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ASYMPTOTIC DIRICHLET PROBLEM FOR THE SCHRÖDINGER OPERATOR ON 2-DIMENSIONAL CARTAN-HADAMARD MANIFOLDS

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ABSTRACT. We solve the asymptotic Dirichlet problem for a certain Schrödinger operator on 2-dimensional Cartan-Hadamard manifolds.

1. Introduction

In this paper, we will prove the existence of nonconstant bounded solutions for the Schrödinger operator $\Delta - V$ on a 2-dimensional Cartan-Hadamard manifold M, where Δ denotes the Laplacian on M and V is a nonnegative function on M. (By a Cartan-Hadamard manifold, we mean a complete simply connected manifold with nonpositive sectional curvature.) Throughout this paper, we shall always assume that every potential V is continuous. This assumption guarantees the continuity of solutions of the Schrödinger equation. More generally, such a result can be extended to potentials in the local Kato class. (See [4].) In the case that V is identically zero, the Schrödinger operator becomes the Laplace-Beltrami operator and its solutions are called harmonic functions.

In [2], Choi proposed the asymptotic Dirichlet problem for the Laplace-Beltrami operator and solved the problem when a Cartan-Hadamard manifold M with sectional curvature bounded above by a strictly negative constant satisfies the convex conic neighborhood condition at the boundary at infinity $M(\infty)$. Choi also proved that such a convexity condition is automatically satisfied when the sectional curvature of a 2-dimensional Cartan-Hadamard manifold bounded above by a strictly negative constant. On the other hand, Ancona [1] proved that if a Cartan-Hadamard manifold has no curvature lower bounds, then the asymptotic Dirichlet problem for the Laplace-Beltrami operator cannot be solvable in general.

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We will solve the following asymptotic Dirichlet problem for the Schrödinger operator:

Theorem 1.1. *Let M be a 2-dimensional Cartan-Hadamard manifold. Suppose that the sectional curvature $K_M(x)$ of M at x is less than or equal to a strictly negative constant $-a^2$ for some $a > 0$ and for each point $x \in M$. Let V be a nonnegative continuous function on M satisfying the following decay rate condition: There exist positive constants C and γ such that for any $x \in M$,*

$$V(x) \leq \frac{C}{(1+r(x))^{2+\gamma}}, \tag{1}$$

where $r(x) = d(o,x)$ denotes the distance from a fixed point $o \in M$. Then for a continuous function f on $M(\infty)$, there exists a unique solution u of Schrödinger operator $\Delta - V$ such that for each $\mathbf{v} \in M(\infty)$,

$$\lim_{x \to \mathbf{v}} u(x) = f(\mathbf{v}),$$

where Δ is the Laplacian on M.

2. Proof of main theorem

Let M be a 2-dimensional Cartan-Hadamard manifold and fix a point $o \in M$. Let us denote $M(\infty)$ to be the boundary at infinity of M which is the set of asymptotic classes of unit speed geodesic rays. Then it is topologized with the cone topology in the sense of Eberlein-O'Neill [3]. Note that we may identify $M(\infty)$ with the unit circle $\mathbb{S}^1 \subset T_oM$.

As mentioned before, Choi [2] solved the following asymptotic Dirichlet problem for harmonic functions:

Theorem 2.1. *Let M be a 2-dimensional Cartan-Hadamard manifold. For each point $x \in M$, suppose that $K_M(x) \leq -a^2$ for some $a > 0$. Then the asymptotic Dirichlet problem for harmonic functions is solvable.*

Remark. For a given continuous function $f : M(\infty) \to \mathbb{R}$, let h be the solution of the asymptotic Dirichlet problem for harmonic function with the boundary data f. Then for each $\mathbf{v} \in M(\infty)$, we have

$$\lim_{x \to \mathbf{v}} h(x) = f(\mathbf{v}). \tag{2}$$

Now consider a C^∞-function w defined by

$$w(x) = \frac{1}{(1+r(x))^\delta} \tag{3}$$

for $x \in M \setminus \{o\}$ and $\delta > 0$, where $r(x) = d(o,x)$ denotes the distance from a fixed point $o \in M$.

Lemma 2.2. *For a given continuous function $f : M(\infty) \to \mathbb{R}$, let h be the solution of the asymptotic Dirichlet problem for harmonic function with the boundary data f and w be given by (3) with $\delta \in (0,1) \cap (0,\gamma)$. Then $h + w$ is* *a supersolution and* $h-w$ *is a subsolution for the Schrödinger operator* $\Delta - V$ *in* $M \setminus \overline{B}_R(o)$*, where* $R = R(a,\delta) > 0$.

