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HOLOMORPHIC MAPS ONTO KÄHLER MANIFOLDS WITH NON-NEGATIVE KODAIRA DIMENSION

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ABSTRACT. This paper studies the deformation theory of a holomorphic surjective map from a normal compact complex space X to a compact Kähler manifold Y. We will show that when the target has non-negative Kodaira dimension, all deformations of surjective holomorphic maps $X \to Y$ come from automorphisms of an unramified covering of Y and the underlying reduced varieties of associated components of $\mathrm{Hol}(X,Y)$ are complex tori. Under the additional assumption that Y is projective algebraic, this was proved in [7]. The proof in [7] uses the algebraicity in an essential way and cannot be generalized directly to the Kähler setting. A new ingredient here is a careful study of the infinitesimal deformation of orbits of an action of a complex torus. This study, combined with the result for the algebraic case, gives the proof for the Kähler setting.

1. Introduction

This paper studies the deformation theory of a holomorphic surjective map from a normal compact complex space to a compact Kähler manifold. For a holomorphic map $f : X \to Y$ between two compact complex spaces, denote by $\mathrm{Hol}(X,Y)$ the space of holomorphic maps from X to Y and by $\mathrm{Hol}_f(X,Y)$ the connected component of $\mathrm{Hol}(X,Y)$ containing the point corresponding to f. Let $\mathrm{Hol}(X,Y)_{red}$ and $\mathrm{Hol}_f(X,Y)_{red}$ be the underlying reduced varieties. We prove:

1.1. Theorem. *Let $f : X \to Y$ be a surjective map between a normal compact complex space X and a compact Kähler manifold Y of nonnegative Kodaira dimension. Then there exists a factorization*

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y, \qquad f = \beta \circ \alpha$$

where

(1) *β is a finite unramified covering and*

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(2) *if* $\mathrm{Aut}^0(Z)$ *is the maximal connected subgroup of the automorphism group of* Z*, then* $\mathrm{Aut}^0(Z)$ *is a complex torus, and the natural morphism*

$$\mathrm{Aut}^0(Z)\Big/\mathrm{Aut}(Z/Y)\cap\mathrm{Aut}^0(Z)\to\mathrm{Hol}_f(X,Y)_{red}$$

is isomorphic.

In particular, all components of $\mathrm{Hol}(X,Y)_{red}$ *corresponding to surjective maps are biholomorphic to complex tori.*

This answers the question raised in [6, p. 769] and [7, Remark 1.4] affirmatively, modulo the unobstructedness of infinitesimal deformations. Our approach does not shed much light on the unobstructedness question.

It is natural to ask whether a similar statement holds for any compact complex manifold Y of nonnegative Kodaira dimension. It is very likely that this holds at least for Y of Fujiki's class $\mathcal{C}$, namely, for Y bimeromorphic to a Kähler manifold. But our argument in this paper uses the Kähler assumption crucially and cannot be generalized to Fujiki's calss $\mathcal{C}$. It is not known whether an analogous statement holds even for a Moishezon manifold Y.

Recall that the tangent space to $\mathrm{Hol}(X,Y)$ at f is $H^0(X, f^*T_Y)$. We say that an element $\sigma\in H^0(X, f^*T_Y)$ is *unobstructed* if there exists a deformation of f whose infinitesimal deformation is σ. For simplicity, we say that a surjective holomorphic map $f: X\to Y$ from a normal compact complex space X to a compact complex manifold Y is *essentially rigid* if there exists Z factorizing f as in (1.1) such that for each unobstructed $\sigma\in H^0(X, f^*T_Y)$,

$$\sigma\in\alpha^*H^0(Z,T_Z).$$

We can reformulate Theorem 1.1:

1.2. Theorem. *Let* X *be a normal compact complex space and* Y *be a compact Kähler manifold of non-negative Kodaira dimension. Then* f *is essentially rigid.*

Note that it suffices to prove Theorem 1.2 for each unobstructed $\sigma\in H^0(X, f^*T_Y)$: there exists an unramified holomorphic covering $h: Z\to Y$ with a holomorphic map $g: X\to Z$ satisfying $f = h\circ g$, such that

$$\sigma\in g^*H^0(Z,h^*T_Y) = g^*H^0(Z,T_Z).$$

Here of course Z and h depend on σ, but it is clear by taking fiber products that one can choose Z and h independent from σ.

In case X and Y are projective, Theorem 1.1, and equivalently, Theorem 1.2, was proved in [7]. The proof of Theorem 1.2 in the projective setting in [7] depends on Miyaoka's semi-positivity theorem whose proof in turn requires the use of characteristic $p>0$ method. So the proof in [7] cannot be generalized directly to compact Kähler manifolds.

