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A GENERAL UNIQUENESS RESULT OF AN ENDEMIC STATE FOR AN EPIDEMIC MODEL WITH EXTERNAL FORCE OF INFECTION

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ABSTRACT. We present a general uniqueness result of an endemic state for an S-I-R model with external force of infection. We reduce the problem of finding non-trivial steady state solutions to that of finding zeros of a real function of one variable so that we can easily prove the uniqueness of an endemic state. We introduce an assumption which was usually used to show stability of a non-trivial steady state. It turns out that such an assumption adopted from a stability analysis is crucial for proving the uniqueness as well, and the assumption holds for almost all cases in our model.

1. Introduction

In this paper we study the uniqueness of a non-trivial steady state of an age-structured S-I-R type epidemic model. Age structured S-I-R models are suitable for most common childhood diseases (measles, chickenpox, rubella), as well as for many sexually transmitted diseases which impart immunity (syphilis, chlamydia), and also for those diseases, like HIV/AIDS, which lead to definitive isolation or death [1, 5, 6, 7, 8, 12, 14].

In [4], the authors have considered an S-I-R model with external force of infection. Although infection of the human disease mainly occurs between humans through physical contacts, there are lots of other ways of infection. For instance, one can be infected with bird flu (avian flu) by direct contact with live birds or bird droppings. Most people who have gotten the virus work directly with poultry or have had close contact with birds.

Mad cow disease is another good example; there is still no evidence that indicates the recently emerged human form of mad cow disease, it is believed that potentially a lot of people are affected by that disease.

It seems that infected animals could be the main source of the infection in those diseases. The thing is that, for some diseases, external force of infection is more important than anything.

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Since the external force was considered in [4], the existence and uniqueness results they obtained were somewhat different from those of the usual S-I-R models without external force: a non-trivial steady state always exists and it is unique if the vertical transmission parameter, q, of the disease is equal to zero.

In this article, as a generalization, we shall get a uniqueness result for the same model with an arbitrary vertical transmission parameter, i.e., $0 \leq q \leq 1$.

Before stating the uniqueness result, we have to begin with the PDE system for the S-I-R model and the parameters used in it:

$$(1)\qquad \begin{cases} \dfrac{\partial s}{\partial t} + \dfrac{\partial s}{\partial a} + \mu(a)s = -\lambda(a,t)s, \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} + \mu(a)i = \lambda(a,t)s - \gamma(a)i, \\ \dfrac{\partial r}{\partial t} + \dfrac{\partial r}{\partial a} + \mu(a)r = \gamma(a)i, \\ s(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a)\big(s(a,t) + r(a,t) + (1-q)i(a,t)\big)\, da, \\ i(0,t) = q \displaystyle\int_0^{a_\dagger} \beta(a)i(a,t)\, da, \\ r(0,t) = 0, \\ i(a,0) = i_0(a), \quad s(a,0) = s_0(a), \quad r(a,0) = r_0(a). \end{cases}$$

Here a is the age of individuals, and t is the time. Also, $s(a,t)$, $i(a,t)$ and $r(a,t)$, respectively, denotes the age-specific density of susceptible, infected, and removed individuals.

The other important parameters are as follows: $\beta(a)$ is the birth rate and $\mu(a)$ is the death rate of the population. The parameter $q \in [0,1]$ is the vertical transmission parameter, i.e., the probability that the disease be transmitted from parent to newborn, $\gamma(\cdot)$ is the removal rate of infected individuals, and $\lambda(a,t)$ is the force of infection. Note that since $r(0,t) = 0$, our model assumes that there is no vertical transmission of immunity.

Our main concern is the existence and uniqueness of an endemic state of the model. (Endemic state is a steady state solution of the model for which the density of infected individuals does not vanish identically.)

Summing the equations in (1) we obtain the following problem for the total population age-density $p(a,t) = s(a,t) + i(a,t) + r(a,t)$,

$$(2)\qquad \begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} + \mu(a)p = 0, \\ p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t)da, \\ p(a,0) = p_0(a), \end{cases}$$

where $p_0(a) = s_0(a) + i_0(a) + r_0(a)$. (This is the standard McKendrick–Von Foester equation [1, 2, 13].)

