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REMARKS ON $\mathcal{K}$-STARCOMPACT SPACES

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ABSTRACT. In this note, we construct an example of a Hausdorff $\mathcal{K}$-starcompact (hence,$1\frac{1}{2}$-star-compact) space X having a regular closed G_δ-subset which is not $1\frac{1}{2}$-starcompact (hence, not $\mathcal{K}$-starcompact).

1. Introduction

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For a subfamily $\mathcal{V}$ of $\mathcal{U}$, we define

$$St(\cup\mathcal{V},\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap (\cup\mathcal{V}) \neq \emptyset\}.$$

As usual, we write $St(x,\mathcal{U})$ for $St(\{x\},\mathcal{U})$.

Definition 1.1 ([4]). A space X is $1\frac{1}{2}$-*starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U}) = X$.

Definition 1.2 ([3, 4]). A space X is $\mathcal{K}$-*starcompact* if for every open cover $\mathcal{U}$ of X, there exists a compact subset K of X such that $St(K,\mathcal{U}) = X$.

In [1], a $1\frac{1}{2}$-starcompact space is called starcompact. From the above definitions, It is clear that every $\mathcal{K}$-starcompact space is $1\frac{1}{2}$-starcompact.

In [4], Matveev constructed a Tychonoff $1\frac{1}{2}$-starcompact space X having a regular closed subspace which is not $1\frac{1}{2}$-starcompact, and asked the following two questions:

Question 1 ([4, Question 20]). Is $1\frac{1}{2}$-starcompactnesss preserved by closed G_δ-sets (in particular, zero-set)?

Question 2 ([4, Question 21]). Is $1\frac{1}{2}$-starcompactnesss preserved by a subspace which is both regular closed and G_δ (or zero-set) in the whole space?

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In [6], Song showed that there exists a $1\frac{1}{2}$-starcompact Hausdorff space having a zero-set which is not pseudocompact (and hence, not $1\frac{1}{2}$-starcompact) and a Tychonoff example under certain set-theoretic assumption, which answered negatively Question 1. In [7], Song showed that a regular closed subset of a $\mathcal{K}$-starcompact space need not be $\mathcal{K}$-starcompact. Thus it is natural for us to consider the following questions:

Question 3. Is $\mathcal{K}$-starcompactnesss preserved by closed G_δ-sets (in particular, zero-set)?

Question 4. Is $\mathcal{K}$-starcompactnesss preserved by a subspace which is both regular closed and G_δ (or zero-set) in the whole space?

The purpose of this note is to construct an example stated in the abstract which give negative answers to the Question 2, Question 3 and Question 4 in the class of Hausdorff spaces.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. Other terms and symbols that we do not define here will be used as in [2].

2. An example on $\mathcal{K}$-starcompact spaces

In this note, we construct a Hausdorff $\mathcal{K}$-starcompact (hence, $1\frac{1}{2}$-starcompact) space X having a regular closed G_δ-subset which is not $1\frac{1}{2}$-starcompact (hence, not $\mathcal{K}$-starcompact). In order to show the example, we need the following Lemma from [4, Theorem 28]. Here, we include the proof of the Lemma for the sake of completeness.

Lemma 2.1. *If a regular space X contains a discrete closed subspace Y such that $|X| = |Y| = \tau \geq \omega$, then X is not $1\frac{1}{2}$-starcompact.*

Proof. Let $\mathcal{S}$ be the set of all finite subsets of X. Then, $|\mathcal{S}| = \tau$. First, we show that there exists a bijection $\varphi : \mathcal{S} \to Y$ such that $\varphi(K) \notin K$ for each $K \in \mathcal{S}$. Since $|Y| = \tau$, we can enumerate Y as $\bigcup\{Y_i : i \in \omega\}$ such that $|Y_i| = \tau$ for each $i \in \omega$ and $Y_i \cap Y_j = \emptyset$ if $i \neq j$. For each $i \in \omega$, let

$$\mathcal{S}_i = \{K \in \mathcal{S} : K \cap Y_i = \emptyset \text{ and } K \cap Y_j \neq \emptyset \text{ for all } j < i\}.$$

Then, $\mathcal{S} = \bigcup\{\mathcal{S}_i : i \in \omega\}$, $|\mathcal{S}_i| = \tau$ for each $i \in \omega$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ if $i \neq j$. For each $i \in \omega$, since $|\mathcal{S}_i| = |Y_i|$, then there exists a bijection $\varphi_i : \mathcal{S}_i \to Y_i$. It is clear that $\varphi_i(K) \notin K$ for each $K \in \mathcal{S}_i$ by the definition of $\mathcal{S}_i$. Let $\varphi(K) = \varphi_i(K)$ for each $K \in \mathcal{S}_i$. Then, we get the desired bijection.

