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STABILITY OF TWO-PHASE FLOW MODELS

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ABSTRACT. In this paper, we study two-phase flow models. The chunk mix model of the two-phase flow equations is analyzed by a characteristic analysis. The model discussed herein has real characteristic values for all physically acceptable states and except for a set of measure zero has a complete set of characteristic vectors in state space.

1. Introduction

In this paper, we study two-phase flow models. Multiphase flows display a wealth of detail which is not reproducible, neither experimentally nor in simulations. Generally speaking, this detail is not relevant, and fortunately, only the statistical averages of the detail are of importance. Thus direct numerical simulation (DNS) of mix, as discussed in [4, 6, 15, 22], gives more information than is needed, and information which cannot be reproducible in detail. Since we really want the averages of the DNS, the natural question is to find averaged equations which will compute the averaged quantities directly, without use of the difficult intermediate DNS step.

Averaging equations [5, 16] arise in many areas of science. Generally, when the original equations are nonlinear, or when the coefficients of a linear term are to be averaged, lengthy discussions of how to formulate the averaged equations ensue. The issue is that nonlinearities do not commute with averaging, so the average of a nonlinear function is not equal to the function evaluated at the average value of its argument. We wish to average over each phase, and end up with multi-phase flow equations. The nonlinear closure terms will then reflect the forces, etc. exerted between the two phases.

For mix at a molecular level, all the nonlinear closure issues occur in the equation of state, which must describe the pressure and other thermodynamic functions of an atomic mixture of multiple species. In this case all species have common velocities and temperatures. If the mixing is less fine grained, we call the problem chunk mix. The complete first order multiphase averaging of the microphysical equations leads to such a model, in which each species

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has separate velocities and thermodynamics (pressure and temperature). We have recently found a closure of this type which preserves all requirements of an obvious physical nature: required boundary conditions at the edges of the mixing zone, conservation of species mass, total momentum, total energy and for smooth flows, phase entropy [13, 14]; see also earlier work [8, 9, 10, 11]. The only parameters to be fixed in this closure are determined by the growth rates for the edges of the mixing zone. Also we have compared the various models, which are distinguished by their choice of closures, and we have studied their compatibility for two-phase flow models of mixing layers [12].

The purpose of this paper is to analyze the chunk mix model of the two-phase flow equations by a characteristic analysis. The model we consider has distinct phase pressures and leads to hyperbolic models, eliminating mathematical difficulties of complex characteristics associated with single pressure flow models. For systems of partial differential equations of first order, stability in the sense of von Neumann is essentially equivalent to the condition that the model be hyperbolic. Here we show that the models have real characteristic values for all physically acceptable states and except for a set of measure zero have a complete set of characteristic vectors in state space. Therefore, these models are hyperbolic a.e. (almost everywhere) in state space. Also, they are stable in the sense of von Neumann a.e. in state space even without inclusion of viscosity terms.

Compared to the single-pressure model, the two-pressure model approximate additional physical features and is shown to be a viable approach for the case of separated flow. The single-pressure models have complex characteristic values within the range of interest of the dependent variables. Thus the models are physically unacceptable and lead to ill-posed initial value problems. The complex characteristic values of the single pressure models appear to result from the unrealistic assumption called the hydrostatic assumption that the pressures are in equilibrium. Whereas, the two-pressure models allow for the possibility that the flow is not hydrostatic and thus do not include the unrealistic assumption which apparently leads to the complex characteristic values. A careful inspection of all of the approximations and assumptions in the development of the single-pressure model led to the conclusion that the only reasonable change leading to a hyperbolic model is the change in the hydrostatic assumption to allow the model to become a truly two-pressure model. Development of the two-pressure models has been discussed in [18, 21].

1.1. A two-phase flow model for a fluid mixing layer

Jin *et al.* [13, 14] recently proposed a two-phase flow model for fluid mixing using a formalism that is described by Drew [5]. In this section, we present this model and specify improved constitutive laws for the material coupling terms.

