

## CHAIN RECURRENCE AND ATTRACTORS IN GENERAL DYNAMICAL SYSTEMS

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ABSTRACT. We introduce here notions of chain recurrent sets, attractors and its basins for general dynamical systems and prove important properties including (i) the chain recurrent set  $CR(f)$  of  $f$  on a metric space  $(X, d)$  is the complement of the union of sets  $B(A) - A$  as  $A$  varies over the collection of attractors and (ii) genericity of general dynamical systems.

### 1. Introduction and preliminaries

The main purpose of this paper is to extend the following two Theorem A, Theorem B for ordinary dynamical systems to general dynamical systems.

**Theorem A** (Conley's theorem, [5]). *If  $X$  is a compact metric space and  $f : X \rightarrow X$  is continuous, then the chain recurrent set  $CR(f)$  is the complement of  $\cup[B(A) - A]$ , where the union is taken over all attractors  $A$  of  $f$ , and  $B(A)$  (the basin of  $A$ ) is the set of points whose orbits tend asymptotically to  $A$ .*

**Theorem B** (Takens's theorem, [12]). *A generic system  $\Phi$  has the following properties*

- (1)  $\Phi$  has no  $C^0$ -explosions;
- (2)  $\Phi$  is continuous point of maps  $WPer, W\Omega, CR$ ;
- (3)  $\Omega(\Phi) = WPer(\Phi) = W\Omega(\Phi) = CR(\Phi)$ .

We prove here

**Theorem A'** (Conley's theorem for general dynamical systems). *The chain recurrent set  $CR(f)$  of a general dynamical system  $f$  on a metric space  $(X, d)$  is the complement of the union of sets  $B(A) - A$  as  $A$  varies over the collection of attractors of  $f$ , i.e.,  $X - CR(f) = \cup_A(B(A) - A)$ .*

**Theorem B'** (Takens's theorem for general dynamical systems). *A generic general dynamical system  $\Phi$  has the following properties*

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- (1)  $\Phi$  has no  $C^0$ -explosions;
- (2)  $\Phi$  is continuous point of maps  $WPer, W\Omega, CR$ ;
- (3)  $\Omega(\Phi) = WPer(\Phi) = W\Omega(\Phi) = CR(\Phi)$ .

Chain recurrent sets and attractors are key notions in the study of the qualitative theory of dynamical systems. These widely studied by many researchers in different contexts. We extend here widely their properties on general dynamical systems to the well known result due to Conely [5]. Conley's result extended for noncompact spaces by Hurely [7]. For many resulted works are see [1, 3, . . . , 12].

Generalized theory of dynamical systems [i.e.,  $\Phi : X \times \mathbb{R} \rightarrow X$  continuous,  $\Phi(x, 0) = x$  and  $\Phi(x, t + s) = \Phi(\Phi(x, t), s)$ ] was introduced by Sibirsky. (see chapter VI [14]).

We are many similarities when we study ordinary dynamical systems and general dynamical systems. But, there are sharp difference also while we study these properties, for instance, invariance, minimality so on. To clarify this fact it suffices to note that  $q \in \Phi(p, \mathbb{R})$  does not imply the inclusion  $\Phi(q, \mathbb{R}) \subset \Phi(p, \mathbb{R})$  [14].

We now introduce notions and definitions necessary for our works.

Let  $(X, d)$  be a metric space and  $F(X)$  be the set of all nonempty compact subsets of  $X$ . For  $(A, B) \in F(X) \times F(X)$ , define

$$h(A, B) = \sup\{d(x, B) : x \in A\}$$

and let

$$D(A, B) = \max\{h(A, B), h(B, A)\}.$$

Then  $D$  is a metric called as the *Hausdorff metric* in  $F(X)$ .

**Definition 1.1.** Let  $S : X \rightarrow F(X)$  be a function. Then  $S$  is called

- (1) *upper semicontinuous* at  $x$  if for any neighborhood  $U$  of  $S(x)$ , there exists a neighborhood  $V$  of  $x$  such that  $y \in V$  implies  $S(y) \subset U$ .
- (2) *lower semicontinuous* at  $x$  if for any open set  $U$  with  $S(x) \cap U \neq \emptyset$ , there exists a neighborhood  $V$  of  $x$  such that  $y \in V$  implies  $S(y) \cap U \neq \emptyset$ .
- (3) *continuous* at  $x$  if it is both upper semicontinuous and lower semicontinuous at  $x$ .

We now introduce the notion of general dynamical system.

**Definition 1.2** ([14]). Let  $X$  be a given metric space and suppose that to each point  $p$  of this space and moment of time  $t \in \mathbb{R}$  there is a set into correspondence a nonempty closed compact subset  $f(p, t)$  of  $X$  whereby, if we let  $f(A, K) \equiv \bigcup_{p \in A, t \in K} f(p, t)$ , then

- (1)  $f(p, 0) = \{p\}$  for every point  $p \in X$
- (2) if  $t_1 t_2 > 0$ , then  $f(f(p, t_1), t_2) = f(p, t_1 + t_2)$
- (3) if  $q \in f(p, t)$ , then  $p \in f(q, -t)$
- (4)  $\lim_{p \rightarrow p_0, t \rightarrow t_0} D(f(p, t), f(p_0, t_0)) = 0$ , where  $D$  is a Hausdorff metric.  
(This means that  $f$  is continuous.)

If this is the situation, then we say that a *general (or dispersive) dynamical system* has been defined.