Theorem 1.2 was proved for a compact Kähler manifold Y with trivial canonical class in [6]. The proof in [6], which generalizes the earlier work of [8], uses the differential geometry of a Ricci-flat metric. The proof in [6] seems difficult to generalize to prove Theorem 1.2, because the non-negativity of Kodaira dimension alone is too weak to give a nice Kähler metric.

The key idea of this paper is that the following weaker version of Theorem 1.2 can be proved without characteristic $p > 0$ or differential geometric techniques.

1.3. Theorem. *Let X be a normal compact complex space and Y be a compact Kähler manifold of non-negative Kodaira dimension. Given a surjective holomorphic map $f : X \to Y$ and a section*

$$\sigma \in df(H^0(X, T_X)) \subset H^0(X, f^*T_Y),$$

there exists an unramified holomorphic covering $h : Z \to Y$ with a holomorphic map $g : X \to Z$ satisfying $f = h \circ g$, such that

$$\sigma \in g^*H^0(Z, h^*T_Y) = g^*H^0(Z, T_Z).$$

Theorem 1.3 will be proved in Section 3. The main idea is to use the infinitesimal deformation of holomorphic maps from a complex torus to Y arising from the section σ and f. The unramified covering is constructed by showing that an essential part of the space of these holomorphic maps is smooth.

Once Theorem 1.3 is established, we will show that Theorem 1.2 in the projective case and Theorem 1.3 implies Theorem 1.2 (Section 4). In this sense, the proof of Theorem 1.1 does depend on the method of characteristic $p > 0$.

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2. Preliminaries

2.1. Proposition. *Let $f : X \to Y$ be a surjective holomorphic map between a normal compact complex space X and a compact Kähler manifold Y. Then there exists a factorization*

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y, \quad f = \beta \circ \alpha$$

where β is a finite unramified covering such that any factorization

$$X \xrightarrow{\alpha'} Z' \xrightarrow{\beta'} Y, \quad f = \beta' \circ \alpha',$$

with unramified β' factors through a finite unramified map $Z \to Z'$.

Proof. It is easy to see that given two factorizations

$$X \xrightarrow{\alpha_1} Z_1 \xrightarrow{\beta_1} Y \quad \text{and} \quad X \xrightarrow{\alpha_2} Z_2 \xrightarrow{\beta_2} Y \quad (f = \beta_1\circ\alpha_1 = \beta_2\circ\alpha_2)$$

with unramified β_1 and β_2, there exists another factorization

$$X \xrightarrow[\alpha_3]{} Z_3 \xrightarrow[\beta_3]{} Y, \quad f = \beta_3 \circ \alpha_3$$

with unramified β_3 and unramified maps $\gamma_1 : Z_3 \to Z_1$ and $\gamma_2 : Z_3 \to Z_2$ such that

$$\beta_1 \circ \gamma_1 = \beta_2 \circ \gamma_2 = \beta_3.$$

A repeated application of this shows the existence of the factorization Z with the desired universal property. □

It is clear that the factorization

$$X \xrightarrow[\alpha]{} Z \xrightarrow[\beta]{} Y, \quad f = \beta \circ \alpha$$

of Proposition 2.1 is uniquely determined up to automorphisms. This is called *the maximal unramified fatorization* of f. When X, Y are algebraic varieties, this concept was introduced in [9]. The proof of [9, Theorem 1.7], works verbatim to give the following.

2.2. Proposition. *In the notation of Proposition 2.1, there exists a natural proper surjective holomorphic map* $\mathrm{Hol}_\alpha(X, Z)_{red} \to \mathrm{Hol}_f(X, Y)_{red}$.

An immediate consequence is the following proposition.

2.3. Proposition. *A surjective holomorphic map $f : X \to Y$ from a normal compact complex space to a compact Kähler manifold is essentially rigid if there exists a factorization*

$$X \xrightarrow[\alpha]{} Z \xrightarrow[\beta]{} Y, \quad f = \beta \circ \alpha$$

with β unramified and α essentially rigid.

2.4. Proposition. *Let Y be a compact Kähler manifold and $q : \tilde{Y} \to Y$ be a finite unramified covering. Suppose all surjective holomorphic maps from normal compact complex spaces onto $\tilde{Y}$ are essentially rigid. Then so are all surjective holomorphic maps onto Y.*

Proof. Given a surjective holomorphic map $f : X \to Y$, the fiber product gives a surjective holomorphic map $\tilde{f} : \tilde{X} \to \tilde{Y}$ and an unramified map $p : \tilde{X} \to X$. Since $\tilde{f}$ is essentially rigid by assumption, $q \circ \tilde{f} = f \circ p$ is essentially rigid by Proposition 2.3. This implies that f is essentially rigid. □

We need to recall a few results about actions of complex tori on compact Kähler manifolds.

