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SYMMETRIC SPACE, GEOMETRY AND TOPOLOGY

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Abstract. In this note we survey some recent results on symmetric space, and related topics like rigidity, flexibility and geometry-topology aspect of symmetric space, specially in the line of hyperbolic 3-manifolds.

1. Introduction

Recently there has been a tremendous progress on the Kleinian group theory which had been initiated by pioneers like Ahlfors, Bers, Kra, Sullivan, and has been developed and directed by W. P. Thurston last 30 years. Thurston proposed a best-known frame, known as *geometrization program* for the classification of 3-manifolds. The most remarkable feature of this classification is that he first introduced the Riemannian metric for a topological classification. Inspired by the uniformization theorem for Riemann surface, he proposed 8 homogeneous geometries for the classification of 3-manifolds, and he worked on hyperbolic geometry tremendously. His deep insight and results on Kleinian group theory inspired many mathematicians and unified the realm of geometry, topology, dynamics and complex analysis. During the past few years, big conjectures like tameness, density and ending lamination conjecture have been settled down by his descendants. Also his geometrization program is completed by the contribution of Perelman and Hamilton using analysis and non-linear partial differential equations. We want to survey the origin of this research area, namely symmetric space and related areas. This goes back to early 20th century. This area has a root in semi-simple Lie group theory, ergodic theory and Riemannian geometry as well. Symmetric space is a special kind of Riemannian manifold which is homogeneous and with many symmetries as the name suggests. Though it is a special Riemannian metric, it has rich enough structures on which one can do Lie group theory, ergodic theory, geometry and even topology, number theory and representation theory. It is initiated by Poincare, Harish-Chandra, Lie, and many more. Recently by Mostow and

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Margulis, this field has been popularized and became an important branch of mathematics. Tied up with a low dimensional topology, specially with a 3-dimensional topology, it presents a successful application to topology, as a powerful tool for untouchable area of topology. This survey is far from being thorough and complete, we just touch the one aspect of the gigantic glacier, which is familiar with the author and which is related to author's work.

2. Definitions and terminologies

A *symmetric space* of noncompact type is a Hadamard homogeneous Riemannian manifold whose curvature tensor is parallel under the parallel transport. The identity component of the isometry group is a semi-simple Lie group of noncompact type. Let G be the identity component of the isometry group of the symmetric space X of noncompact type. If K is a isotropy group of a point x_0 in X, then X is identified with G/K. Denote the Lie algebra of G (respectively K) by $\mathfrak{g}$ (respectively $\mathfrak{k}$). Then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with a Cartan relation $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

The killing form B defined by

$$B(Y, Z) = Tr(adY \circ adZ)$$

is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Identifying $\mathfrak{p}$ with a tangent space of X at a point x_0, one obtain G-invariant Riemannian metric on X. The sectional curvature is given by $curv(Y, Z) = -||[Y, Z]||^2$ and it is always less than or equal to zero. Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$. One calls the dimension of $\mathfrak{a}$ the *rank* of the symmetric space X. If the rank is 1, then the symmetric space is strictly negatively curved. There are four kinds of rank one symmetric spaces of noncompact type: real, complex, quaternionic hyperbolic spaces and octonionic hyperbolic 2-plane. Their sectional curvature is pinched between -4 and -1. If the rank is at least 2, then it is called a higher rank symmetric space. The famous example is $SL(n, \mathbb{R})/SO(n)$. Note $(\exp \mathfrak{a})x_0$ is a maximal flat in X through $x_0 \in X$.

By the standard theory of Lie algebra, one obtains the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

The set $\mathfrak{a}_{sing} = \{H \in \mathfrak{a} | \exists \alpha \in \Lambda, \alpha(H) = 0\}$ of singular vectors divides $\mathfrak{a}$ into the finite number of components called *Weyl chambers.* Fixing a component $\mathfrak{a}^+$ amounts to choosing positive roots

$$\Lambda^+ = \{\alpha \in \Lambda | \alpha(H) > 0, \forall H \in \mathfrak{a}^+\}.$$

Furthermore if one sets

$$\mathfrak{n}^\pm = \sum_{\alpha \in \Lambda^\pm} \mathfrak{g}_\alpha$$

one obtains the *Iwasawa* decomposition KAN where $N = \exp \mathfrak{n}^+$. It is worthwhile to note that MAN fixes $A^+x_0(\infty)$ pointwise where M is the subgroup of K fixing Ax_0 pointwise. The *Weyl group* W is M'/M where M' is the subgroup of K fixing Ax_0 globally and it acts transitively on the Weyl chambers of Ax_0. For more details see [2].