Throughout this paper  $(X, f)$  denotes a general dynamical system on a compact metric space  $(X, d)$ . The notion of integral continuity condition was introduced. This has important role in the theory of general dynamical systems.

**Theorem 1.3** ([13], Integral continuity theorem for general dynamical system). *For any point  $x \in X$ , and any compact subset  $K$  of  $\mathbb{R}$  and any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $D(f(x, t), f(y, t)) < \varepsilon$  for all  $y \in X$  and  $t \in K$  which satisfy the inequality  $d(x, y) < \delta$ .*

*Proof.* From the continuity of  $f$  at  $(x, t)$ , there exists a  $\delta_t > 0$  such that if  $d(x, y) < \delta_t$  and  $|t - s| < \delta_t$ , then  $D(f(x, t), f(y, s)) < \varepsilon/2$ .

Let  $V_t \equiv (t - \delta_t, t + \delta_t)$ . Since the collection  $\{V_t : t \in K\}$  is an open cover of a compact set  $K$ , we can choose finitely many elements  $t_1, t_2, \dots, t_n \in K$  such that  $K \subset \cup_{i=1}^n V_{t_i}$ .

Clearly  $\delta \equiv \min\{\delta_{t_i} : 1 \leq i \leq n\} > 0$ . Now observe that for any  $t \in K$  we have a  $t_i$  such that  $t \in V_{t_i}$ . Thus, if  $d(x, y) < \delta$  then

$$\begin{aligned} D(f(x, t), f(y, t)) &\leq D(f(x, t), f(x, t_i)) + D(f(x, t_i), f(y, t)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

## 2. Attractors and chain recurrent sets

Let us denote by  $P(X)$  the family of all positive real valued continuous functions defined on  $X$ . For any  $\varepsilon \in P(X)$ , we define positive real valued function  $m(\varepsilon, f)$  (respectively  $M(\varepsilon, f)$ ) on  $X \times \mathbb{R}$  by  $m(\varepsilon, f) = \min\{\varepsilon(f(x, t)) : (x, t) \in X \times \mathbb{R}\}$ , (respectively  $M(\varepsilon, f) = \max\{\varepsilon(f(x, t)) : (x, t) \in X \times \mathbb{R}\}$ ). We now prove that these are continuous functions.

**Lemma 2.1.** *The above defined maps  $m(\varepsilon, f)$  and  $M(\varepsilon, f)$  are continuous.*

*Proof.* We only prove continuity of  $m(\varepsilon, f)$ . The continuity of  $M(\varepsilon, f)$  can be obtained similarly. Let  $(x, t) \in X \times \mathbb{R}$ . Then  $\varepsilon(y) = m(\varepsilon, f)(x, t)$  for some  $y \in f(x, t)$ . Let  $\gamma > 0$ . Since  $\varepsilon$  is continuous at  $y$ , there is a neighborhood  $U$  of  $y$  such that if  $z \in U$ , then  $|\varepsilon(y) - \varepsilon(z)| < \gamma$ . Since  $U \cap f(x, t) \neq \emptyset$  and  $f$  is lower semicontinuous at  $(x, t)$ , there are neighborhoods  $V$  of  $x$  and  $I$  of  $t$  respectively, such that if  $p \in V$  and  $s \in I$ , then  $U \cap f(p, s) \neq \emptyset$ . Choose a point  $q \in U \cap f(p, s)$ . Since  $|\varepsilon(y) - \varepsilon(q)| < \gamma$ ,  $m(\varepsilon, f)(p, s) \leq \varepsilon(q) < \varepsilon(y) + \gamma = m(\varepsilon, f)(x, t) + \gamma$  holds. From  $m(\varepsilon, f)(x, t) - \gamma < m(\varepsilon, f)(x, y) \leq \varepsilon(a)$  for  $a \in f(x, t)$  and the continuity of  $\varepsilon$  at  $a$ , there exists a neighborhood  $V_a$  of  $a$  such that if  $b \in V_a$ , then  $m(\varepsilon, f)(x, t) - \gamma < \varepsilon(b)$ .

Set  $V \equiv \cup\{V_a : a \in f(x, t)\}$ . Then,  $V$  is a neighborhood of  $f(x, t)$ . By the upper semicontinuity of  $f$  at  $(x, t)$ ,  $p \in W$ ,  $s \in J$  implies that  $f(p, s) \subset V$  for some neighborhoods  $W$  of  $x$  and  $J$  of  $t$  respectively. Thus, for any  $q \in$

$f(p, s) \subset V$ ,  $q \in V_a$  for some  $a \in f(x, t)$ . Since  $m(\varepsilon, f)(x, t) - \gamma < \varepsilon(q)$ , we get  $m(\varepsilon, f)(x, t) - \gamma < m(\varepsilon, f)(p, s)$ . Therefore,  $|m(\varepsilon, f)(x, t) - m(\varepsilon, f)(p, s)| < \gamma$  for all  $p \in V \cap W$  and for all  $s \in I \cap J$ , where  $V \cap W$  and  $I \cap J$  are neighborhoods of  $x$  and  $t$ , respectively.

This proves that  $m(\varepsilon, f)$  is continuous at  $(x, t)$ .  $\square$

We introduce the following Dower's Lemma which was well known in analysis.