2.5. Proposition. *Let Y be a compact Kähler manifold with a complex torus G acting effectively as a group of holomorphic transformations. Then there exists a compact Kähler manifold $\tilde{Y}$ with a free G-action and a G-equivariant finite unramified covering $\tilde{Y} \to Y$.*

Proof. Let $\alpha : Y \to A := \mathrm{Alb}(Y)$ be the Albanese map. By [3, Theorem 5.5], the induced group homomorphism $G \to \mathrm{Aut}(A)$ has finite kernel G'. This renders A a structure of G/G'-principal bundle. In particular, there exists a finite unramified covering $\tilde{A} \to A$ with a G-principal bundle structure. Let $\tilde{Y} \to Y$ be the fiber product induced from $\tilde{A} \to A$. There exists a natural G-action on $\tilde{Y}$, which is free because there exists a G-equivariant map $\tilde{Y} \to \tilde{A}$ and the G-action on $\tilde{A}$ is free. □

2.6. Proposition. *Let Y be a compact Kähler manifold with a complex torus G acting freely on Y as a group of holomorphic transformations. Then there exists a compact Kähler manifold Y/G and a submersion $q : Y \to Y/G$ which is a G-principal fiber bundle. In particular, Y and Y/G have the same Kodaira dimension.*

The proof is found in [4, Satz 21]. We call $q : Y \to Y/G$ the G-quotient of Y.

2.7. Proposition. *In the situation of Proposition 2.6, let $Z = Y/G$. Suppose furthermore that there is an unramified covering $s : \tilde{Z} \to Z$ and set $\tilde{Y} = Y \times_Z \tilde{Z}$ with projection $\tilde{q} : \tilde{Y} \to \tilde{Z}$. Then we have an exact sequence*

$$0 \to H^0(\tilde{Y}, T_{\tilde{Y}/\tilde{Z}}) \to H^0(\tilde{Y}, T_{\tilde{Y}}) \to H^0(\tilde{Y}, \tilde{q}^* T_{\tilde{Z}}) \to 0.$$

Proof. It suffices to notice that the connecting homomorphism, i.e., the Kodaira-Spencer map,

$$H^0(Y, \tilde{q}^* T_{\tilde{Z}}) \to H^1(\tilde{Y}, T_{\tilde{Y}/\tilde{Z}})$$

vanishes, since fibers of $\tilde{q}$ are biholomorphic to G. □

2.8. Proposition. *Let $q : Y \to Z$ be a G-quotient of Y as in Proposition 2.6. Then a surjective holomorphic map $f : X \to Y$ is essentially rigid if $q \circ f : X \to Y/G$ is essentially rigid.*

Proof. Since $p := q \circ f$ is essentially rigid, there is an unramified covering $s : \tilde{Z} \to Z$ and a factorization

$$X \xrightarrow{r} \tilde{Z} \xrightarrow{s} Z, \qquad p = s \circ r$$

such that all unobstructed elements of $H^0(X, p^* T_Z)$ belong to $r^* H^0(\tilde{Z}, T_{\tilde{Z}})$. Let $\tilde{Y} = Y \times_Z \tilde{Z}$ with projection $\tilde{q} : \tilde{Y} \to \tilde{Z}$. Let $\tilde{f} : X \to \tilde{Y}$ be the canonical map. Consider the following diagram where the exactness of the first row comes from Proposition 2.7.

$$\begin{array}{ccccccc}
H^0(\tilde{Y}, T_{\tilde{Y}/\tilde{Z}}) & \hookrightarrow & H^0(\tilde{Y}, T_{\tilde{Y}}) & \to & H^0(\tilde{Y}, \tilde{q}^*T_{\tilde{Z}}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(X, \tilde{f}^*T_{\tilde{Y}/\tilde{Z}}) & \hookrightarrow & H^0(X, \tilde{f}^*T_{\tilde{Y}}) & \to & H^0(X, \tilde{f}^*\tilde{q}^*T_{\tilde{Z}}) & \to & H^1(X, \tilde{f}^*T_{\tilde{Y}/\tilde{Z}})
\end{array}$$

Note that $T_{Y/Z}$ is spanned by the G-vector fields and therefore is trivial. Thus $T_{\tilde{Y}/\tilde{Z}}$ is trivial and the first vertical arrow is an isomorphism. Given an unobstructed element $\sigma \in H^0(X, \tilde{f}^*T_{\tilde{Y}})$, its image in $H^0(X, \tilde{f}^*\tilde{q}^*T_{\tilde{Z}})$ is unobstructed, too. Thus by the assumption that $q \circ f$ is essentially rigid, there exists an element of $H^0(\tilde{Y}, T_{\tilde{Y}})$ whose image $\sigma' \in H^0(X, \tilde{f}^*T_{\tilde{Y}})$ satisfies $\sigma - \sigma' \in H^0(X, \tilde{f}^*T_{\tilde{Y}/\tilde{Z}})$. This shows that σ is in the image of $H^0(\tilde{Y}, T_{\tilde{Y}})$. □

We will need the following two results of C. Horst. Proposition 2.9 is [5, Theorem 0.2.1] and Proposition 2.10 is [5, Corollary 5.1.1]. Note that in these two propositions, the factorizing map h is *not* required to be unramified, which is the essential difference between [5] and our Theorem 1.3.

