on $[0,1]$ and let $r \in [0,1]$ and $g(r,p) = \frac{r\log p + (1-r)\log(1-p)}{r \log a + (1-r)\log(1-a)}$. Then*
(1) *for $0 < p < a$*

$$\liminf_{k\to\infty} \frac{n_0(x|k)}{k} = r \iff \liminf_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r,p),$$

(2) *for $a < p < 1$*

$$\liminf_{k\to\infty} \frac{n_0(x|k)}{k} = r \iff \limsup_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r,p),$$

(3) *for $0 < p < a$*

$$\limsup_{k\to\infty} \frac{n_0(x|k)}{k} = r \iff \limsup_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r,p),$$

(4) *for $a < p < 1$*

$$\limsup_{k\to\infty} \frac{n_0(x|k)}{k} = r \iff \liminf_{k\to\infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r,p).$$

Proof. It follows from the same arguments in the proof of the lemma 1 in [1]. □

Theorem 2. *Let $g(r,p) = \frac{r\log p + (1-r)\log(1-p)}{r \log a + (1-r)\log(1-a)}$ and let $r \in [0,1]$. Then*
(1) $\underline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$ *if $0 < p < a$,*
(2) $\underline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$ *if $a < p < 1$,*

(3) $\overline{F}(r) = \overline{E}^{(p)}_{g(r,p)}$ *if* $0 < p < a$,
(4) $\overline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ *if* $a < p < 1$.

Proof. It is immediate from the above Lemma. □

Corollary 3. *Let* $\delta(p) = \frac{p\log p+(1-p)\log(1-p)}{p\log a+(1-p)\log(1-a)}$. *Then*
(1) $\underline{F}(p) = \underline{E}^{(p)}_{\delta(p)}$ *if* $0 < p < a$,
(2) $\underline{F}(p) = \overline{E}^{(p)}_{\delta(p)}$ *if* $a < p < 1$,
(3) $\overline{F}(p) = \overline{E}^{(p)}_{\delta(p)}$ *if* $0 < p < a$,
(4) $\overline{F}(p) = \underline{E}^{(p)}_{\delta(p)}$ *if* $a < p < 1$.

Proof. It is immediate from the above Theorem with $r = p$. □

Since the family of our cylinders are a bounded Vital covering ([4, 7]) of the unit interval, we can apply the cylinder density theorem to the following Corollary.

Corollary 4. *Let* $\delta(p) = \frac{p\log p+(1-p)\log(1-p)}{p\log a+(1-p)\log(1-a)}$. *Then*
(1) $\dim(\underline{F}(p)) = \dim(\overline{F}(p)) = \delta(p)$ *and* $\mathrm{Dim}(\overline{F}(p)) = \delta(p)$ *if* $0 < p < a$,
(2) $\dim(\underline{F}(p)) = \dim(\overline{F}(p)) = \delta(p)$ *and* $\mathrm{Dim}(\underline{F}(p)) = \delta(p)$ *if* $a < p < 1$,
(3) $\dim(\underline{F}(a)) = \dim(\overline{F}(a)) = 1$ *and* $\mathrm{Dim}(\underline{F}(a)) = \mathrm{Dim}(\overline{F}(a)) = 1$.

Proof. It follows from the same arguments in the proof of the corollary 5 in [1] together with the theorem 4.3 with its dual results of the remark 4.5 in [7]. □

Theorem 5. $\dim(\underline{F}(0)) = 0$, $\mathrm{Dim}(\overline{F}(0)) = 0$, $\mathrm{Dim}(\underline{F}(1)) = 0$ *and* $\dim(\overline{F}(1)) = 0$.

Proof. It follows from the same arguments in the proof of the theorem 6 in [1]. □

Remark 1. Noting that $\dim(E) \leq \mathrm{Dim}(E)$ for every set E, we easily see that $\dim(\overline{F}(0)) = 0$ and $\dim(\underline{F}(1)) = 0$.

Remark 2. $\underline{F}(0) \cap \overline{F}(1)$ in $F_{\frac{1}{2}}$ is comeager in $[0, 1]$ ([8]). In this case Olsen ([8]) used non-terminating expansion. However $\underline{F}(0) \cap \overline{F}(1)$ is the same set with ours since the points which can be expressed as both non-terminating and terminating expansion are in $F(0)$ of Olsen's definition and $F(1)$ in our definition respectively which are disjoint with $\underline{F}(0) \cap \overline{F}(1)$ of either of the definitions.

Theorem 6. $\mathrm{Dim}(\underline{F}(0)) = 1$ *and* $\mathrm{Dim}(\overline{F}(1)) = 1$.