**Lemma 2.2** (Dower's Lemma). *Suppose that  $\alpha$  and  $\omega$  are, respectively, upper and lower semicontinuous functions defined on a paracompact space  $X$  with  $\alpha(x) < \omega(x)$  for every  $x \in X$ . Then there is a continuous real valued function  $\tau$  on  $X$  such that  $\alpha(x) < \tau(x) < \omega(x)$  for all  $x \in X$ .*

**Theorem 2.3.** *Let  $\varepsilon \in P(X)$  and  $T > 0$ . Then there exists a  $\delta \in P(X)$  such that if  $d(x, y) < \delta(x)$ , then  $D(f(x, t), f(y, t)) < m(\varepsilon, f)(x, t)$  for all  $0 \leq t \leq T$ .*

*Proof.* For any  $x$  in  $X$ , set  $M_x \equiv \min\{m(\varepsilon, f)(x, t) : 0 \leq t \leq T\}$  and for any  $y \in X$  define  $\rho(x, y) \equiv \max\{D(f(x, t), f(y, t)) : 0 \leq t \leq T\}$ . Also, define  $\beta : X \rightarrow (0, \infty)$  by

$$\beta(x) = \sup\{\eta > 0 : \text{there exists } 0 < \alpha < 1 \\ \text{such that if } d(x, y) < \eta \text{ then } \rho(x, y) < \alpha M_x\}.$$

Since  $0 < \alpha < 1$ ,  $\alpha M_x > 0$ . By integral continuity theorem, there exists an  $\eta > 0$  such that if  $d(x, y) < \eta$ , then  $\rho(x, y) < \alpha M_x$ . Therefore,  $\beta(x) \geq \eta > 0$ . Next, we shall show that  $\beta$  is lower semicontinuous. Let  $x \in X$  and  $h < \beta(x)$ . There is an  $\eta > 0$  with  $h < \eta$  such that  $d(x, y) < \eta$  implies  $\rho(x, y) < \alpha M_x$  for some  $0 < \alpha < 1$ . We claim that for any  $\xi > 0$ , there is a neighborhood  $V$  of  $x$  such that if  $y \in V$  and  $t \in [0, T]$ , then  $|m(\varepsilon, f)(x, t) - m(\varepsilon, f)(y, t)| < \xi$ . Since  $m(\varepsilon, f)$  is continuous at  $(x, s)$  for any  $s \in [0, T]$ , there are neighborhoods  $V_s$  of  $x$  and  $I_s$  of  $s$  such that  $|m(\varepsilon, f)(x, s) - m(\varepsilon, f)(y, t)| < \xi/2$  for all  $y \in V_s$  and  $t \in I_s$ . It is clear that  $\{I_s : 0 \leq s \leq T\}$  is an open cover of  $[0, T]$ . In view of the compactness of  $[0, T]$ , there are finitely many  $s_1, s_2, \dots, s_n \in [0, T]$  satisfying  $[0, T] \subset \cup_{i=1}^n I_{s_i}$ . Then  $V \equiv \cap_{i=1}^n V_{s_i}$  is a neighborhood of  $x$ . Let  $y \in V$  and  $t \in [0, T]$ . Then  $t \in I_{s_i}$  for some  $i$ . Since  $x, y \in V_{s_i}$ , we obtain that

$$|m(\varepsilon, f)(x, t) - m(\varepsilon, f)(y, t)| \leq |m(\varepsilon, f)(x, t) - m(\varepsilon, f)(x, s_i)| \\ + |m(\varepsilon, f)(x, s_i) - m(\varepsilon, f)(y, t)| < \xi.$$

Choose  $\Theta_1$  and  $\Theta_2$  such that  $0 < \alpha < \Theta_1 < \Theta_2 < 1$ . Since  $(\Theta_2 - \Theta_1)/\Theta_2 M_x > 0$ , by integral continuity theorem there exists a neighborhood  $V$  of  $x$  such that if  $y \in V$  and  $t \in [0, T]$ , then

$$m(\varepsilon, f)(y, t) > m(\varepsilon, f)(x, t) - (\Theta_2 - \Theta_1)/\Theta_2 M_x \\ \geq M_x - (\Theta_2 - \Theta_1)/\Theta_2 M_x \\ = \Theta_1/\Theta_2 M_x.$$

Thus, we get  $\Theta_1 M_x < \Theta_2 M_y$ . Since  $(\Theta_1 - \alpha)M_x > 0$ , by the integral continuity theorem, there is a  $\xi > 0$  such that  $\rho(x, y) < (\Theta_1 - \alpha)M_x$  whenever  $d(x, y) < \xi$ .

Choose  $\zeta$  with  $h < \zeta < \eta$ . There exists  $0 < b < \min\{\eta - \zeta, \xi\}$  such that  $B(x, b) \subset V$ . Take  $U \equiv B(x, b)$ . Then  $U$  is a neighborhood of  $x$ . Let  $y \in U$ . If  $d(y, z) < \zeta$ , then we have  $d(x, z) \leq d(x, y) + d(y, z) < b + \zeta < \eta - \zeta + \zeta = \eta$ .