2.9. Proposition. *Let $f : X \to Y$ be a finite surjective map between a normal compact complex space X and a compact complex manifold Y of nonnegative Kordaira dimension. Let $Z \subset \mathrm{Hol}_s(X, Y)$ be a compact subvariety of the space of surjective holomorphic maps. Then there exists a factorization*

$$X \xrightarrow{k} Y' \xrightarrow{h} Y, \quad f = h \circ k$$

such that

$$Z \subset h \circ \mathrm{Aut}^0(Y') \circ k.$$

The second proposition replaces the compactness assumption by a condition on the branch locus.

2.10. Proposition. *Let $f : X \to Y$ be a finite surjective map between a normal compact complex space X and a compact complex manifold Y. Let $Z \subset \mathrm{Hol}_f(X, Y)$ be an irreducible closed subvariety containing $[f]$. For each $g \in Z$, denote by $B_g \subset Y$ the branch locus of g. Suppose that B_g and $g^{-1}(B_g)$ are independent of g for all $g \in Z$. Then there exists a factorization*

$$X \xrightarrow{k} Y' \xrightarrow{h} Y, \quad f = h \circ k$$

such that

$$Z \subset h \circ \mathrm{Aut}^0(Y') \circ k.$$

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. So we consider the differential

$$df : T_X \to f^*T_Y$$

and the associated map

$$df : H^0(X, T_X) \to H^0(Y, f^*T_Y).$$

We consider

$$0 \neq v \in H^0(f^*T_Y)$$

and assume that v is of the form

$$v = df(v')$$

with

$$v' \in H^0(X, T_X).$$

By integration the automorphism group of X is positive-dimensional, and since X is not uniruled, the identity component is a torus. First we show

3.1. Proposition. *Let X be a normal compact complex variety and G be a complex torus acting on X effectively by a holomorphic map*

$$\Phi : X \times G \to X.$$

Let Y be a compact complex manifold and $f : X \to Y$ be a finite surjective holomorphic map. Denote by $F : X \times G \to Y$ the composition of Φ and f. For $x \in X$, let

$$F_x : G \to Y$$

be the map defined by $F_x(g) := F(x, g)$ for $g \in G$. Assume that there exists a subvariety E of codimension ≥ 2 in X which is preserved by G, namely, $\Phi(E \times G) = E$, such that for each $x \in X \setminus E$, the pull-back $F_x^*T_Y$ is a trivial vector bundle on G.

Then there exists an unramified covering $Y_1 \to Y$ factoring f such that the G-action descends to Y_1.

Proof. Note that $F(E \times G) = f(E)$ is of codimension ≥ 2 in Y. We have a natural holomorphic map

$$\chi : X \to \mathrm{Hol}(G, Y)$$

defined by $\chi(x) = F_x$. Let $B \subset \mathrm{Hol}(G, Y)$ be the image $\chi(X)$. For the identity element $e \in G$, the map $\Phi(\cdot, e) : X \to X$ is the identity map of X. Thus when $b = \chi(x)$, we have

$$b(e) = f \circ \Phi(x, e) = f(x)$$

and

$$\{b(e) \in Y, b \in B\} = \{f(x), x \in X\} = Y.$$

This implies that the irreducible variety B has dimension $n = \dim Y$.

For $x \in X \setminus E$, let $b := \chi(x) : G \to Y$. Then $b^*T_Y = F_x^*T_Y$ which is a trivial bundle by the assumption. It follows that the tangent space (e.g. [10, I.2.16]) to $\mathrm{Hol}(G, Y)$ at b, $H^0(G, b^*T_Y) = H^0(G, \mathcal{O}_G^n)$, has dimension n. Thus b is a smooth point of B and B must be an irreducible component of $\mathrm{Hol}(G, Y)$.