For each $x \in Y$, there exists an open neighborhood V_x of x in X such that $V_x \cap Y = \{x\}$, since Y is discrete and closed in X. For each $x \in X \setminus Y$ there also exists an open neighborhood W_x of x in X such that $x \in W_x \subseteq \overline{W_x} \subseteq X \setminus Y$. For each $x \in Y$, set

$$U_x = V_x \setminus \overline{\bigcup\{V_z : z \in \varphi^{-1}(x)\}}.$$

For each $x \in X \setminus Y$, set $U_x = W_x$. Then, $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X. Let $\mathcal{U}_0 = \{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ be an arbitrary finite subset of $\mathcal{U}$. Let $K = \{x_1, x_2, \ldots, x_n\}$. Then, we have $\varphi(K) \notin St(\cup\mathcal{U}_0, \mathcal{U})$ by the construction of $\mathcal{U}$. This shows that X is not $1\frac{1}{2}$-starcompact. □

Example 2.2. There exists a Hausdorff $\mathcal{K}$-starcompact (hence, $1\frac{1}{2}$-starcompact) space X having a regular closed G_δ-subset which is not $1\frac{1}{2}$-starcompact (hence, not $\mathcal{K}$-starcompact).

Proof. Let D be a countable discrete space and $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of D such that $|\mathcal{R}| = \mathfrak{c}$. Let $S_1 = D \cup \mathcal{R}$ be the Isbell-Mrówka space as in [5].

Let

$$A = \{a_\alpha : \alpha < \mathfrak{c}\} \text{ and } B = \{b_n : n \in \omega\}$$
$$Y = \{\langle a_\alpha, b_n\rangle : \alpha < \mathfrak{c}, n \in \omega\},$$

and let

$$S_2 = Y \cup A \cup \{a\}, \text{ where } a \notin Y \cup A.$$

We topologize S_2 as follows: every point of Y is isolated; a basic neighborhood of a point $a_\alpha \in A$ for each $\alpha < \mathfrak{c}$ takes the from

$$U_{a_\alpha}(n) = \{a_\alpha\} \cup \{\langle a_\alpha, b_m\rangle : m > n\} \text{ for } n \in \omega$$

and a basic neighborhood of a takes the from

$$U_a(F) = \{a\} \cup \cup\{\langle a_\alpha, b_n\rangle : a_\alpha \in A \setminus F, n \in \omega\} \text{ for a finite subset } F \text{ of } A\ .$$

□

Lemma 2.3. *The following properties hold:*

(1) *S_1 is regular;*
(2) *S_2 is Hausdorff, but not regular;*
(3) *S_2 is $\mathcal{K}$-starcompact.*

Proof. (1) Since every point of S_1 has an open and closed neighborhood base, S_1 is regular.

(2) Clearly, S_2 is a Hausdorff space by the construction of the topology of S_2. However, S_2 is not regular, since the point a can not be separated from the closed subset A by disjoint open subsets of S_2.

(3) Now, we show that S_2 is $\mathcal{K}$-starcompact. For this end, let $\mathcal{U}$ be an open cover of S_2. Without loss of generality, we assume that $\mathcal{U}$ consists of basic open sets of S_2. Since $\mathcal{U}$ is an open cover of S_2, there exists a $U_a \in \mathcal{U}$ such that $a \in U_a$. By assumption, there exists a finite subset $F$of A such that

$$U_a = U_a(F) = \{a\} \cup \{\langle a_\alpha, b_n\rangle : a_\alpha \in A \setminus F, n \in \omega\}$$

by the definition of the topology of S_2, thus we have

$$U_a \subseteq St(a, \mathcal{U}).$$

For each $a_\alpha \in F$, let

$$B_{a_\alpha} = \{a_\alpha\} \cup \{\langle a_\alpha, b_n\rangle : n \in \omega\}.$$