This means that  $\rho(x, z) < \alpha M_x$ . Since  $d(x, y) < b < \xi$ , we have  $\rho(x, y) < (\Theta_1 - \alpha)M_x$ . Thus  $\rho(y, z) \leq \rho(x, y) + \rho(x, z) < (\Theta_1 - \alpha)M_x + \alpha M_x = \Theta_1 M_x$ . Also, we obtain that  $\Theta_1 M_x < \Theta_2 M_y$  using  $y \in U \subset V$ . Thus  $\rho(y, \zeta) < \Theta_2 M_y$ . Hence  $\beta(y) \geq \zeta > h$ . Therefore,  $\beta$  is lower semicontinuous. By Lemma 2.2, there is a continuous map  $\delta : X \rightarrow (0, \infty)$  satisfying  $0 < \delta(x) < \beta(x)$  for all  $x \in X$ . Also, there is an  $\eta > 0$  with  $\delta(x) < \eta$  such that  $d(x, y) < \eta$  implies  $\rho(x, y) < \alpha M_x$  for some  $0 < \alpha < 1$ . Thus, we obtain that  $D(f(x, t), f(y, t)) \leq \rho(x, y) < \alpha M_x \leq \alpha m(\varepsilon, f)(x, t) < m(\varepsilon, f)(x, t)$  whenever  $d(x, y) < \delta(x) < \eta$  for all  $t \in [0, T]$ . This completes the proof of the lemma.  $\square$

We now define notions of chain recurrence and attractors in general dynamical systems. Observe that  $d(x, f(x, t))$  varies for points on the set  $f(x, t)$ . Moreover, in our definitions for  $\varepsilon$ -chain and  $\varepsilon$ -weak chain, we use functions  $m(\varepsilon, f)$  and  $M(\varepsilon, f)$ . These variable components generate striking differences between our notions and the existing notions on ordinary dynamical systems.

**Definition 2.4.** Let  $x, y \in X, \varepsilon \in P(X), T > 0$ . An  $(\varepsilon, T)$ -chain for  $f$  from  $x$  to  $y$  means a collection  $\{x = x_1, x_2, \dots, x_n, x_{n+1} = y, t_1, t_2, \dots, t_n\}$  of points  $x_i \in X$  and numbers  $t_i \in \mathbb{R}$  such that  $d(x_{i+1}, f(x_i, t_i)) < m(\varepsilon, f)(x_i, t_i)$  and  $t_i \geq T$  for all  $i = 1, 2, \dots, n$ . Without loss of generality, we suppose that  $t_i < 2T$  for all  $i$ . A point  $x \in X$  is *chain recurrent* if and only if there exists a  $(\varepsilon, T)$ -chain from  $x$  to  $x$  for all  $\varepsilon \in P(X)$  and  $T > 0$ . We denote the set of all chain recurrent points in a general dynamical system  $f$  by  $CR(f)$ .

**Lemma 2.5.** Let  $\varepsilon \in P(X)$ . Then there exists a  $\delta \in P(X)$  such that if  $d(x, y) < \delta(x)$  then  $\varepsilon(y) > \varepsilon(x)/2$ .

*Proof.* Define  $\beta : X \rightarrow (0, \infty)$  by

$$\beta(x) = \sup\{\eta > 0 : d(x, y) < \eta \text{ implies } \alpha\varepsilon(x) < \varepsilon(y) \text{ for some } 1/2 < \alpha < 1\}.$$

Let  $1/2 < \alpha < 1$ . Then  $\alpha\varepsilon(x) < \varepsilon(x)$ . By the continuity of  $\varepsilon$ , there is an  $\eta > 0$  such that  $\varepsilon(y) > \alpha\varepsilon(x)$  whenever  $d(x, y) < \eta$ . Thus  $\beta(x) \geq \eta > 0$ . We shall show that  $\beta$  is lower semicontinuous.

Let  $h < \beta(x)$ . There exists an  $\eta > 0$  with  $h < \eta$  such that  $d(x, y) < \eta$  implies  $\varepsilon(y) > \alpha\varepsilon(x)$  for some  $1/2 < \alpha < 1$ . Choose a constant  $r$  such that  $1/2 < r < \alpha < 1$ . Then  $(\alpha - r)/r\varepsilon(x) > 0$ . There is a neighborhood  $V$  of  $x$  such that if  $y \in V$ , then  $\varepsilon(y) < \varepsilon(x) + (\alpha - r)/r\varepsilon(x) = \alpha/r\varepsilon(x)$ . Thus  $r\varepsilon(y) < \alpha\varepsilon(x)$ . Choose a constant  $\zeta$  with  $h < \zeta < \eta$ , and  $B(x, b) \subset V$  for some  $0 < b < \eta - \zeta$ . Let  $y \in B(x, b)$  and  $z \in B(y, \zeta)$ . Then we have

$$d(x, z) \leq d(x, y) + d(y, z) < b + \zeta < \eta - \zeta + \zeta = \eta.$$

Thus  $z \in B(x, \eta)$ , implying  $B(y, \zeta) \subset B(x, \eta)$ . If  $d(y, z) < \zeta$ , then  $z \in B(y, \zeta) \subset B(x, \eta)$  and so  $d(x, z) < \eta$ . This implies  $\varepsilon(z) > \alpha\varepsilon(x)$ . Also, we have  $r\varepsilon(y) < \alpha\varepsilon(x)$  since  $y \in B(x, b) \subset V$ . Therefore  $\varepsilon(z) > \alpha\varepsilon(x) > r\varepsilon(y)$ . Hence  $\beta(y) \geq \zeta > h$ . It follows that  $\beta$  is lower semicontinuous. By Lemma 2.2, there is a continuous function  $\delta : X \rightarrow (0, \infty)$  such that  $0 < \delta(x) < \beta(x)$ . Since  $\delta(x) < \beta(x)$ , there is an  $\eta > 0$  with  $\delta(x) < \eta$  such that if  $d(x, y) < \eta$ , then  $\varepsilon(y) > \alpha\varepsilon(x)$  for some  $1/2 < \alpha < 1$ . When  $d(x, y) < \delta(x)$ , we have  $d(x, y) < \eta$ . Consequently,  $\varepsilon(y) > \alpha\varepsilon(x) > 1/2\varepsilon(x)$ , which is what was required to be proved.  $\square$