Now let $\Psi : B \times G \to Y$ be the restriction of the evaluation map

$$\mathrm{Hol}(G, Y) \times G \to Y.$$

Given $b \in B$ and $p \in G$, let

$$\phi(p, b) : T_{B,b} = H^0(G, b^*T_Y) \to b^*T_{Y,p}$$

be the evaluation at p. As in [10, II.3.4], the differential

$$d\Psi_{b,p} : T_{B,b} \times T_{G,p} \to T_{Y,y},\ y := \Psi(b, p),$$

is $\phi(p, b) + db(p)$. Suppose $b = \chi(x), x \in X \setminus E$, By the assumption on the triviality of b^*T_Y, we see that $\phi(p, b)$ is surjective. It follows that Ψ is submersive at b. (This part is analogous to [10, II.3.5]). In particular, the fiber $\Psi^{-1}(y)$ is smooth when $y \in Y \setminus f(E)$. Now let $j : B \times G \to Y_1$ and $h : Y_1 \to Y$ be the Stein factorization of Ψ. Then h must be unramified over $Y \setminus f(E)$, hence unramified everywhere. Since j has connected fibers, the natural G-action on $B \times G$ must descend to a G-action on Y_1. Composing

$$X \times \{e\} \subset X \times G \to B \times G \to Y_1 \to Y,$$

we get a factorization of $f : X \to Y$ with the desired properties. □

We have to verify that the technical assumption in Proposition 3.1 is indeed true:

3.2. Proposition. *Let X be a normal compact complex variety and G be a complex torus acting on X. Let Y be a compact complex manifold with non-negative Kodaira dimension and $f : X \to Y$ be a finite holomorphic map. Let Φ, F, F_x be as in the previous proposition. Then there exists a subvariety $E \subset X$ of codimension ≥ 2 which is preserved by G, such that for any $x \in X \setminus E$, $F_x^*T_Y$ is a trivial vector bundle on G.*

Proof. Since X is normal, we may put all singular points of X in our E. Thus we may consider only smooth points of X. Let $x \in X$ be a smooth point. Choose a small neighborhood U of x where T_X is a trivial bundle. Choose a holomorphic frame $v_1, \ldots, v_n$ of $T_X|_U$ and regard them as vector fields on $U \times G$ by the projection $U \times G \to U$. Let

$$u_i := dF(v_i) \in H^0(U \times G, F^*T_Y), i = 1, \ldots, n.$$

We will use these sections of F^*T_Y on $U \times G$ to show that $F_x^*T_Y$ is trivial for suitable choices of x.

First, we will show this for any smooth point $x \in X$ such that $\Phi(x, g) = g \cdot x \notin Ram(f)$ for some $g \in G$, where $Ram(f)$ denotes the underlying reduced divisor of the ramification of f. In the construction of the sections $u_1, \ldots, u_n$, we may assume that $g \cdot U$ is disjoint from $Ram(f)$ by shrinking U. Then $u_1, \dots, u_n$ will be pointwise independent at every point of $U \times \{g\}$. This implies that $u_1 \wedge \cdots \wedge u_n$ defines a section of $F^*K_Y^{-1}$ on $U \times G$ whose zero divisor Z is disjoint from $U \times \{g\}$. Since the Kodaira dimension of Y is non-negative, $F_z^*K_Y^{-1}$ cannot have a non-zero section with non-empty zero for general $z \in U$. Thus Z is disjoint from $\{z\} \times G$ for general $z \in U$. It follows that Z is empty and $u_1, \dots, u_n$ are pointwise independent everywhere on $U \times G$. Thus $u_1, ..., u_n$ give a trivialization of $F_x^*T_Y$.

Let $Ram(f)'$ be the complex analytic subset of $Ram(f)$ defined by

$$Ram(f)' := \{x \in Ram(f), g \cdot x \in Ram(f) \text{ for each } g \in G\}.$$

We have established that $F_x^*T_Y$ is trivial unless $x \in Ram(f)'$. If $Ram(f)'$ contains no component of codimension 1, then we are done by setting E to be the union of the singular locus of X and $Ram(f)'$. Assume that there exists a component R of codimension 1 in $Ram(f)'$. R is preserved by the G-action, namely, $\Phi(R \times G) = R$. Let $B = f(R) = F(R \times G)$ be the reduced irreducible divisor on Y.

Let x be a smooth point of R with the following conditions.

(1) f has rank $n-1$ at $g \cdot x$ for some $g \in G$.
(2) R is the only irreducible component of $f^{-1}(B)$ which contains the G-orbit $G \cdot x$.

Note that the set of points of R which does not satisfy (1) or (2) must be of codimension ≥ 2 and is preserved by G. Thus if we show $F_x^*T_Y$ is trivial for x as above, the proof of Proposition 3.2 is complete.