Proof. Clearly $\underline{F}(0) \cap \overline{F}(1)$ in F_a is homeomorphic to $\underline{F}(0) \cap \overline{F}(1)$ in $F_{\frac{1}{2}}$. Since $\underline{F}(0) \cap \overline{F}(1)$ in $F_{\frac{1}{2}}$ is comeager in $[0, 1]$ ([8]), we see that $\underline{F}(0) \cap \overline{F}(1)$ in F_a is also comeager in $[0, 1]$. It follows from the Exercise (1.8.4) in [5]. □

Remark 3. If α is given as the cylinder density of γ_p, then we can solve $r \in [0,1]$ of the equation $\alpha = g(r,p)$. Then $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$) can be expressed as one of $\underline{F}(r)$ and $\overline{F}(r)$. If we apply Theorem 2 to this information, then we have Hausdorff and packing dimensions of $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$). This gives the following Corollary.

Corollary 7. *Let* $p \in (0,1)$ *and* $p \neq a$. *Let* α *be in* $[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)}]$ *or* $[\frac{\log(1-p)}{\log(1-a)}, \frac{\log p}{\log a}]$. *For the solution* r *of the equation* $\alpha = \frac{r\log p+(1-r)\log(1-p)}{r\log a+(1-r)\log(1-a)}$ *and* $\delta(r) = \frac{r\log r+(1-r)\log(1-r)}{r\log a+(1-r)\log(1-a)}$,

(1) $\dim(\underline{E}_\alpha^{(p)}) = \dim(\overline{E}_\alpha^{(p)}) = \dim(E_\alpha^{(p)}) = \dim(F(r)) = \delta(r)$,

(2) $\mathrm{Dim}(E_\alpha^{(p)}) = \mathrm{Dim}(F(r)) = \delta(r)$,

(3) $\mathrm{Dim}(\underline{E}_\alpha^{(p)}) = \delta(r)$ *if* $0 < p < a$ *with* $a \le r \le 1$ *or* $a < p < 1$ *with* $0 \le r \le a$,

(4) $\mathrm{Dim}(\overline{E}_\alpha^{(p)}) = \delta(r)$ *if* $0 < p < a$ *with* $0 \le r \le a$ *or* $a < p < 1$ *with* $a \le r \le 1$,

(5) $\dim(\underline{E}_1^{(a)}) = \dim(\overline{E}_1^{(a)}) = \mathrm{Dim}(\underline{E}_1^{(a)}) = \mathrm{Dim}(\overline{E}_1^{(a)}) = 1$.

Proof. It follows from the same arguments in the proof of the corollary 7 in [1]. □

Remark 4. $F(= F_a)$ is completely decomposed into classes by the lower and upper distribution sets as $F = \cup_{0\le p\le 1}\underline{F}(p)$ and $F = \cup_{0\le p\le 1}\overline{F}(p)$. Similarly $F(= F_a)$ is completely decomposed into classes by the lower and upper local cylinder densities of self-similar measure γ_p as

$$F = \cup_{\alpha\in[\frac{\log(1-p)}{\log(1-a)}, \frac{\log p}{\log a}]}\underline{E}_\alpha^{(p)} \quad \text{if } 0 < p < a,$$

$$F = \cup_{\alpha\in[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)}]}\underline{E}_\alpha^{(p)} \quad \text{if } a < p < 1,$$

$$F = \cup_{\alpha\in[\frac{\log(1-p)}{\log(1-a)}, \frac{\log p}{\log a}]}\overline{E}_\alpha^{(p)} \quad \text{if } 0 < p < a,$$

$$F = \cup_{\alpha\in[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)}]}\overline{E}_\alpha^{(p)} \quad \text{if } a < p < 1.$$

Remark 5. In the view of [2, 3], we see that $\mathrm{Dim}(\underline{F}(p)) = 1$ if $0 < p < a$, and $\mathrm{Dim}(\overline{F}(p)) = 1$ if $a < p < 1$.

References

[1] I. S. Baek, *Relation between spectral classes of a self-similar Cantor set*, J. Math. Anal. Appl. **292** (2004), no. 1, 294–302.

[2] ______, *Another complete decomposition of a self-similar Cantor set*, preprint.

[3] I. S. Baek, L. Olsen, and N. Snigireva, *Divergence points of self-similar measures and packing dimension*, Adv. Math. **214** (2007), no. 1, 267–287.

[4] C. D. Cutler, *A note on equivalent interval covering systems for Hausdorff dimension on* $\mathbb{R}$, Internat. J. Math. & Math. Sci. **11** (1988), 643–650.

[5] G. Edgar, *Integral, Probability and Fractal measures*, Springer Verlag, 1998.

[6] K. J. Falconer, *Techniques in fractal geometry*, John Wiley and Sons, 1997.

[7] H. H. Lee and I. S. Baek, *A note on equivalent interval covering systems for packing dimension of* $\mathbb{R}$, J. Korean Math. Soc. **28** (1991), 195–205.

[8] L. Olsen, *Extremely non-normal numbers*, Math. Proc. Camb. Phil. Soc. **137** (2004), 43–53.

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