**Definition 2.6.** A set  $M$  is said to be *invariant* if  $f(M, t) \subset M$  for any  $t \in \mathbb{R}^+$ . A map  $\Phi : \mathbb{R} \rightarrow X$  is called a *trajectory* of  $f$  through  $x \in X$  if  $\Phi(0) = x$  and  $\Phi(t) \in f(\Phi(s), t - s)$  whenever  $s < t$ . For  $x \in X$ , the set of all trajectories of  $f$  through  $x$  is denoted by  $\Psi(x)$ .

**Theorem 2.7** ([13]). *Let  $s < t$ . If  $y \in f(x, s)$  and  $z \in f(x, t)$ , then there exists  $\Phi \in \Psi(x)$  such that  $\Phi(s) = y$  and  $\Phi(t) = z$ .*

**Theorem 2.8.** *The set  $CR(f)$  is invariant.*

*Proof.* Let  $x \in CR(f)$ ,  $y \in f(x, s)$ ,  $s > 0$ . For any  $\varepsilon \in P(X)$  and  $t > 0$ , there exists  $\zeta \in P(X)$  with  $\zeta < \varepsilon$  such that if  $d(x, y) < \zeta(x)$ , then

$$D(f(x, s), f(y, s)) < m(\varepsilon, f)(x, s).$$

Since  $x$  belongs to  $CR(f)$ , there exists a  $(\zeta, s + t)$ -chain  $\{x_0, x_1, \dots, x_{n-1}, x_n; t_0, t_1, \dots, t_{n-1}\}$  from  $x$  to  $x$ .

Since

$$f(y, t_0 - s) \subset f(f(x, s), t_0 - s) = f(x, t_0),$$

we have

$$\begin{aligned} d(x_1, f(y, t_0 - s)) &\leq d(x_1, f(x, t_0)) < m(\zeta, f)(x, t_0) \\ &\leq m(\zeta, f)(y, t_0 - s) < m(\varepsilon, f)(y, t_0 - s). \end{aligned}$$

Also,  $d(x, f(x_{n-1}, t_{n-1})) < m(\zeta, f)(x_{n-1}, t_{n-1})$ , implies

$$d(x, z) = d(x, f(x_{n-1}, t_{n-1})) < m(\zeta, f)(x_{n-1}, t_{n-1}) \leq \zeta(z)$$

for some  $z \in f(x_{n-1}, t_{n-1})$ . Hence, we obtain

$$D(f(x, s), f(z, s)) < m(\varepsilon, f)(z, s).$$

Now, it follows that  $\varepsilon(p) = m(\varepsilon, f)(x_{n-1}, t_{n-1} + s)$  for some  $p \in f(x_{n-1}, t_{n-1} + s)$ . By Theorem 2.7, there exists  $\Phi \in \Psi(x_{n-1})$  such that  $\Phi(t_{n-1}) = z$  and  $\Phi(t_{n-1} + s) = p$ . From the facts

$$f(z, s) \subset f(f(x_{n-1}, t_{n-1}), s) = f(x_{n-1}, t_{n-1} + s)$$

and

$$y \in f(x, s) \subset B(f(z, s), m(\varepsilon, f)(z, s)),$$

respectively, we have

$$d(y, f(x_{n-1}, t_{n-1} + s)) \leq d(y, f(z, s)) < m(\varepsilon, f)(z, s).$$

As  $p = \Phi(t_{n-1} + s) \in f(\Phi(t_{n-1}), s) = f(z, s)$ , it follows that

$$\begin{aligned} d(y, f(x_{n-1}, t_{n-1} + s)) &\leq d(y, f(z, s)) < m(\varepsilon, f)(z, s) \leq \varepsilon(p) \\ &= m(\varepsilon, f)(x_{n-1}, t_{n-1} + s). \end{aligned}$$

Therefore,

$$\{y, x_1, \dots, x_{n-1}, y; t_0, t_1, \dots, t_{n-1}\}$$

is an  $(\varepsilon, t)$ -chain from  $y$  to  $y$ . Hence  $y$  belongs to  $CR(f)$ . This completes the proof of the theorem.  $\square$

**Definition 2.9.** A nonempty open subset  $U$  of  $X$  is said to *preattractor* for  $f$  if there is  $T \geq 0$  such that  $Clf(U \times [T, \infty)) \subset U$ . A nonempty closed subset  $A$  of  $X$  is said to *attractor* defined by  $A = \bigcap_{t \geq T} Clf(U \times [t, \infty))$  for a preattractor  $U$ . The *basin* of  $A$ , denoted by,  $B(A)$ , consists of all points  $x$  with the property that some point on the forward orbit of  $x$  lies in a preattractor  $U$  that determines  $A$ .