Among the symmetric spaces, the most well-known space is a hyperbolic 3-space $\mathbb{H}^3$ due to W. Thurston. It has a close relation to a geometry and topology of 3-dimensional manifolds. Hyperbolic space $\mathbb{H}^3$ is a complete simply connected Riemannian manifold of constant curvature -1. The Poincaré ball gives a model for hyperbolic space as the unit ball in $\mathbb{R}^3$ with the metric

$$ds^2 = \frac{4dx^2}{(1-r^2)^2}.$$

The boundary of Poincaré ball models is the *sphere at infinity* S^2_∞ for hyperbolic space and the isometries of $\mathbb{H}^3$ prolong to conformal maps on the boundary. The sphere at infinity can be identified with the Riemann sphere $\widehat{\mathbb{C}}$, providing an isomorphism between the orientation preserving group $Isom^+(\mathbb{H}^3)$ and the group of fractional linear transformations $Aut(\mathbb{C}) \cong PSL_2\mathbb{C}$.

A *Kleinian group* Γ is a discrete (torsion free) subgroup of $Isom(\mathbb{H}^3)$. A hyperbolic 3-manifold N is the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a Kleinian group Γ. The *domain of discontinuity* or *discontinuity domain* $\Omega(\Gamma)$ of Γ is the largest Γ-invariant open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously. If Γ is not abelian, then $\Omega(\Gamma)$ inherits a hyperbolic metric, called the Poincaré metric, on which Γ acts as a group of isometries. One may then consider $\bar{N} = N \cup \Omega(\Gamma)/\Gamma$ and $\partial_c N = \partial\bar{N} = \Omega(\Gamma)/\Gamma$ to be the *conformal boundary at infinity* of the hyperbolic 3-manifold N. See [1] or [30].

A *compression body* N is a compact oriented 3-manifold which is the boundary connected sum of solid tori and trivial interval bundles over closed surfaces of genus at least 2. A trivial interval bundle over a closed surface is not considered as a compression body. A compression body N is *small* if it is the connected sum along the boundary of either two trivial interval bundles over closed surfaces or an interval bundle over a closed surface and a solid torus. Otherwise it is called *large*. The boundary ∂N of a compression body N has a unique compressible component which is called the *exterior boundary* $\partial_e N$. The inclusion $i_e : \partial_e N \hookrightarrow N$ induces a surjective homomorphism $\pi_1(\partial_e N) \to \pi_1(N)$. The other components except exterior boundary form the interior boundary $\partial_{int} N$.

Given a compression body N, there is a convex cocompact representation $\rho : \pi_1(N) \to PSL_2\mathbb{C}$ such that the interior of N is homeomorphic to $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ which induces ρ. We say that ρ uniformizes N [23]. The image of ρ is a *function group*, i.e. there is an invariant component of the discontinuity domain Ω_ρ under the action of $\rho(\pi_1(N))$ on $\widehat{\mathbb{C}}$. We can identify the Riemann surface $\Omega_\rho/\rho(\pi_1(N))$ with the boundary ∂N of N. Under this identification the exterior boundary $\partial_e N$ corresponds to a unique invariant component of the discontinuity domain. For more on the topology of compression bodies, see [4] or [24].

3. General background

In 1960's, Mostow [26] proved a striking rigidity result about the lattices in semisimple Lie groups.

Theorem 3.1. *If Γ_1 and Γ_2 are isomorphic lattices in semisimple Lie groups of noncompact type G_1 and G_2 respectively, then $G_1 = G_2$ and Γ_1 and Γ_2 are conjugate.*

In geometric terms, if M_1 and M_2 are locally symmetric manifolds with isomorphic fundamental groups, then they are isometric. A bit later, Margulis [22] proved the super rigidity theorem for higher rank symmetric space.

Theorem 3.2. *If Γ is a lattice in a higher rank semisimple Lie group, it is arithmetic.*

Most recently, there is an entropy rigidity theorem for rank 1 symmetric space by Besson-Courtois-Gallot [3].

Theorem 3.3. *Let M^n be a compact negatively curved locally symmetric manifold and N^n a compact negatively curved Riemannian manifold. Suppose there is a map*

$$f : M \to N$$

so that the degree of f is nonzero. Then

$$deg(f)h(M)^n vol(M) \leq h(N)^n vol(N),$$

where h denotes the volume entropy. The equality holds if and only if they are locally isometric.