Effective equations of motion are derived by performing single-phase averages of the microphysical model over an infinite ensemble of microscopic flow realizations. We assume a mixing zone homogeneity in a specified flow regime characterized by large scale coherent mixing structures (bubbles of light fluid, *etc.*), on the order of the thickness of the mixing zone, and by short time scales, so that relaxation terms are omitted.

The two-phase flow model obtained by ensemble averaging within each fluid is then

$$\frac{\partial \beta_k}{\partial t} + v^* \frac{\partial \beta_k}{\partial z} = 0 \ , \tag{1}$$

$$\frac{\partial(\beta_k \rho_k)}{\partial t} + \frac{\partial(\beta_k \rho_k v_k)}{\partial z} = 0 \ , \tag{2}$$

$$\frac{\partial(\beta_k \rho_k v_k)}{\partial t} + \frac{\partial(\beta_k \rho_k v_k v_k)}{\partial z} + \frac{\partial(\beta_k p_k)}{\partial z} = p^* \frac{\partial \beta_k}{\partial z} + \beta_k \rho_k g \tag{3}$$

for the advection of the volume fraction and for conservation of mass and momentum. We also have one and only one of the energy equations

$$\frac{\partial(\beta_k \rho_k E_k)}{\partial t} + \frac{\partial(\beta_k \rho_k v_k E_k)}{\partial z} + \frac{\partial(\beta_k p_k v_k)}{\partial z} = (pv)^* \frac{\partial \beta_k}{\partial z} + \beta_k \rho_k v_k g \ , \tag{4a}$$

$$\frac{\partial(\beta_k \rho_k e_k)}{\partial t} + \frac{\partial(\beta_k \rho_k v_k e_k)}{\partial z} + p_k \frac{\partial(\beta_k v_k)}{\partial z} = (pv)^* \frac{\partial \beta_k}{\partial z} \ , \tag{4b}$$

$$\frac{\partial(\beta_k \rho_k S_k)}{\partial t} + \frac{\partial(\beta_k \rho_k v_k S_k)}{\partial z} = 0 \tag{4c}$$

for the volume fraction β_k, velocity v_k, density ρ_k, pressure p_k, entropy S_k, internal energy e_k and total energy E_k of phase k. Here $g = g(t) > 0$ is the gravity and we assume $\rho_2 > \rho_1$. The quantities v^*, p^* and $(pv)^*$ are the averaged quantities at the interface,

$$v^* = \frac{\langle \mathbf{v} \cdot \nabla X_k \rangle}{\langle \mathbf{n}_3 \cdot \nabla X_k \rangle} \ , \quad p^* = \frac{\langle p\mathbf{n}_3 \cdot \nabla X_k \rangle}{\langle \mathbf{n}_3 \cdot \nabla X_k \rangle} \ , \quad (pv)^* = \frac{\langle p\mathbf{v} \cdot \nabla X_k \rangle}{\langle \mathbf{n}_3 \cdot \nabla X_k \rangle} \ , \tag{5}$$

where $\mathbf{n}_3$ is the unit normal vector in the preferred direction and $\langle \cdot \rangle$ is an average over the x, y symmetry plane and in principle an ensemble average.

The definitions (5) are fundamental to all that follows. They are mathematically exact consequences of the averages of the microphysical equations and specify the quantities (the RHS of (5)) that are to be approximated in a definition of closure to complete the averaged equations (1)-(4c). In §2, we reexamine the mathematically exact expression for each q^*, $q = v,\ p,\ pv$, independently of any closure assumptions. The derivation leads to a natural formulation closures for the constitutive laws. See [13, 14] for more details.

There is a choice of averaging the total energy [19], internal energy [3, 9] or entropy [17, 20] equations (4); and only one is to be used. These averages give distinct equations, which differ by triple correlations only, and so they should have similar solutions. The triple correlations which mark the difference between the three sets of equations occur in the energy equation. Obviously, the total energy closure (4a) and entropy closure (4c) show total energy and phase entropy conservation, respectively. But the entropy in the total energy closure (4a), the total energy and entropy in the internal energy closure (4b) and the total energy in the entropy closure (4c) are not obviously conserved. We have discussed conservation of the energy or entropy for the three cases and derived conservation constraints in [12].

