

HAUSDORFF DISTANCE BETWEEN THE OFFSET CURVE OF QUADRATIC BEZIER CURVE AND ITS QUADRATIC APPROXIMATION

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ABSTRACT. In this paper, we present the exact Hausdorff distance between the offset curve of quadratic Bézier curve and its quadratic GC^1 approximation. To illustrate the formula for the Hausdorff distance, we give an example of the quadratic GC^1 approximation of the offset curve of a quadratic Bézier curve.

1. Preliminaries

Quadratic Bézier curves and their offset curves are widely used in CAD/CAM or Computer Graphics. But offset curve of quadratic Bézier curve cannot be expressed in Bézier form. In the recent twenty years, many papers about the quadratic Bézier approximation [2, 7, 13] or the offset curve [6, 8, 10, 11, 15] have been published. Especially as the error measurement between the target curve and approximation curve, the Hausdorff distance is used in CAD/CAM or Approximation Theory. The Hausdorff distance $d_H(\mathbf{p}, \mathbf{q})$ between two curves $\mathbf{p}(s)$, $s \in [a, b]$ and $\mathbf{q}(t)$, $t \in [c, d]$, is given by

$$d_H(\mathbf{p}, \mathbf{q}) = \max\left\{ \max_{s \in [a, b]} \min_{t \in [c, d]} |\mathbf{p}(s) - \mathbf{q}(t)|, \max_{t \in [c, d]} \min_{s \in [a, b]} |\mathbf{p}(s) - \mathbf{q}(t)| \right\}.$$

(For more knowledge about the Hausdorff distance, refer to [1, 14, 12]) By the way, it is not easy to find the Hausdorff distance between planar curve and quadratic Bézier curve. The Hausdorff distance is obtained from the maximum distance between $\mathbf{p}(s_0)$ and $\mathbf{q}(t_0)$ satisfying

$$(1.1) \quad \mathbf{p}'(s_0) \parallel \mathbf{q}'(t_0) \quad \text{and} \quad \mathbf{p}'(s_0) \perp \overrightarrow{\mathbf{p}(s_0)\mathbf{q}(t_0)}$$

when they have the same end points, as shown in Figure 1. Thus to get the Hausdorff distance it requires to solve nonlinear system of two variables such as in Equation (1.1).

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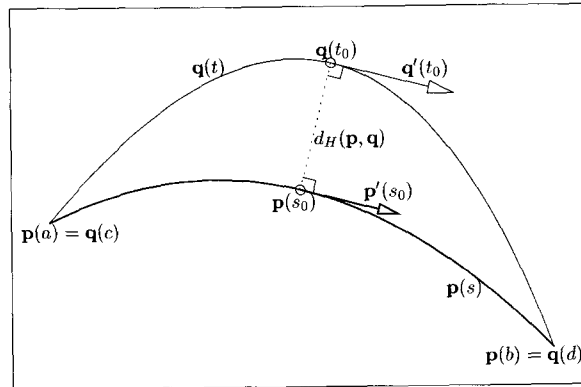


FIGURE 1. Hausdorff distance between two curves $\mathbf{p}(s)$, $s \in [a, b]$, (thick lines) and $\mathbf{q}(t)$, $t \in [c, d]$ (thin lines).

In this paper we find the formula for the Hausdorff distance between the offset of quadratic Bézier curve and the quadratic GC^1 Bézier approximation. Using the formula for the Hausdorff distance, we can easily obtain the error of the quadratic GC^1 Bézier approximation of the offset of quadratic Bézier curve.

2. Hausdorff distance between offset of quadratic Bézier curve and its quadratic approximation

In this section we present the error of the quadratic GC^1 (geometric continuity of order one) end-points interpolation of offset curve of quadratic Bézier curve.

