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UNIMODULAR GROUPS OF TYPE $\mathbb{R}^3 \rtimes \mathbb{R}$

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ABSTRACT. There are 7 types of 4-dimensional solvable Lie groups of the form $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ which are unimodular and of type (R). They will have left-invariant Riemannian metrics with maximal symmetries. Among them, three nilpotent groups ($\mathbb{R}^4$, $\mathrm{Nil}^3 \times \mathbb{R}$ and Nil^4) are well known to have lattices.

All the compact forms modeled on the remaining four solvable groups $\mathrm{Sol}^3 \times \mathbb{R}$, Sol_0^4, Sol'^4_0 and Sol_λ^4 are characterized: (1) $\mathrm{Sol}^3 \times \mathbb{R}$ has lattices. For each lattice, there are infra-solvmanifolds with holonomy groups 1, $\mathbb{Z}_2$ or $\mathbb{Z}_4$. (2) Only some of Sol_λ^4, called $\mathrm{Sol}_{m,n}^4$, have lattices with no non-trivial infra-solvmanifolds. (3) Sol'^4_0 does not have a lattice nor a compact form. (4) Sol_0^4 does not have a lattice, but has infinitely many compact forms. Thus the first Bieberbach theorem fails on Sol_0^4. This is the lowest dimensional such example. None of these compact forms has non-trivial infra-solvmanifolds.

1. Introduction

We study certain class of 4-dimensional solvable Lie groups of the form $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$, where $\varphi : \mathbb{R} \to \mathrm{GL}(3, \mathbb{R})$ is a continuous homomorphism. The homomorphism φ yields a Lie algebra homomorphism $\psi : \mathbb{R} \to \mathfrak{gl}(3, \mathbb{R})$ so that $\varphi(t) = e^{\psi(t)}$. Therefore, $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ is completely determined by the matrix $\mathcal{A} = \psi(1)$ only.

A connected Lie group G is *of type* (R) if for every $X \in \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues. G is *of type* (E) if $\exp : \mathfrak{g} \to G$ is surjective. It is *unimodular* if $\mathrm{ad}(X)$ has trace 0 for every $X \in \mathfrak{g}$. For our $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ when it is unimodular and of type (R), up to conjugation and scalar multiple, there are 7 classes; only one class contains a parameter. We tabulate the isomorphism classes of 4-dimensional unimodular, type (R) Lie algebras of the form $\mathbb{R}^3 \rtimes_\mathcal{A} \mathbb{R}$ and the associated simply connected Lie groups:

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$\mathcal{A}$ for Lie algebra $\mathbb{R}^3 \rtimes_{\mathcal{A}} \mathbb{R}$		Associated simply connected Lie group $\mathbb{R}^3 \rtimes_{\varphi(s)} \mathbb{R}$	
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		$\mathbb{R}^4$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		$\mathrm{Nil}^3 \times \mathbb{R}$; $\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		Nil^4; $\begin{bmatrix} 1 & s & \frac{1}{2}s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		$\mathrm{Sol}^3 \times \mathbb{R}$; $\begin{bmatrix} e^s & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$		Sol_0^4; $\begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix}$	
$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$	$(\lambda > 1)$	Sol_λ^4; $\begin{bmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-(1+\lambda)s} \end{bmatrix}$	$(\lambda > 1)$
$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$		$\mathrm{Sol'}_0^4$; $\begin{bmatrix} e^s & se^s & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix}$	

The group law of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ is

$$(\mathbf{x}, s)(\mathbf{y}, t) = (\mathbf{x} + \varphi(s)\mathbf{y}, s + t),$$

and it can be embedded in $\mathrm{Aff}(4)$ as

$$G = \left\{ \begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset \mathrm{Aff}(4) \subset \mathrm{GL}(5, \mathbb{R}),$$

where $\varphi(s) \in \mathrm{GL}(3, \mathbb{R})$, $s \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$ is a column vector.

For a Lie group G with a left-invariant metric, $\mathrm{Isom}(G)$ denotes the group of isometries, $\mathrm{Isom}_0(G)$ its connected component of the identity. Also $\mathrm{Aut}(G)$ denotes the group of automorphisms of G. If Π is a discrete subgroup of $\mathrm{Isom}(G)$ acting freely and properly discontinuously on G, the quotient space $\Pi\backslash G$ is a *compact form*. If Π is a discrete cocompact subgroup of $G \rtimes K$ (where K a compact subgroup of $\mathrm{Aut}(G)$) and $\Pi \subset G$ is a lattice of G, acting freely and properly discontinuously on G, the quotient space $\Pi\backslash G$ is an *infra-homogeneous space*. If, in particular, $\Pi \subset G$, then $\Pi\backslash G$ is a *homogeneous space*. If G is $\mathbb{R}^n$, nilpotent, solvable, then the homogeneous space (infra-homogeneous space) is called a torus (flat manifold), nilmanifold (infra-nilmanifold), solvmanifold (infra-solvmanifold), respectively.