2. Spatial homogeneity closure and stability

In [2, 13, 14] an exact expression for the interface quantities q^*, $q = v,\ p,\ pv$, has been derived by manipulation of the governing equations (1)-(4a) in the absence of any closure assumption. Based on this expression, closures have been proposed for the constitutive law $d_k^q(t)$. Here we recover these formulas as well as the fractional linear form for the convex coefficients and a natural assumption on the constitutive law to close the fractional linear form. We regard the closure equations as a new and independent constraint, which are restrictions on the physical flow regime described by the model. The detailed relations of the equations are repeated in §2.1 and §2.2. In §2.3 we discuss stability of the two-phase flow equations (1)-(4c) by a characteristic analysis.

2.1. The v^* and p^* closures

In [2, 7, 13, 14] the interfacial terms v^* and p^* have been derived exactly from (1)-(3) independently of any closure assumption. The results are summarized in the following theorem.

Theorem 2.1. *The interface quantities v^* and p^* have the exact formula*

$$q^* = \mu_1^q q_2 + \mu_2^q q_1 \ , \quad q = v,\ p, \tag{6}$$

where the mixing coefficients have the fractional linear form

$$\mu_k^q = \frac{\beta_k}{\beta_k + d_k^q \beta_{k'}} \ . \tag{7}$$

The constitutive factor d_k^q is also expressed in the exact form

$$d_k^v(z,t) = \frac{\dfrac{\partial v_{k'}}{\partial z} + \dfrac{1}{\rho_{k'}}\dfrac{D_{k'}\rho_{k'}}{Dt}}{\dfrac{\partial v_k}{\partial z} + \dfrac{1}{\rho_k}\dfrac{D_k\rho_k}{Dt}} \ , \tag{8}$$

$$d_k^p(z,t) = \frac{\frac{\partial p_{k'}}{\partial z} - \rho_{k'}\left(g - \frac{D_{k'}v_{k'}}{Dt}\right)}{\frac{\partial p_k}{\partial z} - \rho_k\left(g - \frac{D_k v_k}{Dt}\right)} \ , \tag{9}$$

where $D_k/Dt \equiv \partial/\partial t + v_k \partial/\partial z$ is the phase k convective derivative.

The factor $d_k^v(z,t)$ in (8) is a ratio of logarithmic rates of volume creation for the two phases. The coefficient $d_k^p(z,t)$ represents a ratio of the forces accelerating the two fluids. As observed earlier [8, 9], p^* contains drag, added mass and buoyancy effects commonly included in phenomenological closure models. Thus p^* and ultimately the μ_k^p and d_k^p determine these effects in the present model.

A closure condition of spatial homogeneity assumes an integral formula for the closure $d_k^q(t)$, $q = v,\ p$. Refer to [2, 7, 13]. These closures are logically and physically independent of and distinct from (8) and (9), respectively. Thus a zero parameter model for the constitutive law $d_k^q(t)$ has been proposed, provided the mixing zone edge positions $Z_k(t)$ or velocities $V_k(t) = \dot{Z}_k$. The relation $d_1^q(t) d_2^q(t) = 1$ is equivalent to $\mu_1^q + \mu_2^q = 1$ from (7). It is proved that $\mu_k^q \geq 0$ for $0 \leq \beta_k \leq 1$ if and only if $d_k^q(t) \geq 0$.

In [1, 2, 13] the spatial homogeneity closures are compared in a validation study to spatial averages of DNS, i.e., simulation solutions of the microphysical equations. When $d_k^q(t) > 0$, the closed and the unclosed q^*, $q = v, p$ show excellent agreement with data. In case that $d_k^q(t) < 0$, a zero in the denominator of (7) occurs. This singularity can be removed through selection of a special choice of $d_k^q(t) = d_k^q(z^*, t)$. In this case, the choice of q^* is insensitive to the choice of the positive minimum d_k^q, and it is insensitive to d_k^q altogether. For this reason, we may avoid problems with cancelation of zeros in a definition of d_k^q and force d_k^q to be positive. For the 3D Rayleigh-Taylor and circular 2D Richtmyer-Meshkov data, we have seen excellent validation agreement for the closures proposed. See [2, 13].