We make the following usual hypotheses for this problem,

$$\beta(\cdot) \in L^\infty([0,a_\dagger)), \qquad \beta(a) \geq 0 \text{ in } [0,a_\dagger), \tag{3}$$

$$\mu(\cdot) \in L^1_{\text{loc}}([0,a_\dagger)), \qquad \mu(a) \geq 0 \text{ in } [0,a_\dagger), \tag{4}$$

$$\int_0^{a_\dagger} \mu(a)d\sigma = \infty. \tag{5}$$

Here $a_\dagger$ is the maximum age an individual of the population may reach and it may be either finite or infinite. If $a_\dagger = \infty$, we also assume that

$$\text{there exists } A > 0 \text{ such that } \beta(a) = 0 \text{ for } a \geq A. \tag{6}$$

Furthermore, in order to deal with a steady state population, we assume that the net reproductive rate of the population is equal to one and that the total population is at an equilibrium. This means that

$$\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma}da = 1, \qquad p(a,t) = p_\infty(a) = b_0\pi(a), \tag{7}$$

where

$$\pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}.$$

Note that the function $\pi(a)$ is the probability that an individual at age 0 can survive until age a. Since no individual may live past age $a_\dagger$, (5) is needed.

We take initial data such that

$$s_0(a) \geq 0, \quad i_0(a) \geq 0, \quad r_0(a) \geq 0, \quad s_0(a) + i_0(a) + r_0(a) = p_\infty(a),$$

which gives

$$b_0 = \frac{\int_0^{a_\dagger} s_0(a)da + \int_0^{a_\dagger} i_0(a)da + \int_0^{a_\dagger} r_0(a)da}{\int_0^{a_\dagger} \pi(a)da}.$$

We assume that

$$\gamma(\cdot) \in L^\infty([0,a_\dagger)), \qquad \gamma(a) \geq 0 \text{ in } [0,a_\dagger)$$

and consider the following form for the force of infection:

$$\lambda(a,t) = \kappa(a)\int_0^{a_\dagger} h(\sigma)i(\sigma,t)\,d\sigma + g(a),$$

where h is the age-specific infectiousness, κ the age-specific contagion rate, and g is the external force of infection [4]. They satisfy the following conditions:

$$h(\cdot),\ \kappa(\cdot),\ g(\cdot) \in L^\infty([0,a_\dagger)) \ \text{ and } \ h(a),\ \kappa(a),\ g(a) \geq 0, \ \text{ on } [0,a_\dagger).$$

We also assume that none of $\beta(\cdot)$, $\mu(\cdot)$, $\gamma(\cdot)$, $h(\cdot)$, $\kappa(\cdot)$, $g(\cdot)$ is identically zero.

We note that by assumption (7), the fourth equation in (1) becomes

$$s(0,t) = \int_0^{a_\dagger} \beta(a)\,p_\infty(a)\,da - q\int_0^{a_\dagger} \beta(a)\,i(a,t)\,da = b_0 - q\int_0^{a_\dagger} \beta(a)\,i(a,t)\,da,$$

so that the equations involving the variable $r(a,t)$ in (1) can be disregarded since $s(a,t)$ and $i(a,t)$ are sufficient to determine the evolution of the whole system. Thus we will be concerned with the following reduced system:

$$(8)\qquad \begin{cases} \dfrac{\partial s}{\partial t}(a,t) + \dfrac{\partial s}{\partial a}(a,t) + \mu(a)s(a,t) = -\lambda(a,t)s(a,t), \\ \dfrac{\partial i}{\partial t}(a,t) + \dfrac{\partial i}{\partial a}(a,t) + \mu(a)i(a,t) = \lambda(a,t)s(a,t) - \gamma(a)i(a,t), \\ s(0,t) = b_0 - i(0,t), \\ i(0,t) = q \displaystyle\int_0^{a_\dagger} \beta(a)i(a,t)da, \\ i(a,0) = i_0(a), \quad s(a,0) = s_0(a). \end{cases}$$

We will analyze (8) by using the techniques developed in [4]. However, we will modify them so that they can be applied to our case. So, in the next section we will reduce the problem of finding endemic states to that of finding zeros of a real function. Then, in sections 3 we will give our main result. Finally, in section 4, concluding remarks will be given.