**Lemma 2.10.** *Let  $A$  be an attractor. If  $p \in CR(f) \cap B(A)$  then  $p \in A$ .*

*Proof.* Let  $U$  be a preattractor which determines  $A$ . There is a  $T \geq 0$  such that

$$Clf(U \times [T, \infty)) \subset U.$$

We claim that there is an  $\varepsilon \in P(X)$  such that  $\varepsilon \leq 1$  and  $B(f(x, t), m(\varepsilon, f)(x, t)) \subset U$  for all  $x \in U$  and  $t \geq T$ . Define  $\delta \in P(X)$  by

$$\delta(x) = 1/2(d(x, Clf(U \times [T, \infty))) + d(x, X - U)).$$

Since  $x \in Clf(U \times [T, \infty)) \subset U$ , we have  $\delta(x) > 0$ . Let  $x \in U$  and  $t \geq T$ . Note that  $f(x, t) \subset f(U \times [T, \infty)) \subset Clf(U \times [T, \infty))$ , gives

$$d(p, Clf(U \times [T, \infty))) = 0$$

for any  $p \in f(x, t)$ . Hence we obtain  $\delta(p) = 1/2d(p, X - U)$ . Let

$$y \in B(f(x, t), m(\delta, f)(x, t)).$$

From

$$\begin{aligned} d(y, f(x, t)) &< m(\delta, f)(x, t) = \min\{\delta(p) : p \in f(x, t)\} \\ &= 1/2 \min\{d(p, X - U) : p \in f(x, t)\}, \end{aligned}$$

we have  $2d(y, f(x, t)) < \min\{d(p, X - U) : p \in f(x, t)\}$ . Note that there exists a  $z \in f(x, t)$  such that  $d(y, z) = d(y, f(x, t))$ . Thus  $\min\{d(p, X - U) : p \in f(x, t)\} \leq d(z, X - U) \leq d(y, z) + d(y, X - U) = d(y, f(x, t)) + d(y, X - U)$ , we obtain  $2d(y, f(x, t)) < d(y, f(x, t)) + d(y, X - U)$ , gives  $d(y, X - U) > d(y, f(x, t)) \geq 0$ , and gives  $y \in U$ .

Consequently,  $B(f(x, t), m(\delta, f)(x, t)) \subset U$  and  $\varepsilon \equiv \min\{\delta, 1\}$  is the desired function.

Let  $t \geq T$  and  $n$  be a positive integer. Since  $p \in CR(f)$ , there is an  $(\varepsilon/n, t)$ -chain  $\{x_1, x_2, \dots, x_k, x_{k+1}; t_1, t_2, \dots, t_k\}$  from  $p$  to  $p$ .

Note that

$$d(x, f(x_1, t_1)) < (1/n)m(\varepsilon, f)(x_1, t_1) \leq m(\varepsilon, f)(x_1, t_1),$$

gives

$$x_2 \in B(f(x_1, t_1), m(\varepsilon, f)(x_1, t_1)) \subset U.$$

$$d(x_3, f(x_2, t_2))(1/n)m(\varepsilon, f)(x_2, t_2) \leq m(\varepsilon, f)(x_2, t_2)$$

implies  $x_3 \in B(f(x_2, t_2), m(\varepsilon, f)(x_2, t_2)) \subset U$ . Continuing in this process, we obtain  $x_k \in U$  satisfying

$$d(p, f(x_k, t_k)) < (1/n)m(\varepsilon, f)(x_k, t_k) \leq 1/n$$

which implies

$$d(p, f(U \times [t, \infty))) \leq d(p, f(x_k, t_k)) < 1/n.$$

Taking  $n \rightarrow \infty$ , we obtain  $d(p, f(U \times [t, \infty))) = 0$ . This proves

$$p \in \bigcap_{t \geq T} Clf(U \times [t, \infty)) = A.$$

Now, let  $Y = \{t \in \mathbb{R}^+ : f(p, t) \subset U\}$ . Observe that  $p \in B(A)$  makes  $Y$  a nonempty set. If  $f(p, t) \subset U$ , then,  $f(p, t) \subset A$ . Thus  $Y$  is the same set  $Z = \{t \in \mathbb{R}^+ : f(p, t) \subset A\}$ . By the continuity of  $f$ ,  $Y$  is open in  $\mathbb{R}^+$ , while  $Z$  is closed. Finally connectedness of  $\mathbb{R}^+$  gives  $Y = Z = \mathbb{R}^+$ . In particular,  $0 \in Z$  and hence it follows that  $p \in A$ .  $\square$

**Definition 2.11.** Let  $x, y \in X$ ,  $\varepsilon \in P(X)$ ,  $T > 0$ . A *weakly*  $(\varepsilon, T)$ -chain for  $f$  from  $x$  to  $y$  means a collection  $\{x = x_1, x_2, \dots, x_n, x_{n+1} = y; t_1, t_2, \dots, t_n\}$  of points  $x_i \in X$  and numbers  $t_i \in \mathbb{R}$  such that  $d(x_{i+1}, f(x_i, t_i)) < M(\varepsilon, f)(x_i, t_i)$  for  $t_i \geq T$  and for all  $i = 1, 2, \dots, n$ .

We now prove the characterization of the chain recurrent set in terms of attractors and their basins.