2.2. The $(pv)^*$ closure

The exact form for $(pv)^*$ is derived from the total energy closure (4a). It is based on the entropy equation derived from (4a). Thus total energy is automatically conserved. A spatial homogeneity assumption gives a closure for the constitutive law d_k^{pv}. For details, refer to [2, 12, 14].

Using (4a), we yield the exact expression

$$
\begin{aligned}
(pv)^* &= p^* \frac{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} v_2 + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt} v_1}{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt}} \\
&\quad + v^* \frac{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} p_2 + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt} p_1}{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt}} \\
&\quad - \frac{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} p_2 v_2 + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt} p_1 v_1}{\beta_1 \rho_1 T_1 \frac{D_1 \mathbb{S}_1}{Dt} + \beta_2 \rho_2 T_2 \frac{D_2 \mathbb{S}_2}{Dt}} \\
&\equiv p^* \left(\mu_1^{pv} v_2 + \mu_2^{pv} v_1 \right) + v^* \left(\mu_1^{pv} p_2 + \mu_2^{pv} p_1 \right) - \left(\mu_1^{pv} p_2 v_2 + \mu_2^{pv} p_1 v_1 \right) .
\end{aligned} \tag{10}
$$

Here the macro entropy

$$
\mathbb{S}_k = \mathbb{S}_k(\mathfrak{e}_k, \rho_k) = \mathbb{S}_k \left(\frac{\langle X_k \rho E \rangle}{\langle X_k \rho \rangle} - \frac{1}{2} \frac{\langle X_k \rho v_z \rangle^2}{\langle X_k \rho \rangle^2} ,\ \frac{\langle X_k \rho \rangle}{\langle X_k \rangle} \right) \tag{11}
$$

expressed *via* the equation of state from the macro (averaged) energy is not the same as the macro entropy, $S_k \equiv \langle X_k \rho S\rangle / \langle X_k \rho\rangle$, expressed directly as an average of the micro entropy [14]. The mixing coefficients in (10) satisfy the fractional form

$$\mu_k^{pv} = \frac{\beta_k}{\beta_k + d_k^{pv}\beta_{k'}} \; . \tag{12}$$

The exact form (10) for $(pv)^*$ have no approximations and they are mathematically equivalent because they are derived from the equivalent Eq. (4a).

The identity (10) suggests possible distinct closure relation for the constitutive law,

$$d_k^{pv}(t) = \frac{\displaystyle\int_{Z_k}^{Z_{k'}} \rho_{k'} T_{k'} \frac{D_{k'}\mathbb{S}_{k'}}{Dt} dz}{\displaystyle\int_{Z_k}^{Z_{k'}} \rho_k T_k \frac{D_k \mathbb{S}_k}{Dt} dz} \; . \tag{13}$$

The ratio $d_k^{pv}(t)$ is formed by spatially averaging the numerator and denominators of the exact form $d_k^{pv}(z,t)$ based on (10), and it represents the ratio of the incremental amount of heat added to the system for two fluids. The mixing coefficients satisfy the relations $\mu_1^{pv} + \mu_2^{pv} = 1$ and $\mu_k^{pv} \geq 0$, which are equivalent to the relations $d_1^{pv}(t) d_2^{pv}(t) = 1$ and $d_k^{pv}(t) \geq 0$. The direct validation of the closure $(pv)^*$, based on analysis of the simulation data is presented in [2].

2.3. Characteristic analysis

In this section we analyze the chunk mix model of the two-phase flow equations (1)-(4c) in characteristic analysis and show it to be hyperbolic. We observe that the hyperbolicity of the model is independent of the choice of the averaged total energy, internal energy or entropy equations (4a)-(4c). We here present specific results for each of the three 7×7 systems (1)-(4c).