2. Equations of real variables

We now consider the problem of the existence and uniqueness of steady states of system (8). Consequently, we are concerned with the following problem:

$$(9)\qquad \begin{cases} \text{i)} & \dfrac{\partial s}{\partial a} + \mu(a)s(a) = -\big(J\kappa(a) + g(a)\big)\, s(a), \\ \text{ii)} & \dfrac{\partial i}{\partial a} + \mu(a)i(a) = \big(J\kappa(a) + g(a)\big)\, s(a) - \gamma(a)\, i(a), \\ \text{iii)} & J = \displaystyle\int_0^{a_\dagger} h(a)\, i(a)\, da, \\ \text{iv)} & s(0) = b_0 - i(0), \\ \text{v)} & i(0) = q \displaystyle\int_0^{a_\dagger} \beta(a)\, i(a)\, da. \end{cases}$$

It is easy to see that the problem admits the disease-free equilibrium $s^*(a) = p_\infty(a)$ and $i^*(a) \equiv 0$ if and only if $g(a) \equiv 0$. Since we are assuming that g is not identically zero, there is no disease-free equilibrium. To investigate the existence and uniqueness of an endemic state, we modify problem (9) by taking the following new variables called the age profiles respectively of infected and susceptible individuals:

$$u(a) = \frac{i(a)}{p_\infty(a)}; \qquad v(a) = \frac{s(a)}{p_\infty(a)}.$$

With these new variables, problem (9) becomes

$$
(10)\qquad \begin{cases}
\text{i)} & \dfrac{dv}{da} = -\big(J\kappa(a) + g(a)\big)\, v(a), \\
\text{ii)} & \dfrac{du}{da} = \big(J\kappa(a) + g(a)\big)\, v(a) - \gamma(a)\, u(a), \\
\text{iii)} & J = b_0 \displaystyle\int_0^{a_\dagger} h(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma, \\
\text{iv)} & v(0) = 1 - X, \\
\text{v)} & X = q \displaystyle\int_0^{a_\dagger} \beta(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma.
\end{cases}
$$

From (10.i) and (10.ii) we get the following:

$$
(11)\qquad v(a) = (1 - X)e^{-\int_0^a (J\kappa(\sigma)+g(\sigma))d\sigma}.
$$

$$
(12)\qquad \begin{aligned} u(a) = Xe^{-\int_0^a \gamma(\sigma)d\sigma} &+ (1 - X) \\ &\times \int_0^a \big(J\kappa(\sigma) + g(\sigma)\big)e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds}\, d\sigma. \end{aligned}
$$

Finally, substituting (11) and (12) into (10.iii) and (10.v) we are led to the following relations,

$$
(13)\qquad \begin{cases} J = XG + \big(1 - X\big)\big(JM(J) + D(J)\big), \\ X = XR + \big(1 - X\big)\big(JL(J) + C(J)\big), \end{cases}
$$

where we have introduced the following notation:

$$
(14)\qquad \begin{aligned}
G &= b_0 \int_0^{a_\dagger} h(a)\pi(a)\Gamma(a)\, da, \\
R &= q \int_0^{a_\dagger} \beta(a)\pi(a)\, \Gamma(a)\, da, \\
M(J) &= b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, F(a, J)\, da, \\
L(J) &= q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, F(a, J)\, da, \\
D(J) &= b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, H(a, J)\, da, \\
C(J) &= q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, H(a, J)\, da,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Gamma(a) = e^{-\int_0^a \gamma(\sigma)d\sigma}, \\
&F(a,J) = \int_0^a \kappa(\sigma) e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma N(s,J)\,ds} d\sigma, \\
&H(a,J) = \int_0^a g(\sigma) e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma N(s,J)\,ds} d\sigma, \\
&N(s,J) = J\kappa(s) + g(s).
\end{aligned}
\tag{15}
$$

We seek solutions of (13) such that $J \geq 0$ and $0 \leq X \leq 1$. (Note that $X = u(0)$ and $0 \leq u(0) \leq 1$.) In fact, any such a pair (X, J) provides a nonnegative solution of (10) via (11) and (12).