Let $\mathbf{b}(t)$ be the quadratic Bézier curve with the control points \mathbf{b}_0 , \mathbf{b}_1 and \mathbf{b}_2 , i.e.,

$$\mathbf{b}(t) = \sum_{i=0}^2 B_i^n(t) \mathbf{b}_i \quad t \in [0, 1]$$

(refer to [9]) where $B_i^n(t)$ is the Bernstein polynomial of degree n , i.e.,

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

Let $T(t)$, $N(t)$ and $k(t)$ be the unit tangent, unit normal vector and curvature of $\mathbf{b}(t)$, in order, i.e.,

$$T(t) = \mathbf{b}'(t)/|\mathbf{b}'(t)|, \quad N(t) = T'(t)/|T'(t)|, \quad k(t) = |\mathbf{b}'(t) \times \mathbf{b}''(t)|/|\mathbf{b}'(t)|^3.$$

Let $N_i = N(i)$ for $i = 0, 1$. We denote the offset curve of $\mathbf{b}(t)$ for the offset distance $d \in \mathbb{R}$ by $\mathbf{b}^d(t)$, i.e.,

$$\mathbf{b}^d(t) = \mathbf{b}(t) + dN(t) \quad t \in [0, 1].$$

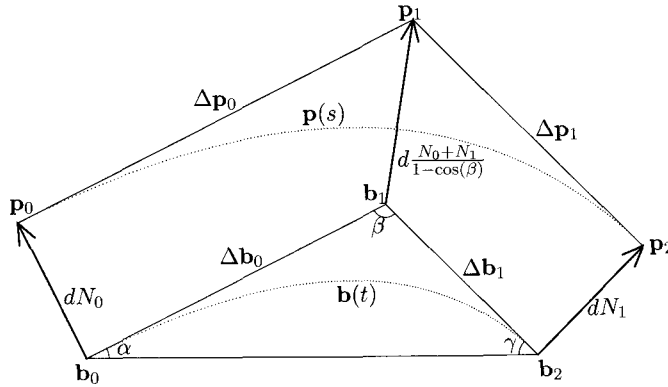


FIGURE 2. Quadratic GC^1 interpolation $\mathbf{p}(s)$ of offset of quadratic Bézier curve $\mathbf{b}(t)$ with offset distance d . The quadratic Bézier curves (dotted lines) $\mathbf{b}(t)$ and $\mathbf{p}(s)$ have the control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ and $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$, respectively.

We assume that the control points are not collinear. Note [10] that if

$$\min(1/k(t)) > d,$$

then offset has neither cusp nor loop and the same tangent direction with $\mathbf{b}(t)$ at end points. The necessary and sufficient condition that the quadratic Bézier curve has the local extreme of curvature is well-known [4].

In the following proposition the control polygon of the quadratic GC^1 end-points interpolation $\mathbf{p}(s) = \sum_{i=0}^2 B_i^2(s)\mathbf{p}_i$ of the offset curve $\mathbf{b}^d(t)$ is obtained explicitly.

Proposition 2.1. For $d < \min(1/k(t))$, the quadratic Bézier curve $p(t)$ having the control points

$$\begin{aligned} \mathbf{p}_0 &= \mathbf{b}_0 + d N_0 \\ \mathbf{p}_1 &= \mathbf{b}_1 + d \frac{N_0 + N_1}{1 + N_0 \circ N_1} \\ \mathbf{p}_2 &= \mathbf{b}_2 + d N_1 \end{aligned}$$

is GC^1 end-points interpolation of the offset curve $\mathbf{b}^d(t)$.

Proof. It suffices to show that two quadratic Bézier curve $\mathbf{p}(t)$ and $\mathbf{b}(t)$ have the same tangent direction at both end points. For simplification, let $\Delta\mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$, as shown in Figure 2.

$$\Delta\mathbf{p}_0 = \mathbf{p}_1 - \mathbf{p}_0 = \Delta\mathbf{b}_0 + \frac{d}{1 + N_0 \circ N_1} (N_1 - N_0(N_0 \circ N_1)).$$

Since $N_1 - N_0(N_0 \circ N_1)$ is the same direction with $T(0)$, $\Delta \mathbf{p}_0$ and $\Delta \mathbf{b}_0$ have the same direction. Thus $\mathbf{p}(t)$ and $\mathbf{b}(t)$ have the same tangent direction at $t = 0$. By same reason, $\mathbf{p}(t)$ and $\mathbf{b}(t)$ also have the same tangent direction at $t = 1$. □

The angles in the triangle $\Delta \mathbf{b}_0 \mathbf{b}_1 \mathbf{b}_2$ are denoted by α, β and γ , in order, as shown in Figure 2. If $\beta = \pi/2$, then $\mathbf{p}_1 = \mathbf{b}_1 + d(N_0 + N_1)$, if $\beta \rightarrow 0$ then $\mathbf{p}_1 \rightarrow \infty$, and if β tends to ∞ then \mathbf{p}_1 tends to $\mathbf{b}_1 + dN_0$.