**Theorem 2.12** (Conley's theorem for general dynamical systems). *The chain recurrent set  $CR(f)$  of  $f$  on a metric space  $(X, d)$  is the complement of the union of sets  $B(A) - A$  as  $A$  varies over the collection of attractors of  $f$ , i.e.,  $X - CR(f) = \bigcup_A (B(A) - A)$ .*

*Proof.* Let  $p \notin CR(f)$ . Then, there exist  $\varepsilon \in P(X)$  and  $T > 0$  such that there is no  $(\varepsilon, T)$ -chain from  $p$  to  $p$ , also we can find a  $\delta \in P(X)$  such that  $M(\delta, f)(x, t) \leq m(\varepsilon, f)(x, t)$  for all  $(x, t) \in X \times \mathbb{R}$ . Clearly, there is no weakly  $(\delta, T)$ -chain from  $p$  to  $p$ . We consider a set  $U$  consists of  $x \in X$  with the property that there is a weakly  $(\delta, T)$ -chain from  $p$  to  $x$ . Then  $U$  is an open set. In fact, there is a weakly  $(\delta, T)$ -chain  $\{x_1, x_2, \dots, x_n, x_{n+1}; t_1, t_2, \dots, t_n\}$  from  $p$  to  $x$  and  $B(x, M(\delta, f)(x_k, t_k) - d(x, f(x_k, t_k))) \subset U$ . Let us show that  $Clf(U \times [T, \infty)) \subset U$ . There exists a  $\zeta \in P(X)$  with  $\zeta \leq \delta/2$  such that  $\delta(y) > 1/2\delta(x)$  whenever  $d(x, y) < \zeta(x)$  by Lemma 2.5. Let  $y \in Clf(U \times [T, \infty))$ . Notice that  $B(y, \zeta(y)) \cap f(U \times [T, \infty)) \neq \emptyset$ . Choose a point  $z \in B(y, \zeta(y)) \cap f(x, t)$  for some  $x \in U$  and  $t \in [T, \infty)$ . Then we have a weakly  $(\delta, T)$ -chain



$\{x_1, x_2, \dots, x_n, x_{n+1}; t_1, t_2, \dots, t_n\}$  from  $p$  to  $x$ . From  $d(y, z) < \zeta(y)$ , we get  $\delta(z) > 1/2\delta(y)$ . Also for

$$d(y, f(x, t)) \leq d(y, z) < \zeta(y) \leq 1/2\delta(y) < \delta(z) \leq M(\delta, f)(x, t)$$

gives that  $\{x_1, x_2, \dots, x_n, x_{n+1}, y; t_1, t_2, \dots, t_n, t\}$  is a weakly  $(\delta, T)$ -chain from  $p$  to  $y$ . This means  $y \in U$  and so  $U$  is a preattractor. Suppose that  $A$  is an attractor determined by  $U$ . Then  $p \in B(A) - A$  via  $f(p, T) \subset U$ . Conversely, let  $p \in B(A) - A$ . If  $p \in CR(f)$ , then, by Lemma 2.10,  $p \in A$ , which is a contradiction. Hence,  $p \in X - CR(f)$ . Consequently, we obtain  $X - CR(f) = \cup_A(B(A) - A)$ , which is what was required to be proved.  $\square$

### 3. Genericity

In this section we establish genericity of general dynamical systems. Let us begin with the following definition.

Let  $M_0$  be the set of all general dynamical systems on a compact metric space  $X$ . Define a metric  $\rho : M_0 \times M_0 \rightarrow \mathbb{R}$  by setting

$$\rho(f, g) = \sup\{\min\{\max\{D(f(x, t), g(x, t)) : x \in X, -T \leq t \leq T\}, \frac{1}{T}\} : T > 0\}$$

for  $f, g \in M_0$ .

**Theorem 3.1.**  $\rho$  is a metric on  $M_0$ .

*Proof.* Suppose that  $f \neq g$ . Then,  $f(y, s) \neq g(y, s)$  for some  $y \in X$  and  $s \in \mathbb{R}$ . We obtain that  $D(f(y, s), g(y, s)) > 0$ . Put

$$M_T(f, g) \equiv \max\{D(f(x, t), g(x, t)) : x \in X, |t| \leq T\}.$$

Hence,

$$\rho(f, g) \geq \min\{M_{|s|}(f, g), 1/|s|\} \geq \min\{D(f(y, s), g(y, s)), 1/|s|\} > 0.$$

Let  $\varepsilon > 0$ . Since  $\rho(f, g) - \varepsilon < \rho(f, g)$ , we have  $\min\{M_T(f, g), 1/T\} > \rho(f, g) - \varepsilon$  for some  $T > 0$ . Also,  $D(f(y, s), g(y, s)) = M_T(f, g)$  for some  $y \in X$  and  $s \in [-T, T]$ . Let us check the triangle inequality  $\rho(f, h) + \rho(h, g) \geq \rho(f, g)$ . If  $M_T(f, h) \geq 1/T$ , then we have  $\rho(f, h) + \rho(h, g) \geq \rho(f, h) \geq \min\{M_T(f, h), 1/T\} = 1/T \geq \min\{M_T(f, g), 1/T\} > \rho(f, g) - \varepsilon$ .