Theorem 2.2. *Each of the systems* (1)-(4c) *has all real characteristic values*

$$v^* \; , \; v_k \pm c_k \; , \; v_k \tag{14}$$

for $k = 1, 2$, *where* c_k *is the sound speed of phase* k. *It is hyperbolic a.e. in state space and stable in the sense of von Neumann a.e. in state space.*

The hyperbolicity of the two-pressure model (1)-(4c) is lost at a subset of the points where

$$\eta_k \equiv v_k - v^* = 0 \quad \text{or} \quad \gamma_k \equiv c_k^2 - (v_k - v^*)^2 = 0 \; . \tag{15}$$

This set has measure zero (because it has lower dimension). This result follows from an observation regarding characteristic vectors.

Let

$$U^t = (\beta_2\rho_2, \beta_2\rho_2 v_2, \beta_1\rho_1, \beta_1\rho_1 v_1, \beta_1, S_2, S_1) \tag{16}$$

and let

(17)

$$A(U) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_2^2 - v_2^2 & 2v_2 & 0 & 0 & -P_2^{(\gamma)} & \beta_2 \frac{\partial p_2}{\partial S_2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_1^2 - v_1^2 & 2v_1 & P_1^{(\gamma)} & 0 & \beta_1 \frac{\partial p_1}{\partial S_1} \\ 0 & 0 & 0 & 0 & v^* & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{P_2^{(\eta)}}{\beta_2 \rho_2 T_2} & v_2 & 0 \\ 0 & 0 & 0 & 0 & \frac{P_1^{(\eta)}}{\beta_1 \rho_1 T_1} & 0 & v_1 \end{pmatrix},$$

where U^t denotes the transpose of the vector U,

(18)

$$P_k^{(\gamma)} = p_k - p^* - \rho_k c_k^2 ,$$

(19)

$$P_k^{(\eta)} = \begin{cases} p_k v^* - p_k v_k + p^* v_k - (pv)^* & \text{if the total energy averaged (4a)} \\ p_k v^* - (pv)^* & \text{if the internal energy averaged (4b)} \\ 0 & \text{if the entropy averaged (4c).} \end{cases}$$

Here we denote S_k the directly averaged entropy for the entropy closure (4c) and the macro entropy expressed from the fluid k EOS of directly averaged quantities for the total energy closure (4a) and internal energy closure (4b). Then the two phase flow model may be written as the following

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial z} = 0 . \tag{20}$$

Observe that the characteristic equation has the seven real roots

$$\lambda = v_2 \pm c_2 \ , \ v_1 \pm c_1 \ , v_2 \ , \ v_1 \ , \ v^* \ . \tag{21}$$

Given either that $v^* - \lambda \neq 0$ or that $v^* - \lambda = 0$ and

$$\left(c_2^2 - (v_2 - \lambda)^2\right)\left(c_1^2 - (v_1 - \lambda)^2\right)(v_2 - \lambda)(v_1 - \lambda) \neq 0,$$

the seven right eigenvectors associated with the characteristic values λ are linearly independent. We now consider conditions to guarantee the existence of a complete set of linearly independent characteristic vectors when $v^* - \lambda = 0$ and $\left(c_2^2 - (v_2 - \lambda)^2\right)\left(c_1^2 - (v_1 - \lambda)^2\right)(v_2 - \lambda)(v_1 - \lambda) = 0$. For example, if $\gamma_1 = 0$ and $\gamma_2 = 0$ are satisfied, we know that $\eta_1 \neq 0$ and $\eta_2 \neq 0$ because $c_k^2 \neq 0$. In this case, a necessary and sufficient condition for existence of a complete set of characteristic vectors is that $P_k^{(\gamma)} - \frac{P_k^{(\eta)}}{\eta_k \rho_k T_k} \frac{\partial p_k}{\partial S_k} = 0$ for both phases. We summarize this result as follows

Proposition 2.3. *For all physically acceptable states the chunk mix model* (1)-(4c) *is hyperbolic if and only if one of the following cases is obtained:*

(a) $\gamma_1 = 0, \gamma_2 = 0, \eta_1 \neq 0, \eta_2 \neq 0$ *and* $P_1^{(\gamma)} - \dfrac{P_1^{(\eta)}}{\eta_1 \rho_1 T_1} \dfrac{\partial p_1}{\partial S_1} = 0,$
$P_2^{(\gamma)} - \dfrac{P_2^{(\eta)}}{\eta_2 \rho_2 T_2} \dfrac{\partial p_2}{\partial S_2} = 0 .$