Now, we will find all the solutions (t, s) of the equations

$$(2.1) \quad \mathbf{b}'(t) \parallel \mathbf{p}'(s)$$

$$(2.2) \quad \mathbf{b}'(t) \perp \overrightarrow{\mathbf{p}(s)\mathbf{b}(t)}.$$

Let $\mathbf{b}(t) = (x(t), y(t))$, $\mathbf{p}(s) = (u(s), v(s))$. Equation (2.1) yields

$$u'(s)y'(t) = v'(s)x'(t).$$

Using $\Delta u_0 \Delta y_0 = \Delta v_0 \Delta x_0$ and $\Delta u_1 \Delta y_1 = \Delta v_1 \Delta x_1$, we have the simple equation

$$s = \frac{(\Delta u_0 \Delta y_1 - \Delta v_0 \Delta x_1)t}{(-\Delta u_1 \Delta y_0 + \Delta v_1 \Delta x_0)(1-t) + (\Delta u_0 \Delta y_1 - \Delta v_0 \Delta x_1)t}.$$

Let $E_1 = \Delta u_0 \Delta y_1 - \Delta v_0 \Delta x_1$ and $E_2 = \Delta u_0 \Delta y_1 - \Delta v_0 \Delta x_1$. Then $s = s(t) = \frac{E_1 t}{E_2(1-t) + E_1 t}$, $s(0) = 0$ and $s(1) = 1$. Equation (2.2) yields

$$\begin{aligned} (u(s) - x(t), v(s) - y(t)) \circ (x'(t), y'(t)) &= 0 \quad \text{or} \\ x'(t)(u(s) - x(t)) + y'(t)(v(s) - y(t)) &= 0. \end{aligned}$$

By $1 - s(t) = \frac{E_2(1-t)}{E_2(1-t) + E_1 t}$, we have

$$u(s(t)) = \frac{E_2^2(1-t)^2 u_0 + 2E_1 E_2 t(1-t)u_1 + E_1^2 t^2 u_2}{(E_2(1-t) + E_1 t)^2}$$

and

$$\begin{aligned} u(s(t)) - x(t) &= \frac{1}{(E_2(1-t) + E_1 t)^2} (E_2^2(1-t)^2 u_0 \\ &\quad + 2E_1 E_2 t(1-t)u_1 + E_1^2 t^2 u_2 - (E_2(1-t) + E_1 t)^2 x(t)). \end{aligned}$$

The dominator in the last equation is a quartic polynomial, so that it can be expressed in quartic Bézier form as follows.

$$\begin{aligned} u(s(t)) - x(t) &= \frac{1}{(E_2(1-t) + E_1 t)^2} [E_2^2(u_0 - x_0)B_0^4(t) + F_1 B_1^4(t) \\ &\quad + F_2 B_2^4(t) + F_3 B_3^4(t) + E_1^2(u_2 - x_2)B_4^4(t)], \end{aligned}$$

where

$$\begin{aligned}
 F_1 &= \frac{1}{2}E_2(E_2(u_0 - x_1) + E_1(u_1 - x_0)) \\
 (2.3) \quad F_2 &= \frac{1}{6}(E_2^2(u_0 - x_2) + 4E_1E_2(u_1 - x_1) + E_1^2(u_2 - x_0)) \\
 F_3 &= \frac{1}{2}E_1(E_1(u_2 - x_1) + E_2(u_1 - x_2)).
 \end{aligned}$$