The volume entropy $h(M)$ is defined as follows. Let $\tilde{M}$ be the universal cover of M. Take a base point $x_0 \in \tilde{M}$. $B(x, R)$ denotes the metric ball of radius R around x_0 in $\tilde{M}$. Then

$$h(M) = \lim_{R \to \infty} \frac{\log volB(x, R)}{R}.$$

4. Kleinian group theory, hyperbolic geometry

In 1970's as a part of his geometrization program, Thurston proved the following important theorem. A 3-manifold M is *atoroidal* if it does not contain any irreducible torus except boundary parallel ones. It is *Haken* if it contains an irreducible surface, i.e., an embedded surface S so that $\pi_1(S)$ injects into $\pi_1(M)$.

Theorem 4.1. *If M is an atoroidal, Haken 3-manifold, then it admits a geometrically finite hyperbolic metric.*

In the course of the proof of this theorem, the important case was the mapping torus case, i.e., $M = S \times [0,1]/\sim$ where $(x,0) \sim (\phi(x),1)$ and ϕ is a pseudo-Anasov map on the surface. Thurston used so-called the "double limit theorem" for quasifuchsian manifolds. By Ber's uniformization theorem, a quasifuchsian group is determined by the pair of conformal structure at infinity, so the set of quasifuchsian groups is homeomorphic to $\mathcal{T}(S) \times \mathcal{T}(S)$, where $\mathcal{T}(S)$ is a Teichmüller space of S. This is the set of convex cocompact hyperbolic metrics up to isotopy on $S \times \mathbb{R}$. If a sequence of quasifuchsian metrics (t_n, s_n) is given as in above parametrization, such a sequence converges algebraically if $t_n \to \lambda, s_n \to \beta$ in Thurston's compactification of $\mathcal{T}(S)$ so that two projective laminations λ, β fill up the surface S, i.e.,

$$i(\lambda, c) + i(\beta, c) > 0,$$

for any simple closed curve c. This is a double limit theorem. Right after this monumental theorem, Thurston conjectured that such a theorem should hold for general hyperbolic 3-manifold, specially a hyperbolic 3-manifold with a compressible boundary. Originally he conjectured that if a sequence of convex cocompact hyperbolic metrics is given on a handle body, and if the conformal structures at infinity converge to a projective lamination in Thurston boundary which lies in a Masur domain, then the sequence converges algebraically. Let N be a compression body with exterior boundary $\partial_e N$ and $\rho : \pi_1(N) \to PSL_2\mathbb{C}$ be a discrete and faithful representation with associated quotient manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$. A *meridian* is a homotopically nontrivial simple closed curve m on the exterior boundary $\partial_e N$ which is compressible in N. A meridian may be seen as an element in the space $\mathcal{PML}(\partial_e N)$ of projective classes of measured lamination on the exterior boundary $\partial_e N$. The set of projective classes of weighted multicurves of meridians in $\mathcal{PML}(\partial_e N)$ will be denoted by $\mathcal{M}$ and its closure in $\mathcal{PML}(\partial_e N)$ by $\mathcal{M}'$. For a small compression body which is the boundary connected sum of two trivial surface bundles over closed surfaces or the boundary connected sum of a trivial surface bundle over a closed surface and a solid torus, set

$$\mathcal{O} := \{\lambda \in \mathcal{PML}(\partial_e N) |\ i(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{PML}(\partial_e N) \text{ such that there is } \nu \in \mathcal{M}' \text{ with } i(\mu, \nu) = 0\}.$$

Otherwise, in a large compression body case, set

$$\mathcal{O} := \{\lambda \in \mathcal{PML}(\partial_e N) |\ i(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{M}'\}.$$

The set $\mathcal{O}$ is called the *Masur domain.* We will say that $\lambda \in \mathcal{ML}(\partial_e N)$ is in $\mathcal{O}$ (resp. $\mathcal{M}'$) if its projective class is in $\mathcal{O}$ (resp. $\mathcal{M}'$).

Recently the author together with co-workers settled down this conjecture for general hyperbolic 3-manifold with a compressible boundary [20].