If  $M_T(f, h) < 1/T$ , then we have

$$\begin{aligned} \rho(f, h) + \rho(h, g) &\geq \rho(h, g) \geq \min\{M_T(h, g), 1/T\} = 1/T \\ &\geq \min\{M_T(f, g), 1/T\} > \rho(f, g) - \varepsilon. \end{aligned}$$

On the other hand if  $M_T(f, h) < 1/T$  or  $M_T(h, g) < 1/T$ , then we have

$$\begin{aligned} \rho(f, h) + \rho(h, g) &\geq \min\{M_T(f, h), 1/T\} + \min\{M_T(h, g), 1/T\} \\ &= M_T(f, h) + M_T(h, g) \\ &\geq D(f(y, s), h(y, s)) + D(h(y, s), g(y, s)) \\ &\geq D(f(y, s), g(y, s)) = M_T(f, g) \\ &\geq \min\{M_T(f, g), 1/T\} > \rho(f, g) - \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we have  $\rho(f, h) + \rho(h, g) \geq \rho(f, g)$ . Other conditions for  $\rho$  can be easily verified. This completes the proof of the theorem.  $\square$

**Theorem 3.2.** *The map  $CR : (M_0, \rho) \rightarrow (F(X), D)$  is upper semicontinuous.*

*Proof.* Suppose that  $CR : (M_0, \rho) \rightarrow (F(X), D)$  is not upper semicontinuous at  $f \in M_0$ . Then we can find  $\varepsilon > 0$  such that for any  $n$  in  $\mathbb{N}$  there exists a  $f_n \in B_\rho(f, 1/n)$  satisfying  $CR_\rho(f_n) \not\subseteq B(CR(f), \varepsilon)$ . There exists  $x_n \in CR(f_n) - B(CR(f), \varepsilon)$ . We claim that if  $\rho(f, g) < 1/s$  for all  $f, g \in M_0$  and  $s > 0$ , then  $D(f(x, t), g(x, t)) \leq \rho(f, g)$  for all  $x \in X$  and  $t \in [-s, s]$ . To prove this claim, assume  $M_s(f, g) \geq 1/s$ . Then,

$$\begin{aligned} 1/s > \rho(f, g) &= \sup\{\min\{M_T(f, g), 1/T\} : T > 0\} \\ &\geq \min\{M_s(f, g), 1/s\} \\ &= 1/s, \end{aligned}$$

gives a contradiction. Hence, we obtain

$$\begin{aligned} \rho(f, g) &= \sup\{\min\{M_T(f, g), 1/T\} : T > 0\} \\ &\geq \min\{M_s(f, g), 1/s\} \\ &= M_s(f, g). \end{aligned}$$

Since  $X$  is compact, we can assume  $x_n$  converges to  $x$ . To show that  $x \in CR(f)$ , let  $\alpha > 0$  and  $T > 0$ . By integral continuity theorem, there exists  $0 < \beta \leq \alpha$  such that if  $d(x, y) < \beta$ , then  $D(f(x, t), f(y, t)) < \alpha/3$  for all  $t \in [T, 2T]$ . Since  $f_n \rightarrow f$  and  $x_n \rightarrow x$ ,  $\rho(f, f_n) < \min\{\alpha/3, 1/2T\}$  and  $d(x, x_n) < \beta$  for some  $n$ .

Recall that  $\rho(f, f_n) < 1/2T$ , thus we get  $D(f(x, t), f_n(x, t)) \leq \rho(f, f_n) < \alpha/3$  for all  $x \in X$  and  $t \in [T, 2T]$ .

Moreover  $x_n \in CR(f_n)$  implies there exists an  $(\alpha/3, T)$ -chain

$$\{y_1, y_2, \dots, y_k, y_{k+1}; t_1, t_2, \dots, t_k\}$$

from  $x_n$  to  $x_n$ .

It follows that

$$\begin{aligned} D(y_2, f(x, t_1)) &\leq D(y_2, f_n(x_n, t_1)) + D(f_n(x_n, t_1), f(x_n, t_1)) \\ &\quad + D(f(x_n, t_1), f(x, t_1)) \\ &< \alpha/3 + \alpha/3 + \alpha/3 < \alpha. \end{aligned}$$

Also,

$$\begin{aligned} D(y_{i+1}, f(y_i, t_i)) &\leq D(y_{i+1}, f_n(y_i, t_i)) + D(f_n(y_i, t_i), f(y_i, t_i)) \\ &< \alpha/3 + \alpha/3 < \alpha, \end{aligned}$$

$$\begin{aligned} D(x, f(y_k, t_k)) &\leq d(x, x_n) + D(x_n, f_n(y_k, t_k)) + D(f_n(y_k, t_k), f(y_k, t_k)) \\ &< \alpha/3 + \alpha/3 + \alpha/3 < \alpha \end{aligned}$$

hold. Therefore,  $\{x, y_2, \dots, y_k, x; t_1, t_2, \dots, t_k\}$  is a  $(\alpha, T)$ -chain for  $f$  from  $x$  to  $x$ . Hence,  $x \in CR(f)$ , contracting  $x \notin B(CR(f), \varepsilon)$ . Consequently the

map  $CR : (M_0, \rho) \rightarrow (F(X), D)$  is upper semicontinuous at  $f$ . This completes proof.  $\square$

The importance of lower or upper semicontinuities for the study of generic properties is contained in the following theorem.

**Theorem 3.3.** *Let  $X$  and  $Y$  be metric spaces with  $Y$  compact, and if  $f : X \rightarrow F(M)$  is either lower semicontinuous or upper semicontinuous, then the set of continuity points of  $f$  is a residual subset of  $X$ .*

*Proof.* See [10].  $\square$

By Theorems 3.2, 3.3, we obtain the following Takens's theorem for general dynamical systems. Remark that the remainder results of Takens's theorem for general dynamical systems is similarly checked.

**Theorem 3.4** (Takens's theorem for general dynamical systems). *The set of continuity points of  $CR : (M_0, \rho) \rightarrow (F(X), D)$  is residual in  $M_0$ .*

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