Consider the exact sequence of sheaves of differentials

$$0 \to \Omega^1_{B/Y} \longrightarrow \Omega^1_Y|_B \longrightarrow \Omega^1_B \to 0.$$

Since B is a reduced divisor in the complex manifold Y, $\Omega^1_{B/Y}$ is isomorphic to the invertible sheaf $\mathcal{O}_B(-B)$. The condition (2) guarantees that we can write on $U \times G$,

$$F^*(-B) = -m(R \times G) - H$$

as divisors where m is a positive integer and H is an effective divisor such that $\{x\} \times G$ is not contained in the support of H. Since $\mathcal{O}(R \times G)$ is a trivial invertible sheaf along $\{x\} \times G$,

$$F_x^*\mathcal{O}_B(-B) = F_x^*(-H).$$

This implies

$$\dim H^0(G, F_x^*\Omega^1_{B/Y}) = \dim H^0(\{x\} \times G, \mathcal{O}(F^*(-B))) \leq 1.$$

Pulling back the sequence of differentials by F_x, we get the exact sequence

$$0 \to H^0(G, F_x^*\Omega^1_{B/Y}) \longrightarrow H^0(G, F_x^*\Omega^1_Y) \longrightarrow H^0(G, F_x^*\Omega^1_B).$$

Note that when z is a general point of U, $\dim H^0(G, F_z^*\Omega_Y^1) = n$ because we established previously that $F_z^*T_Y$ is trivial. It follows from the upper-semi-continuity,

$$\dim H^0(G, F_x^*\Omega_B^1) \geq n-1.$$

Taking duals of the exact sequence of differentials, we get

$$0 \to (\Omega_B^1)^* \longrightarrow T_Y|_B \longrightarrow (\Omega_{B/Y}^1)^*.$$

Here $(\Omega_B^1)^*$ is the subsheaf of the tangent sheaf of Y consisting of vector fields tangent to B.

Now let us go back to our construction of sections $u_1, \ldots, u_{n-1}$ of F^*T_Y on $U \times G$. We can choose the neighborhood U and vector fields $v_1, \ldots, v_n$ on U such that all points of $U \cap R$ satisfy (i) and (ii), and $v_1, \ldots, v_{n-1}$ are tangent to $U \cap R$. Then the sections $u_1, \ldots, u_{n-1}$ of F^*T_Y on $U \times G$ will be sections of the subsheaf $F^*(\Omega_B^1)^*$. By (i), B will be smooth at $y := f(g \cdot x)$ and the differential df at $g \cdot x$ will send $T_{R,g\cdot x}$ isomorphically into the subspace $T_{B,y}$ of $T_{Y,y}$. Thus $u_1, \ldots, u_{n-1}$ will be pointwise independent at (x, g) spanning the fiber $T_{B,y}$. Therefore the sections $u_1, \ldots, u_{n-1} \in H^0(G, F_x^*(\Omega_B^1)^*)$ span the fiber at one point.

Together with $\dim H^0(G, F_x^*\Omega_B^1) \geq n-1$, this implies that $u_1, \ldots, u_{n-1}$ as sections of $F_x^*T_Y$ will be pointwise independent at every point of G. Thus they span a trivial subbundle W_x of rank $n-1$ in $F_x^*T_Y$. But we have seen that when z is a general point of U, $u_1, \ldots, u_{n-1}$ span a trivial subbundle W_z of rank $n-1$ in $F_z^*T_Y$ such that the quotient is a trivial line bundle. It follows that the quotient $F_x^*T_Y/W_x$ is also a trivial line bundle. Thus we can write $F_z^*T_Y$ as an extension of the trivial line bundle by a trivial vector bundle W_z for each $z \in U$. For general $z \in U$, we know that this extension is trivial. Thus the extension is trivial for x, too. This completes the proof. □

Coming back to the set-up of the beginning of this section, we find an unramified covering $h : Z \to Y$ and a ramified covering $g : X \to Z$ such that $f = h \circ g$ such that

$$df(v') \in g^*H^0(Z, T_Z) = g^*H^0(Z, h^*T_Y)$$

proving Theorem 1.3.

4. Proof of the main result

To prove Theorem 1.2, after Stein factorization, we may assume that f is finite. In particular, X belongs to the Fujiki's class $\mathcal{C}$ (e.g. [2, Proposition 3.17]). We fix in this section a normal compact complex space X of class $\mathcal{C}$, a compact Kähler manifold Y and a finite surjective map

$$f : X \to Y.$$

This implies that each component of $\mathrm{Hol}(X,Y)_{red}$ is a Zariski open subset in a compact complex space. We always may assume that X and Y are non-algebraic.

4.1. Construction. By [1] there exists an almost holomorphic map

$$q_X : X \dashrightarrow Q_X$$

to a compact Kähler manifold Q_X with the following property: given two very general points $x, y \in X$, we have $q_X(x) = q_X(y)$ if and only if x and y can be joined by a chain of irreducible curves, all of the components of the chain belonging to families of curves which cover X.