Since $x'(t) = 2(1 - t)\Delta x_0 + 2t\Delta x_1$ and $y'(t) = 2(1 - t)\Delta y_0 + 2t\Delta y_1$, we obtain the following equation in quintic Bézier form

$$\begin{aligned}
 &x'(t)(u(s(t)) - x(t))(E_2(1 - t) + E_1t)^2 \\
 = &2\Delta x_0E_2^2(u_0 - x_0)B_0^5(t) + \frac{2}{5}[4\Delta x_0F_1 + \Delta x_1E_2^2(u_0 - x_0)]B_1^5(t) \\
 &+ \frac{1}{5}[6\Delta x_0F_2 + 4\Delta x_1F_1]B_2^5(t) + \frac{1}{5}[4\Delta x_0F_3 + 6\Delta x_1F_2]B_3^5(t) \\
 &+ \frac{2}{5}[\Delta x_0E_1^2(u_2 - x_2) + 4\Delta x_1F_3]B_4^5(t) + 2\Delta x_1E_1^2(u_2 - x_2)B_5^5(t).
 \end{aligned}$$

By same way, we also get the following equation

$$\begin{aligned}
 &y'(t)(v(s(t)) - y(t))(E_2(1 - t) + E_1t)^2 \\
 = &2\Delta y_0E_2^2(v_0 - y_0)B_0^5(t) + \frac{2}{5}[4\Delta y_0G_1 + \Delta y_1E_2^2(v_0 - y_0)]B_1^5(t) \\
 &+ \frac{1}{5}[6\Delta y_0G_2 + 4\Delta y_1G_1]B_2^5(t) + \frac{1}{5}[4\Delta y_0G_3 + 6\Delta y_1G_2]B_3^5(t) \\
 &+ \frac{2}{5}[\Delta y_0E_1^2(v_2 - y_2) + 4\Delta y_1G_3]B_4^5(t) + 2\Delta y_1E_1^2(v_2 - y_2)B_5^5(t),
 \end{aligned}$$

where

$$\begin{aligned}
 G_1 &= \frac{1}{2}E_2(E_2(v_0 - y_1) + E_1(v_1 - y_0)) \\
 (2.4) \quad G_2 &= \frac{1}{6}(E_2^2(v_0 - y_2) + 4E_1E_2(v_1 - y_1) + E_1^2(v_2 - y_0)) \\
 G_3 &= \frac{1}{2}E_1(E_1(v_2 - y_1) + E_2(v_1 - y_2)).
 \end{aligned}$$

Since $\Delta x_0(u_0 - x_0) + \Delta y_0(v_0 - y_0) = \Delta x_1(u_2 - x_2) + \Delta y_1(v_2 - y_2) = 0$, the coefficients combined with $B_0^5(t)$ and $B_5^5(t)$ of $x'(t)(u(s(t)) - x(t)) + y'(t)(v(s(t)) - y(t))$ are zero. Thus it has the factor $t(1 - t)$ so that

$$x'(t)(u(s(t)) - x(t)) + y'(t)(v(s(t)) - y(t)) = 2t(1 - t)\phi(t),$$

where $\phi(t)$ is a cubic polynomial in Bézier form

$$\begin{aligned} \phi(t) &= (4\Delta x_0 F_1 + \Delta x_1 E_2^2(u_0 - x_0) + 4\Delta y_0 G_1 + \Delta y_1 E_2^2(v_0 - y_0))B_0^3(t) \\ &\quad + \frac{1}{3}(6\Delta x_0 F_2 + 4\Delta x_1 F_1 + 6\Delta y_0 G_2 + 4\Delta y_1 G_1)B_1^3(t) \\ &\quad + \frac{1}{3}(4\Delta x_0 F_3 + 6\Delta x_1 F_2 + 4\Delta y_0 G_3 + 6\Delta y_1 G_2)B_2^3(t) \\ &\quad + (\Delta x_0 E_1^2(u_2 - x_2) + 4\Delta x_1 F_3 + \Delta y_0 E_1^2(v_2 - y_2) + 4\Delta y_1 G_3)B_3^3(t). \end{aligned}$$

Proposition 2.2. For $d < \min(1/k(t))$, the Hausdorff distance between two curves $\mathbf{b}^d(t)$ and $\mathbf{p}(s)$ is equal to

$$d_H(\mathbf{b}^d, \mathbf{p}) = \max_{0 < t_i < 1} |\mathbf{b}^d(t_i) - \mathbf{p}(s_i)|,$$

where $t_i \in (0, 1)$, $i = 1, 2, 3$, is the solution of the cubic equation $\phi(t) = 0$ and $s_i = \frac{E_1 t_i}{E_2(1 - t_i) + E_1 t_i} \in (0, 1)$.