Note that if the following conditions are satisfied, then (13) reduces to a single equation with two unknowns J and X:

$$
R = 1 \quad \text{and} \quad L(J) = C(J) = 0 \quad \text{for all } J. \tag{16}
$$

From the definitions in (14) and (15), we can see that (16) is equivalent to the following:

$$
q = 1, \quad a_\beta^+ \leq a_\gamma^-, \quad a_\beta^+ \leq a_\kappa^-, \quad \text{and} \quad a_\beta^+ \leq a_g^-, \tag{17}
$$

where

$$
\begin{aligned}
a_\beta^+ &= \inf\{A : \beta(a) = 0 \text{ a.e. in } [A, a_\dagger)\}, \\
a_g^- &= \sup\{A : g(a) = 0 \text{ a.e. in } [0, A]\}, \\
a_\kappa^- &= \sup\{A : \kappa(a) = 0 \text{ a.e. in } [0, A]\}, \\
a_\gamma^- &= \sup\{A : \gamma(a) = 0 \text{ a.e. in } [0, A]\}.
\end{aligned}
$$

These results lead us to consider the following very special case [4].

Lemma 2.1. *Assume* (17) *holds. Then problem* (10) *has a continuum of nontrivial solutions.*

Thus if (17) is satisfied, (13) has infinitely many solutions and it is not desired to our model. Since we want to rule out such a pathological case, the following assumption is required in the rest of the paper.

(18) All the relations in (17) do not hold simultaneously.

Under the assumption (18) we can further reduce the system (13) to a single equation. In fact, solving the second equation for X we obtain

$$
X = \frac{JL(J) + C(J)}{1 - R + JL(J) + C(J)}, \tag{19}
$$

which, when substituted into the other equation yields:

$$
(1 - R)\big(J - JM(J) - D(J)\big) + \big(J - G\big)\big(JL(J) + C(J)\big) = 0. \tag{20}
$$

Thus, we need to study this equation. Note that any solution $J \geq 0$ of (20) provides a solution of (13) with $X \in [0, 1]$ given by (19).

In order to study equation (20), we will consider the continuous function

$$\phi(J) = (1-R)\big(J - JM(J) - D(J)\big) + (J-G)\big(JL(J) + C(J)\big) \tag{21}$$

and analyze its behavior in the interval $[0, \infty)$.

3. Uniqueness of an endemic state

To establish the uniqueness of an endemic state, we need the following extra assumption:

$$u^*(a_\dagger) < e^{-\int_0^{a_\dagger} \gamma(s)ds}. \tag{22}$$

Actually, this condition is analogous to the following one used in [12]:

$$u^*(a_\dagger) < e^{-\gamma a_\dagger}, \tag{23}$$

which was used to show stability of the solutions for the S-I-R model with $q = 0$ and a constant removal rate γ.

Since a function $\gamma(\cdot)$, rather than a constant, is used in our case, (22) is a natural generalization of (23). Although we are not dealing with stability but uniqueness in this paper, (22) will play a crucial role in proving uniqueness. Fortunately, from the fact that $a_\dagger$ is the maximum age one can reach, $u^*(a_\dagger)$ might be very small or even zero, which implies that in most cases condition (22) does hold.

The following three lemmas are needed to show the uniqueness. Among them, the first lemma is a direct copy of that in [5]. The lemma including the proof is shown here for the readers' convenience, though.

Lemma 3.1. *If* (22) *holds, then*

$$\int_0^a \gamma(\sigma) e^{\int_\sigma^a (N(s,J)-\gamma(s))ds} d\sigma < 1 \textit{ for all } a \in [0, a_\dagger]. \tag{24}$$

Proof. Note that $u(a)$ can be computed as follows.

$$u(a) = Xe^{-\int_0^a \gamma(\sigma)d\sigma} + (1-X)\int_0^a N(\sigma, J)e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma N(s,J)ds} d\sigma.$$

Hence

$$u(a_\dagger) = e^{-\int_0^{a_\dagger} \gamma(\sigma)d\sigma}\left[X + (1-X)\int_0^{a_\dagger} N(\sigma, J)e^{\int_0^\sigma (\gamma(s)-N(s,J))ds} d\sigma\right].$$