Theorem 4.2. *Let M be a compact irreducible atoroidal 3-manifold with boundary, and $\rho_0 : \pi_1(M) \to PSL(2,\mathbb{C})$ a geometrically finite representation that uniformises M. Let (m_n) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ which* *converges in the Thurston compactification to a projective measured lamination* $[\lambda]$ *contained in* $\mathcal{PD}(M)$. *Let* $q : \mathcal{T}(\partial M) \to QH(\rho_0)$ *be the Ahlfors-Bers map, and suppose that* $(\rho_n : \pi_1(M) \to PSL(2,\mathbb{C}))$ *is a sequence of discrete faithful representations corresponding to* $q(m_n)$. *Then passing to a subsequence,* (ρ_n) *converges in* $AH(M)$.

Consider a compact irreducible atoroidal 3-manifold M with boundary. By Thurston's uniformisation theorem for atoroidal Haken manifolds, there is a representation $\rho_0 : \pi_1(M) \to Isom(\mathbb{H}^3)$ with the following properties : $\rho_0(\pi_1(M))$ is geometrically finite, $\mathbb{H}^3/\rho_0(\pi_1(M))$ is homeomorphic to $int(M)$, and any maximal parabolic subgroup of $\rho_0(\pi_1(M))$ is an Abelian group of rank 2. Such a representation is said to *uniformise* M. Any quasi-conformal deformation of $\rho_0(\pi_1(M))$ also uniformises M. By the Ahlfors-Bers theory, the space $QH(\rho_0)$ of quasi-conformal deformations of ρ_0 up to conjugacy by elements of $Isom(\mathbb{H}^3)$ is parametrised by the Teichmüller space of the boundary of M. More precisely, there is a (possibly ramified) covering map, called the *Ahlfors-Bers map* $\mathcal{T}(\partial_{\chi<0}M) \to QH(\rho_0)$ whose covering transformation group is the group of isotopy classes of diffeomorphisms of M which are homotopic to the identity.

The space $QH(\rho_0)$ is a subspace of the *deformation space* $AH(M)$. This deformation space $AH(M)$ is the space of discrete faithful representations $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ up to conjugacy by elements of $PSL_2(\mathbb{C})$. It is endowed with the compact-open topology which is also called an *algebraic topology.* We shall consider sequences of representations given by sequences in the Teichmüller space whose images under the Ahlfors-Bers map diverge in $QH(\rho_0)$ and study their convergence in $AH(M)$.

Thurston introduced in [31] the notion of *doubly incompressible curves.* This can be extended to measured geodesic laminations in the following way:

We say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if :

- $\exists \eta > 0$ such that $i(\lambda, \partial E) > \eta$ for any essential annulus, Möbius band or disc E.

We denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations and by $\mathcal{PD}(M)$ its projection in the projective lamination space $\mathcal{PML}(\partial M)$.

5. Length spectral rigidity

In Riemannian geometry, one of the main theme is to study when two Riemannian manifolds are isometric. One of the tools is to compare closed geodesic lengths. Two manifolds M and N have the *same marked length spectrum* if there is an isomorphism

$$\rho : \pi_1(M) \to \pi_1(N)$$

so that $l_M(\gamma) = l_N(\rho(\gamma))$ for any $\gamma \in \pi_1(M)$ where $l_M(\gamma)$ denotes the geodesic length in M of the homotopy class of γ. It is conjectured that if M and

N are negatively curved and if they have the same marked length spectrum, then they are isometric. This conjecture is known to be true only for a surface case by [28]. When one manifold X is compact locally symmetric negatively curved, it is shown by [3] that any compact manifold whose geodesic flow is C^1-conjugate to that of X is isometric to X. Note in general if two negatively curved manifolds have the same marked length spectrum, their geodesic flows are only C^0-conjugate. Specially if one manifold is compact real hyperbolic, and the other manifold is compact negatively curved then the same marked length spectrum implies two manifolds are isometric [13]. When the manifolds are locally symmetric ones (not necessarily convex cocompact nor topologically tamed in Thurston's sense) which do not have proper totally geodesically embedded submanifolds, it is proved by [14, 15, 17, 9] that the same marked length spectrum implies that two manifolds are isometric. For nonriemannian case, specially for convex real projective structures and affine structures, see [16, 18].

It should be noted that the unmarked length spectrum sometimes does not determine the geometry. Even in surface case, there are numerous examples of the pair of hyperbolic surfaces of genus greater than or equal to 4 with the same unmarked length spectrum which are not isometric. The general construction is due to [29]. In surface case, the unmarked length spectrum determines the Laplace spectrum and vice versa by means of the Selberg Trace Formula. Y. Colin de Verdière [32] showed that the Laplace spectrum give the unmarked length spectrum. Suppose Y is a negatively curved compact manifold and X is a locally symmetric negatively curved manifold and they are at least dimension 3. Then it is conjectured [3] that if they are isospectral and there exists nonzero degree map between them, then they are isometric.