Then, B_{a_α} is a compact subset of S_2 by the definition of the topology of S_2. On the other hand, for each $a_\alpha \in A \setminus F$, there exists a $U_{a_\alpha} \in \mathcal{U}$ such that $a_\alpha \in U_{a_\alpha}$. Thus there exists a $\alpha_n \in \omega$ such that

$$\langle a_\alpha, b_{\alpha_n}\rangle \in U_{a_\alpha}.$$

Let $C = \{\langle a_\alpha, b_{\alpha_n}\rangle : a_\alpha \in A \setminus F\}$. Then, $C \cup \{a\}$ is a compact subset of S_2 by the definition of the topology of S_2 and

$$A \setminus F \subseteq St(C \cup \{a\}, \mathcal{U}).$$

If we put

$$K = \{a\} \cup C \cup \{B_{a_\alpha} : a_\alpha \in F\}.$$

Then, K is a compact subset of S_2 such that

$$S_2 = St(K, \mathcal{U}),$$

which shows that S_2 is $\mathcal{K}$-starcompact. □

We assume that $S_1 \cap S_2 = \emptyset$. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $\varphi : A \to \mathcal{R}$ be a bijection by

$$\varphi(a_\alpha) = r_\alpha \text{ for each } \alpha < \mathfrak{c}.$$

Definition 2.4. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying a_α with r_α for each $\alpha < \mathfrak{c}$.

Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. Since $\mathcal{R}$ is a discrete closed subset of S_1, then S_1 is not $1\frac{1}{2}$-starcompact by Lemma 2.1. Let $Y = \pi(S_1)$. Then, Y is not $1\frac{1}{2}$-starcompact, hence it is not $\mathcal{K}$-starcompact, since every $\mathcal{K}$-starcompact space is $1\frac{1}{2}$-starcompact. But it is homeomorphic to S_2.

Lemma 2.5. *The following statements hold:*

(1) *Y is a regular closed G_δ-subset of X;*

(2) *X is $\mathcal{K}$-starcompact.*

Proof. (1) Clearly, Y is a regular closed subset of X. Let

$$U_n = \pi(S_1 \cup \{\langle a_\alpha, b_m\rangle : m > n, \alpha < \mathfrak{c}\}) \text{ for each } n \in \omega.$$

Then, U_n is open in X and $Y = \cap_{n\in\omega} U_n$. Then, Y is a regular closed G_δ-subset of X.

(2) Now, we show that X is $\mathcal{K}$-starcompact. For this end, let $\mathcal{U}$ be an open cover of X. Thus, it is sufficient to show that there exists a compact subset K of X such that $X = St(K, \mathcal{U})$. By the above proof, S_2 is $\mathcal{K}$-starcompact. Since $\pi(S_2)$ is homeomorphic to S_2, then $\pi(S_2)$ is $\mathcal{K}$-starcompact, hence there exists a compact subset F_1 of $\pi(S_2)$ such that

$$\pi(S_2) \subseteq St(F_1, \mathcal{U}).$$

On the other hand, for every infinite subset F of D, there exists a $r \in \mathcal{R}$ such that $F \cap r$ is infinite, since $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of D. Hence, $\{r\}$ is an accumulation point of F by the construction of the topology of S_1. Since $\pi(S_1)$ is homeomorphic to S_1, then every infinite subset of $\pi(D)$ has an accumulation point in $\pi(S_1)$. Hence, there exists a finite subset F_2 of $\pi(S_1)$ such that

$$\pi(D) \subseteq St(F_2, \mathcal{U}).$$

For if $\pi(D) \nsubseteq St(B, \mathcal{U})$ for any finite subset B of $\pi(D)$, then, by induction, we can define a sequence $\{x_n : n \in \omega\}$ in $\pi(D)$ such that $x_n \notin St(\{x_i : i < n\}, \mathcal{U})$ for each $n \in \omega$. By the property of $\pi(S_1)$ mentioned above, the sequence $\{x_n : n \in \omega\}$ has an accumulation point x_0 in $\pi(S_1)$. Pick $U \in \mathcal{U}$ such that $x_0 \in U$. Choose $n < m < \omega$ such that $x_n \in U$ and $x_m \in U$. Then, $x_m \in St(\{x_i : i < m\}, \mathcal{U})$, which contradicts the definition of the sequence $\{x_n : n \in \omega\}$.

Let $K = F_1 \cup F_2$. Then, K is a compact subset of X such that

$$X = St(K, \mathcal{U}).$$

This shows that X is $\mathcal{K}$-starcompact. Thus, we complete the proof. □

Remark. The author does not know if there exists an example in ZFC (here, it means without any set-theoretic assumption) showing that a regular closed G_δ-subset (or zero-set) of a $\mathcal{K}$-starcompact Tychonoff space is $1\frac{1}{2}$-starcompact.

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