Thus (22) is equivalent to the following:

$$X + (1-X)\int_0^{a_\dagger} N(\sigma, J)e^{\int_0^\sigma (\gamma(s)-N(s,J))ds} d\sigma < 1. \tag{25}$$

But

$$\begin{aligned}
&\int_0^{a_\dagger} N(\sigma,J)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}d\sigma \\
= &\int_0^{a_\dagger} e^{\int_0^\sigma \gamma(s)ds}\frac{d}{d\sigma}\left(-e^{-\int_0^\sigma N(s,J)ds}\right)d\sigma \\
= &\ 1-e^{\int_0^{a_\dagger}(\gamma(s)-N(s,J))ds}+\int_0^{a_\dagger}\gamma(\sigma)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}d\sigma.
\end{aligned}$$

Therefore (25) is equivalent to

$$\int_0^{a_\dagger}\gamma(\sigma)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}d\sigma < e^{\int_0^{a_\dagger}(\gamma(s)-N(s,J))ds}. \tag{26}$$

(Note that $X \neq 1$ by (25).)

Hence, if we let

$$f(a)=\int_0^a \gamma(\sigma)e^{\int_\sigma^a(N(s,J)-\gamma(s))ds}d\sigma,$$

then $f(0)=0$ and $f(a_\dagger)<1$ by (26). Moreover, by a simple calculation, we have

$$f'(a)=\gamma(a)+\{N(a,J)-\gamma(a)\}f(a).$$

Now, either $N(a,J)-\gamma(a)\geq 0$ and consequently $f'(a)\geq 0$, or $N(a,J)-\gamma(a)<0$ and, in this case, if $0\leq f(a)<1$,

$$f'(a)\geq\gamma(a)+N(a,J)-\gamma(s)=N(a,J)\geq 0. \tag{27}$$

Thus, in both cases, $f(a)$ is a non-decreasing function whenever $0\leq f(a)<1$. This is enough to prove (24): in fact, $f(0)=0$ and $f(a_\dagger)<1$ so that if $f(a)\geq 1$ for some a, then there should be a point $a_0>a$ such that $f(a_0)<1$ but $f'(a_0)<0$. This contradicts (27). □

Lemma 3.2. *If* (22) *holds and* $\int_0^a \kappa(s)\,ds>0$ *for some fixed* $a>0$, *then the function*

$$W(J)=\int_0^a\gamma(\sigma)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}d\sigma-e^{\int_0^a(\gamma(s)-N(s,J))ds} \tag{28}$$

is a strictly increasing function of J.

Proof. First note that, by Lemma 3.1, (24) holds and it is equivalent to the following :

$$\int_0^a\gamma(\sigma)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}d\sigma<e^{\int_0^a(\gamma(s)-N(s,J))ds}.$$

Moreover, since $\int_0^a \kappa(s)\,ds > 0$,

$$\begin{aligned} &\int_0^a \gamma(\sigma)\left(\int_0^\sigma \kappa(s)ds\right) e^{\int_0^\sigma(\gamma(s)-N(s,J))ds} d\sigma \\ < &\left(\int_0^a \kappa(s)ds\right) e^{\int_0^a(\gamma(s)-N(s,J))ds}. \end{aligned}$$

Thus we have

$$\begin{aligned} W'(J) &= \left(\int_0^a \kappa(s)ds\right) e^{\int_0^a(\gamma(s)-N(s,J))ds} \\ &\quad -\int_0^a \gamma(\sigma)\left(\int_0^\sigma \kappa(s)ds\right) e^{\int_0^\sigma(\gamma(s)-N(s,J))ds} d\sigma \\ &> 0, \end{aligned}$$

which completes the proof. □

Lemma 3.3. *If* (22) *holds and* $q > 0$, *then* $JL(J)+C(J)$ *is a strictly increasing function of* J.