The *length spectrum* $\Lambda(M)$ of a Riemannian manifold M is the set of lengths of closed geodesics with *multiplicity*. There is an analogous conjecture. The set of negatively curved metrics on a fixed compact manifold with a fixed set of lengths of closed geodesics, forms a compact (or finite) set in the space of Riemannian metrics. One can rephrase this as an *unmarked length rigidity*. There is also very little progress in this direction. First McKkean in 70's showed [25] that there is only finite number of hyperbolic metrics with a given *spectral set*, i.e. the set of closed geodesic lengths with multiplicity. Later Osgood, Phillips and Sarnak [27] showed the compactness of isospectral metrics on a closed surface. Much later Brooks, Perry and Petersen [5] showed the same result for closed 3-manifolds near a metric of constant curvature. More recently, Croke and Sharafutdinov [8] showed a local isospectral rigidity on a negatively curved closed manifold. For Laplacian spectrum and related problems, see [10, 12, 29, 33].

If we restrict our attention to the locally symmetric manifolds, there are several tools available. Specially for hyperbolic 3-manifold case, one can use Kleinian group theory which has been developed last decade. Here [7] is a finiteness result for isospectral infinite volume hyperbolic 3-manifolds, which is a generalization to higher dimensional manifold of the result of [25].

Theorem 5.1. *Let M be a non-elementary convex cocompact real hyperbolic 3-manifold with a length spectrum Λ. Then there are only finite number of hyperbolic 3-manifolds homotopy equivalent to M with the length spectrum Λ.*

For convex cocompact hyperbolic 3-manifold with incompressible boundary, see [19].

6. Local rigidity of lattices in symmetric space

By the seminal work of Corlette, lattices $\Gamma \subset G$ in quaternionic and octonionic space is superrigid, i.e., if $\phi : \Gamma \rightarrow G'$ is a homomorphism into a semisimple Lie group with a Zariski dense image, then ϕ extends to a homomorphism from G to G'.

Let Γ be a lattice in a symmetric space S. Consider a bigger symmetric space S' which contains S as a totally geodesic submanifold. Γ naturally acts on S'. Is it possible to deform Γ in Iso(S')? In some cases, the answer is yes. There are examples of lattices in $PSO(3,1)$ which can be deformed in $PSO(k,1)$ for $k > 3$. But in general this is not always true. For example a uniform lattice in $PSU(n,1)$ cannot be deformed in $PSU(m,1), m > n$ due to K. Corlette [6]. Following the arguments used in [11], one can see that the fundamental group of a convex cocompact rank one locally symmetric manifold of dimension > 3 is isomorphic to that of a CW complex whose complexity is controlled by the volume of the manifold. So for a fixed upper bound for the volume of the convex core of the manifold, there are only finitely many possible fundamental groups. Then suppose that a lattice in a symmetric space S can be deformed to a convex cocompact discrete group in S' where S is totally geodesically embedded. One can perturb so little that the volume of the convex core is bounded above by a uniform constant. Then by the observation above, this is possible only for finitely many lattices. This shows that most of cases, we cannot deform a lattice in a bigger symmetric space.

In this note we give another example of local rigidity [21].

Theorem 6.1. *Let Γ be a uniform lattice in $PSO(4,1)$ which can be regarded as a discrete group in $PSp(n,1), n > 1$ in a canonical way by identifying $\mathbb{H}^4_{\mathbb{R}}$ with a quaternionic line. Then Γ cannot be locally deformed in $PSp(n,1)$.*

The idea of a proof is to use Raghunathan-Matsushima-Murakami result. We can push the result a litter further to deal with non-uniform lattice with an assumption.

Theorem 6.2. *Under the same assumption with a non-uniform lattice, there is no local deformation preserving parabolicity.*

This fact has to do with L^2-norm of forms. Note that Raghunathan-Matsushima-Murakami result holds only for finite L^2-norm forms. We strongly believe that in our case, we do not need the condition of preserving parabolicity. The same statement holds for uniform lattice in $SU(1,1)$ and in $SO(3,1)$ considering $SU(1,1) \subset Sp(1,1) \subset Sp(n,1)$ and $SO(3,1) \subset SO(4,1) \subset Sp(1,1) \subset Sp(n,1)$.

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