Proof. Since M is a 2-dimensional Cartan-Hadamard manifold whose sectional curvature bounded above by $-a^2$, from the Hessian comparison theorem,

$$\Delta r(x) \geq a \coth(ar(x)).$$

Thus for each sufficiently large $r(x) > R_1 = R_1(a)$, we have

$$\Delta r(x) \geq \frac{2}{1+r(x)},$$

hence

$$\begin{aligned} \Delta w(x) &= -\frac{\delta \Delta r(x)}{(1+r(x))^{1+\delta}} + \frac{\delta(1+\delta)|\nabla r(x)|^2}{(1+r(x))^{2+\delta}} \\ &\leq -\frac{2\delta}{(1+r(x))^{1+\delta}(1+r(x))} + \frac{\delta(1+\delta)}{(1+r(x))^{2+\delta}} \\ &\leq -\frac{\delta(1-\delta)}{(1+r(x))^{2+\delta}} < 0 \end{aligned} \tag{4}$$

whenever $0 < \delta < 1$ and $r > R_1$. Since h is bounded, by (1) and (4), we have

$$\begin{aligned} (\Delta - V)(h+w) &= \Delta h - Vh + \Delta w - Vw \\ &= -Vh + \Delta w - Vw \\ &\leq \frac{C}{(1+r(x))^{2+\gamma}} - \frac{\delta(1-\delta)}{(1+r(x))^{2+\delta}} < 0 \end{aligned}$$

if $\delta \in (0,1) \cap (0,\gamma)$ and $r > R = R(a,\delta)$.

Arguing similarly, we have

$$\begin{aligned} (\Delta - V)(h-w) &= \Delta h - Vh - \Delta w + Vw \\ &= -Vh - \Delta w + Vw \\ &\geq -\frac{C}{(1+r(x))^{2+\gamma}} + \frac{\delta(1-\delta)}{(1+r(x))^{2+\delta}} > 0 \end{aligned}$$

if $\delta \in (0,1) \cap (0,\gamma)$ and $r > R = R(a,\delta)$. This completes the proof. □

Proof of Theorem 1.1. Let $f : M(\infty) \to \mathbb{R}$ be a continuous function. There exist nonnegative continuous functions f_1 and f_2 such that $f = f_1 - f_2$. If there exist solutions u_1 and u_2 for the Schrödinger operator $\Delta - V$ on M such that for each $\mathbf{v} \in M(\infty)$,

$$\lim_{x \to \mathbf{v}} u_i(x) = f_i(\mathbf{v}), \quad i = 1, 2,$$

then $u = u_1 - u_2$ becomes the desired solution. Therefore, we have only to solve the problem in the case that f is a nonnegative continuous function on $M(\infty)$. For such an f, choose a constant $0 < \lambda \leq 1$ such that

$$\lambda \sup_M h \leq \frac{1}{(1+R)^\delta},$$

where h is a continuous function on $M \cup M(\infty)$ such that

$$\begin{cases} \Delta h = 0 & \text{on } M; \\ \lim_{x \to \mathbf{v}} h(x) = f(\mathbf{v}) & \text{for each } \mathbf{v} \in M(\infty), \end{cases}$$

and R and δ are given above.

Arguing similarly as Lemma 2.2, $\lambda h + w$ is a supersolution and $\lambda h - w$ is a subsolution for the Schrödinger operator $\Delta - V$, respectively in $M \setminus \bar{B}_{R'(o)}$, where $R' = R'(\delta, \lambda) > 0$.

For each $i \in \mathbb{N}$, define a function $u_i \in C(M)$ such that u_i is a solution for the Schrödinger operator $\Delta - V$ on $B_{2^i R}(o)$ and $u_i \equiv \lambda h$ on $M \setminus B_{2^i R}(o)$. Since $\Delta h = 0$ and $h \geq 0$ on M, $0 \leq u_i \leq \lambda h$ on $B_{2^i R}(o)$. Hence, $\lambda h - w \leq u_i \leq \lambda h$ on $\partial B_R(o) \cup \partial B_{2^i R}(o)$. Applying the maximum principle, we have $\lambda h - w \leq u_i \leq \lambda h$ on $B_{2^i R}(o) \setminus \overline{B}_R(o)$. Since $\{u_i\}$ is monotone decreasing, there is a solution u for the Schrödinger operator $\Delta - V$ on M such that $(1/\lambda) u_i$ converges to u on M. By definition of h, w and (2),

$$\lim_{x \to \mathbf{v}} u(x) = f(\mathbf{v}) \tag{5}$$

for each $\mathbf{v} \in M(\infty)$.

To prove the uniqueness, let u and v be solutions of the Schrödinger equation on M satisfying (5). We can choose sequences $\{\epsilon_n\}$ and $\{r_n\}$ such that $\epsilon_n \to 0$ and $r_n \to \infty$ as $n \to \infty$, $|u - f| < \epsilon_n$ and $|v - f| < \epsilon_n$ on $M \setminus B_{r_n}(o)$. Hence $|u - v| < 2\epsilon_n$ on $\partial B_{r_n}(o)$. Since $(\Delta - V)\epsilon_n \leq 0$, by the maximum principle, $|u - v| \leq 2\epsilon_n$ on $B_{r_n}(o)$. Consequently, $u \equiv v$ on M. □

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