(b) $\gamma_1 = 0, \gamma_2 \neq 0, \eta_1 \neq 0, \eta_2 = 0$ *and* $P_1^{(\gamma)} - \dfrac{P_1^{(\eta)}}{\eta_1 \rho_1 T_1} \dfrac{\partial p_1}{\partial S_1} = 0,$
$P_2^{(\eta)} = 0 .$

(c) $\gamma_1 = 0, \gamma_2 \neq 0, \eta_1 \neq 0, \eta_2 \neq 0$ *and* $P_1^{(\gamma)} - \dfrac{P_1^{(\eta)}}{\eta_1 \rho_1 T_1} \dfrac{\partial p_1}{\partial S_1} = 0 .$

(d) $\gamma_1 \neq 0, \gamma_2 = 0, \eta_1 = 0, \eta_2 \neq 0$ *and* $P_1^{(\eta)} = 0,$
$P_2^{(\gamma)} - \dfrac{P_2^{(\eta)}}{\eta_2 \rho_2 T_2} \dfrac{\partial p_2}{\partial S_2} = 0 .$

(e) $\gamma_1 \neq 0, \gamma_2 = 0, \eta_1 \neq 0, \eta_2 \neq 0$ *and* $P_2^{(\gamma)} - \dfrac{P_2^{(\eta)}}{\eta_2 \rho_2 T_2} \dfrac{\partial p_2}{\partial S_2} = 0 .$

(f) $\gamma_1 \neq 0, \gamma_2 \neq 0, \eta_1 = 0, \eta_2 = 0$ *and* $P_1^{(\eta)} = 0,\ P_2^{(\eta)} = 0 .$

(g) $\gamma_1 \neq 0, \gamma_2 \neq 0, \eta_1 = 0, \eta_2 \neq 0$ *and* $P_1^{(\eta)} = 0 .$

(h) $\gamma_1 \neq 0, \gamma_2 \neq 0, \eta_1 \neq 0, \eta_2 = 0$ *and* $P_2^{(\eta)} = 0 .$

(i) $\gamma_1 \neq 0, \gamma_2 \neq 0, \eta_1 \neq 0, \eta_2 \neq 0.$

Here $P_k^{(\gamma)}$ *and* $P_k^{(\eta)}$ *are given in* (18) *and* (19).

In the compressible case, the system (1)-(4) is missing one condition at each edge $z = Z_k(t)$ of the mixing zone. Each missing condition is associated with a missing characteristic at the Z_k boundary. For the fluid with vanishing β_k, the sonic characteristic entering from the $\beta_k = 0$ side is missing. This missing information is supplied by the edge acceleration $\ddot{Z}_k(t)$. Thus we regard the edge positions $Z_k(t)$ as input, or data, which complete the specification of the model or close it. We appeal to the buoyancy drag model to provide the $Z_k(t)$. See [10] and references therein. In this sense we separate and almost totally decouple the complete two-phase model into distinct edge and interior models, with the edge model completing the closure of the entire model. The closures (6) and (10) are independent of the unclosed model equations and provide new constraints for d_k^q, $q = v, p, pv$. With the mixing zone edge velocities $V_k(t)$ and constitutive laws (8), (9) and (13), the compressible model has no adjustable parameters. Therefore, we have ten independent equations, (1)-(3), (4a), (6) and (10) for the ten variables, β_1, ρ_1, ρ_2, v_1, v_2, E_1, E_2, v^*, p^* and $(pv)^*$. This full system of the compressible equations closes by a count of primitive variables. It is not hyperbolic because the constitutive laws for d_k^q, $q = v, p, pv$, include integral-differential terms. However, its 7×7 subsystems, (1)-(3), (4a), is hyperbolic as discussed in Theorem 2.2.

3. Conclusion

The recently developed two-phase flow model (1)-(3), (4a) with the closures for q^*, $q = v$, p, pv, in [1, 13, 14] and repeated in §2 satisfies hyperbolic stability conditions.

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