Of course, if X is Moishezon, then Q_X is a single point; conversely by [1], if Q_X is a point, then X is Moishezon. If there is no covering family of curves in X; then $Q_X = X$ (up to birational transformation). Notice that q_X is "in general" different from the algebraic reduction of X, the fiber of which not necessarily being algebraic.

In the same way we obtain a map $q_Y : Y \dashrightarrow Q_Y$.

Clearly f maps a general q_X-fiber to a general q_Y-fiber, so that we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\ Q_X & \xrightarrow{\tilde f} & Q_Y \end{array}$$

By

$$\mathrm{Hol}^v_f(X,Y) \subset \mathrm{Hol}_f(X,Y)$$

we will denote the space of "vertical" deformations of f, which means that we deform f with $\tilde f$ fixed. The tangent space to $\mathrm{Hol}^v_f(X,Y)$ at $[f]$ is the subspace

$$H^0(X, f^*T_{Y/Q_Y}) \subset H^0(X, f^*T_Y)$$

consisting of sections of f^*T_Y which are tangent to the general fiber of q_Y (and therefore tangent to all "fibers" of q_Y.).

4.2. Proposition. $\mathrm{Hol}^v_f(X,Y)$ *is compact.*

Proof. We introduce the shorthand $H = \mathrm{Hol}^v_f(X,Y)$. Then H can be considered as subset of the cycle space $\mathcal{C}(X \times Y)$. Since the irreducible components of $\mathcal{C}(X \times Y)$ are compact (see [2] for a discussion and further references), it suffices to prove that H is closed in $\mathcal{C}(X \times Y)$.

First we observe that we may assume X smooth and Kähler. In fact, take a bimeromorphic holomorphic map $\pi : \hat X \to X$ such that $\hat X$ is smooth and Kähler. Let $\hat f = f \circ \pi$ and $\hat H$ be the space of vertical deformations of $\hat f$. Then $H \subset \hat H$ in a natural way, and it is clear that H is closed in $\mathcal{C}(X \times Y)$ once we have proved that $\hat H$ is closed in $\mathcal{C}(\hat X \times Y)$.

To prove that H is closed, we consider a family $(f_t)_{t\in\Delta^*}$ in H over the punctured unit disc Δ such that the graphs $G_t \subset X \times Y$ converge to a cycle G_0. We need to prove that G_0 is the graph of a holomorphic map $f_0 : X \to Y$.

Let us fix Kähler forms ω_X and ω_Y on X and Y. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections.

We show first that G_0 is irreducible. In fact, G_0 has a unique component with multiplicity 1, say G^*, which maps onto X. Moreover $G^* \to X$ has degree 1. Both statement follow by integrating $p_X^*(\omega_X)$ over G_0.

Now G^* defines at least a meromorphic map $f_0 : X \dashrightarrow Y$. Since $\kappa(X_b) \geq 0$ for general $b \in Q_X$, the space

$$\mathrm{Hol}_{f|X_b}(X_b, Y_{\bar{f}(b)})$$

is compact by [7] and therefore f_0 is holomorphic near X_b, i.e., $G^* \cap X_b$ is the graph of $f_0|X_b$. Since $f_0|X_b$ has degree $d = \deg f_t$, we conclude that $G^* \to Y$ has degree d. Introducing the Kähler form

$$\omega = p_X^*(\omega_X) + p_Y^*(\omega_Y),$$

we notice that

$$G^* \cdot \omega^n = G_t \cdot \omega^n,$$

on the other hand $G_t \cdot \omega^n = G_0 \cdot \omega^n$, so that

$$G^* = G_0,$$

and G_0 is irreducible.

We consider the family $\mathcal{G} = (G_t) \to \Delta$ with projection $p : \mathcal{G} \to X \times \Delta$. $\mathcal{G}$ is an irreducible reduced complex space and $\deg p = 1$. Therefore p has connected fibers, i.e., p is bimeromorphic. If p were not biholomorphic, then the purity-of-branch theorem (recall that X is smooth) exhibits a proper subspace $B \subset X \times \Delta$, necessarily contained in $X \times \{0\}$, such that $D = p^{-1}(B)$ has codimension 1 in $\mathcal{G}$, i.e., $\dim D = n$. Hence D is an irreducible component of $p_\Delta^{-1}(0)$. This contradicts the irreducibility of G_0.

Hence p is biholomorphic, and so does $G_0 \to X$. Therefore f_0 is holomorphic and of course $f_0 \in \mathrm{Hol}_f^v(X, Y)$, which completes the proof. □

4.3. Proposition. *Let $f : X \to Y$ be a surjective holomorphic map to a compact Kähler manifold Y of non-negative Kodaira dimension. Suppose there exists a positive dimensional compact subvariety in* $\mathrm{Hol}_f(X, Y)$. *Then f has an unramified factorization $\beta : Z \to Y$ with* $\dim \mathrm{Aut}(Z) > 0$.