Proof. Since for all $t \in [0, 1]$

$$\mathbf{b}'(t) \parallel (\mathbf{b}^d)'(t), \quad \mathbf{b}'(t) \perp (\mathbf{b}^d(t) - \mathbf{b}(t)),$$

(t, s) satisfy Equations (2.1) - (2.2) if and only if (t, s) satisfy

$$\mathbf{p}'(s) \parallel (\mathbf{b}^d)'(t) \quad \text{and} \quad \mathbf{p}'(s) \perp (\mathbf{p}(s) - \mathbf{b}^d(t)).$$

Also, the points $\mathbf{b}(t)$, $\mathbf{b}^d(t)$ and $\mathbf{p}(s)$ are collinear. Thus the Hausdorff distance $d_H(\mathbf{b}^d, \mathbf{p})$ is obtained from the maximum distance between $\mathbf{p}(s)$ and $\mathbf{b}^d(t)$ satisfying Equations (2.1)-(2.2). Since the solutions of Equations (2.1)-(2.2) satisfy the cubic equation $\phi(t) = 0$ and $s = E_1 t / (E_2(1 - t) + E_1 t)$, the assertion follows. \square

3. Example

In this section we present an example for the GC^1 quadratic Bézier interpolation of the offset curve of the quadratic Bézier curve $\mathbf{b}(t)$ (thick lines in Figure 3) given by the control points

$$\mathbf{b}_0 = (0, 0), \quad \mathbf{b}_1 = (2, 1), \quad \mathbf{b}_2 = (2, 0)$$

with offset distance $d = -1$. By Proposition 2.1, the quadratic Bézier approximation $\mathbf{p}(s)$ (dotted lines) has the control points

$$\mathbf{p}_0 = (-1/\sqrt{5}, 2/\sqrt{5}), \quad \mathbf{p}_1 = \left(2 + \frac{-2\sqrt{5} + 5\sqrt{2}}{10 + \sqrt{10}}, 1 + \frac{4\sqrt{5} + 5\sqrt{2}}{10 + \sqrt{10}}\right),$$

$$\mathbf{p}_2 = (2 + 1/\sqrt{2}, 1/\sqrt{2})$$

as shown in Figure 3.

$$E_1 = -3\left(1 + \frac{3\sqrt{2}}{10 + \sqrt{10}}\right) = -3.967, \quad E_2 = -3\left(1 + \frac{3\sqrt{5}}{10 + \sqrt{10}}\right) = -4.529$$

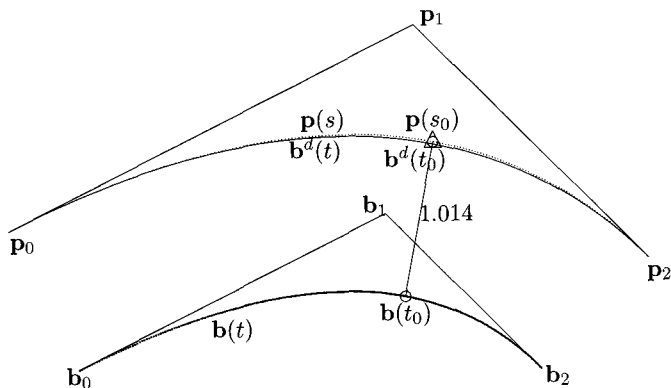


FIGURE 3. The distance between two points $\mathbf{p}(s_0)$ and $\mathbf{b}^d(t_0)$ (triangle) is 0.014, which means that $d_H(\mathbf{p}, \mathbf{b}^d) = 0.014$

$$s(t) = \frac{(3\sqrt{2} + 10 + \sqrt{10})B_1^1(t)}{(3\sqrt{5} + 10 + \sqrt{10})B_0^1(t) + (3\sqrt{2} + 10 + \sqrt{10})B_1^1(t)}$$