Proof. From the equations in (14), we have

$$\begin{aligned} JL(J)+C(J) &= q\int_0^{a_\dagger} \beta(a)\pi(a) \int_0^a N(\sigma,J)e^{-\int_\sigma^a \gamma(s)ds-\int_0^\sigma N(s,J)ds}\, d\sigma\, da \\ &= q\int_0^{a_\dagger} \beta(a)\pi(a)\Gamma(a)\left[-\int_0^a e^{\int_0^\sigma \gamma(s)ds}\frac{d}{d\sigma}e^{-\int_0^\sigma N(s,J)ds}\, d\sigma\right] da \\ &= q\int_0^{a_\dagger} \beta(a)\pi(a)\Gamma(a)\left[1-e^{\int_0^a(\gamma(s)-N(s,J))ds}\right. \\ &\qquad\qquad \left. +\int_0^a \gamma(\sigma)e^{\int_0^\sigma(\gamma(s)-N(s,J))ds}\, d\sigma\right] da. \end{aligned}$$

Now, from Lemma 3.2, together with the fact that $\kappa(\cdot)$ does not vanish identically, we can easily see that $JL(J)+C(J)$ is a strictly increasing function of J. □

In [4], it has been proved that the endemic state for our model always exists. In fact, the function ϕ defined in (21) satisfies the following:

$$\phi(0) \le 0 \text{ and } \lim_{J\to\infty} \phi(J) = \infty.$$

Furthermore, it also has been proved that the endemic state is unique if $q = 0$. (See [4] for more details.)

We need a few more remarks before stating the main theorem.

First, if

$$R = 1 \text{ and } C(\cdot) \equiv 0, \tag{29}$$

then $\phi(J) = J(J-G)L(J)$ has two distinct zeros, 0 and G. (Note that $L(\cdot)$ does not vanish identically in this case, by (18).) But (29) is too restrictive to hold.

Second, if

$$D(\cdot) \equiv 0 \text{ and } C(\cdot) \equiv 0, \tag{30}$$

then

$$\phi(J) = J\left[(1-R)\,(1-M(J)) + (J-G)L(J)\right], \tag{31}$$

so that finding zeros of (31) is equivalent to finding zeros of

$$\psi(J) = (1-R)\,(1-M(J)) + (J-G)L(J), \tag{32}$$

except for the trivial zero, $J=0$. Since the function ψ in (32), is nothing but a function from usual S-I-R model without external force, we are not interested in that case. (See [6] for more details and uniqueness results there.)

Thus we assume that neither (29) nor (30) holds in the rest of the paper, that is, we may assume that

$$\phi(0) = -(1-R)D(0) - GC(0) < 0. \tag{33}$$

Now we state our main theorem.

Theorem 3.4. *If* (22) *holds, then the endemic state is unique.*

Proof. Since the uniqueness result for $q=0$ is proved in [4], we may assume that $q>0$. Moreover, since $\phi(0)<0$ by (33) and $\lim_{J\to\infty}\phi(J)=\infty$ [4], it is enough to prove that the function ϕ is strictly increasing. First note that if $0<J<G$, finding zeros of

$$\phi(J) = J\left[(1-R)\left(1-M(J)-\frac{D(J)}{J}\right) - (G-J)\left(L(J)+\frac{C(J)}{J}\right)\right]$$

is equivalent to finding zeros of

$$\psi(J) = (1-R)\left(1-M(J)-\frac{D(J)}{J}\right) - (G-J)\left(L(J)+\frac{C(J)}{J}\right). \tag{34}$$

From (34), we can conclude that ψ is strictly increasing. In fact,

$$1-M(J)-\frac{D(J)}{J}$$

is increasing for all $J>0$, and

$$-(G-J)\left(L(J)+\frac{C(J)}{J}\right)$$

is negative, and the absolute value of it is decreasing for all $J\in(0,G)$.

Now, for $J\geq G$,

$$\phi(J) = (1-R)J\left(1-M(J)-\frac{D(J)}{J}\right) + (J-G)\,(JL(J)+C(J))$$

is again strictly increasing by Lemma 3.3.

Thus ϕ is strictly increasing for any $J > 0$, which completes the proof. □

4. Conclusions

In this article, we have considered an inter-cohort S-I-R epidemic model with external force of infection. The uniqueness has been shown for all $q \in [0,1]$ under the assumption that the following is true:

$$u^*(a_\dagger) < e^{-\int_0^{a_\dagger} \gamma(s)ds}. \tag{22}$$

The uniqueness result we have found is a generalization compared with that of [4] in which the uniqueness was shown for only $q = 0$.

Although (22) holds for almost all cases so that we may say that the uniqueness *is* proved, a uniqueness result without such an assumption is still open.

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