Proof. By Proposition 2.9, we know that there exists a (not necessarily unramified) factorization $h : Y' \to Y$ with $\dim \mathrm{Aut}(Y') > 0$. Since $\kappa(Y) \geq 0$, the identity component $\mathrm{Aut}^0(Y')$ is a torus G. Applying Theorem 1.3 to h, we get an unramified factorization $\beta : Z \to Y$ with an effective G-action on Z. This gives an unramified factorization of f with $\dim \mathrm{Aut}(Z) > 0$. □

4.4. Proof of Theorem 1.2. To prove Theorem 1.2, we may assume by Proposition 2.3, by Proposition 2.5, by Proposition 2.8 and by induction on $\dim Y$, that f has no unramified factorization $\beta : Z \to Y$ with $\dim \mathrm{Aut}(Z) > 0$. Then by Proposition 4.3, it follows that $\mathrm{Hol}_f(X, Y)$ has no positive dimensional compact subvariety. Consequently by Proposition 4.2, $\dim Hol_f^v(X, Y) = 0$.

Now consider a closed irreducible analytic set (not necessarily compact)

$$Z \subset \mathrm{Hol}_f(X, Y)$$

(e.g., $Z = \mathrm{Hol}_f(X, Y)$), which is constructible in the cycle space $\mathcal{C}(X \times Y)$, and argue as in [12, Section 3]. For $g \in Z$ consider the ramification locus $B_g \subset Y$. If $B_g \subset Y$ (resp. $g^{-1}(B_g) \subset X$) moves, consider a general point $y \in Y$ (resp. $x \in X$) and form the 1-codimensional subvariety $Z(x) \subset Z$ consisting of those g with $y \in B_g$ (resp. $x \in g^{-1}(B_g)$). Repeating this process we obtain the following alternative.

- $\dim Z = 1$;
- there exists a positive-dimensional Z such that B_g and $g^{-1}(B_g)$ are independent of g for all $g \in Z$.

In the first case we consider for fixed general x the closed curve $C = \{g(x) | g \in Z\}$. Since $\mathrm{Hol}_f(X, Y)$ is Zariski-open in the cycle space (see e.g. [3], [11], at least in the case f is an automorphism), we can take closure and obtain a compact curve $\overline{C}$ parametrizing cycle such that the general cycle is the graph of a deformation of f. Now take $x \in X$ general and consider the closure of the curve

$$C_x = \{g(x) | g \in C\}.$$

Thus we obtain a covering family of compact curves in X and since the deformations of f are not vertical, these curves are not contained in fibers of q_X. This contradicts the construction of q_X.

In the second case we may apply Proposition 2.10 and Z is (isomorphic to) a closed subvariety of the automorphism group of X, which is a torus, a contradiction.

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References

[1] F. Campana, *Coréduction algébrique d'un espace analytique faiblement kählérien compact*, Invent. Math. **63** (1981), no. 2, 187–223.

[2] F. Campana and Th. Peternell, *Cycle spaces*, Several complex variables, VII, 319–349, Encyclopaedia Math. Sci., 74, Springer, Berlin, 1994.

[3] A. Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), no. 3, 225–258.

[4] H. Holmann, *Quotienten komplexer Räume*, Math. Ann. **142** (1960/1961), 407–440.

[5] C. Horst, *Compact varieties of surjective holomorphic mappings*, Math. Z. **196** (1987), no. 2, 259–269.

[6] J.-M. Hwang, *Rigidity of surjective holomorphic maps to Calabi-Yau manifolds*, Math. Z. **249** (2005), no. 4, 767–772.

[7] J.-M. Hwang, S. Kebekus, and Th. Peternell, *Holomorphic maps onto varieties of non-negative Kodaira dimension*, J. Algebraic Geom. **15** (2006), no. 3, 551–561.

[8] M. Kalka, B. Shiffman, and B. Wong, *Finiteness and rigidity theorems for holomorphic mappings*, Michigan Math. J. **28** (1981), no. 3, 289–295.

[9] S. Kebekus and T. Peternell, *A refinement of Stein factorization and deformations of surjective morphisms*, math.AG/0508285.

[10] J. Kollár, *Rational curves on algebraic varieties*, A Series of Modern Surveys in Mathematics, 32, Springer-Verlag, Berlin, 1996.

[11] D. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kahler manifolds*, Fonctions de plusieurs variables complexes, III (Sem. Francois Norguet, 1975–1977), pp. 140–186, Lecture Notes in Math., 670, Springer, Berlin, 1978.

[12] J. Noguchi, *Meromorphic mappings into compact hyperbolic complex spaces and geometric Diophantine problems*, Internat. J. Math. **3** (1992), no. 2, 277–289.

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