$$= \frac{-3.967B_1^1(t)}{-4.529B_0^1(t) + -3.967B_1^1(t)}$$

as shown in Figure 4. By Equations (2.3)-(2.4), we have

$$F_1 = -5.358, \quad F_2 = .3036, \quad F_3 = 6.223$$

$$G_1 = 18.83, \quad G_2 = 19.49, \quad G_3 = 17.61$$

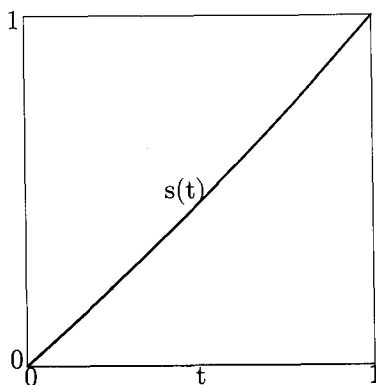


FIGURE 4. $s(t)$ (thick lines).

and the cubic polynomial in Bézier form given by

$$\phi(t) = 9.883B_0^3(t) + 17.96B_1^3(t) - 55.32B_2^3(t) + 3.158B_3^3(t)$$

which has zeros at $t = -.2889, .6306$ and 17.17 . The zero of $\phi(t)$ at $t_0 = .6306$ is the unique solution of $\phi(t) = 0$ inside the open interval $(0, 1)$ combined with $s_0 = s(.6306) = .5993$. Since distance between two points $\mathbf{b}(.6306)$ (small circle) and $\mathbf{p}(.5993)$ (small circle) is 1.014 as shown in Figure 3, finally we obtain the Hausdorff distance between two curve $\mathbf{b}^d(t)$ and its approximation curve $\mathbf{p}(s)$

$$d_H(\mathbf{b}^d, \mathbf{p}) = 0.014.$$

References

- [1] Y. J. Ahn, *Conic Approximation of planar curves*, Comp. Aided Desi. **33** (2001), no. 12, 867–872.
- [2] ———, *Helix approximation with conic and quadratic Bézier curves*, Comp. Aided Geom. Desi. **22** (2005), no. 6, 551–565.
- [3] Y. J. Ahn and H. O. Kim, *Approximation of circular arcs by Bézier curves*, J. Comp. Appl. Math. **81** (1997), 145–163.
- [4] ———, *Curvatures of the quadratic rational Bézier curves*, Comp. Math. Appl. **36** (1998), no. 9, 71–83.
- [5] Y. J. Ahn, H. O. Kim, and K. Y. Lee, *G^1 arc spline approximation of quadratic Bézier curves*, Comp. Aided Desi. **30** (1998), no. 8, 615–620.
- [6] Y. J. Ahn, Y. S. Kim, and Y. S. Shin, *Approximation of circular arcs and offset curves by Bezier curves of high degree*, J. Comp. Appl. Math. **167** (2004), 181–191.
- [7] G. Elber, I. K. Lee, and M. S. Kim, *Comparing offset curve approximation method*, IEEE Comp. Grap. Appl. **17** (1997), no. 3, 62–71.
- [8] G. Farin, *Curvature continuity and offsets for piecewise conics*, ACM Tran. Grap. (1989), 89–99.
- [9] ———, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, San Diego, CA, 1993.
- [10] R. T. Farouki, *Conic Approximation of Conic Offsets*, J. Symb. Comp. **23** (1997), 301–313.
- [11] R. T. Farouki and J. K. Johnstone, *The bisector of a point and a plane parametric curve*, Comp. Aided Geom. Desi. **11** (1994), no. 2, 117–151.
- [12] R. T. Farouki and C. A. Neff, *Analytic properties of plane offset curves*, Comp. Aided Geom. Desi. **7** (1990), 83–99.
- [13] R. T. Farouki and T. W. Sederberg, *Analysis of the offset to a parabola*, Comp. Aided Geom. Desi. **8** (1995), 639–645.
- [14] S. H. Kim and Y. J. Ahn, *Approximation of circular arcs by quartic Bezier curves*, Comp. Aided Desi. **39** (2007), no. 6, 490–493.
- [15] I. K. Lee, M. S. Kim, and G. Elber, *Planar curve offset based on circle approximation*, Comp. Aided Desi. **28** (1996), 617–630.

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