Remark 1.1. Our list consists of $\mathbb{R}^4$, $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, Sol_0^4, Sol'^4_0 and Sol_λ^4, while the Lie groups in Filipkiewicz's list in [12] consist of $\mathbb{R}^4$, $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}$, $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, Sol_0^4, Sol_1^4 and $\mathrm{Sol}_{m,n}^4$.

Note that $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}$ and Sol_1^4 are not in our list since they are not of the type $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$. Note also that Sol_1^4 has nil-radical Nil^3. Our Sol'^4_0 is not in Filipkiewicz's list (with a unknown reason). We shall show the following:

(1) Sol'^4_0 does not have a lattice nor a compact-form, see Proposition 2.2 and Theorem 4.2.
(2) Sol_λ^4 has a lattice if and only if it is of the form $\mathrm{Sol}_{m,n}^4$. Otherwise, there is no compact-form. See Proposition 2.1 and Theorem 4.2.
(3) Sol_0^4 does not have a lattice, yet it has infinitely many compact-forms, see Proposition 2.2 and Theorem 4.3. This is a type (R) counter-example to the generalized Bieberbach's first Theorem.

2. Existence of lattices

For a simply connected solvable Lie group G, a *lattice* of G is a discrete cocompact subgroup of G. As is well known, the three nilpotent groups and the first solvable group $\mathrm{Sol}^3 \times \mathbb{R}$ have lattices. We study the remaining three solvable cases. We shall prove Sol_0^4 and Sol'^4_0 do not have lattices; and the group Sol_λ^4 has generically no lattice except for countably many values of λ's.

Proposition 2.1 ([12]). *The group Sol_λ^4 has a lattice if and only if there exist integers m, n such that the equation $x^3 - mx^2 + nx - 1 = 0$ has 3 distinct positive real roots $\alpha_1 > \alpha_2 > \alpha_3$ (with $\lambda = \frac{\ln \alpha_1}{\ln \alpha_2}$). We call such Sol_λ^4 as $\mathrm{Sol}_{m,n}^4$. There are only countably many such λ's.*

Proof. Recall $G = \mathrm{Sol}_\lambda^4 = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$, where

$$\varphi(s) = \begin{bmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-(1+\lambda)s} \end{bmatrix}, \quad (\lambda > 1).$$

Suppose Γ is a lattice. Since $\mathbb{R}^3$ is the *nil-radical* (i.e., the maximal connected normal nilpotent subgroup) of G, $\Gamma \cap \mathbb{R}^3 \cong \mathbb{Z}^3$ must be a lattice in $\mathbb{R}^3$. Thus Γ is of the form $\mathbb{Z}^3 \rtimes \mathbb{Z}$, where a generator of $\mathbb{Z}$ acts on $\mathbb{Z}^3$ via $A \in \mathrm{GL}(3,\mathbb{Z})$. Let P be a matrix diagonalizing A. Then

$$\varphi(s_0) = PAP^{-1}.$$

Let

$$\chi_A(x) = x^3 - mx^2 + nx - 1$$

be the characteristic polynomial of A (so $m, n \in \mathbb{Z}$). Since A and $\varphi(s_0)$ have the same characteristic polynomial, we have

$$(2\text{–}1)\qquad \begin{cases} m &= e^{\lambda s_0} + e^{s_0} + e^{-(1+\lambda)s_0}, \\ n &= e^{-\lambda s_0} + e^{-s_0} + e^{(1+\lambda)s_0}. \end{cases}$$

Note that $m, n > 0$. For example, we know the companion matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{bmatrix}$$

has the characteristic polynomial $\chi_A(x)$.

The function $\chi_A(x)$ has two critical points

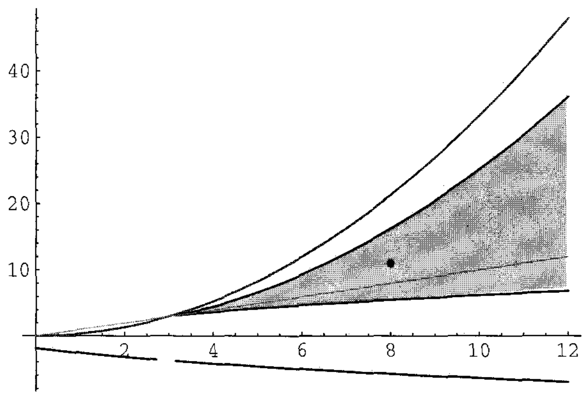
$$\tfrac{1}{3}(m \pm \sqrt{m^2 - 3n}).$$

Then $\chi_A(x) = 0$ has 3 distinct positive real roots if and only if $m^2 > 3n$ and

$$\chi_A(\tfrac{1}{3}(m - \sqrt{m^2 - 3n})) > 0,$$
$$\chi_A(\tfrac{1}{3}(m + \sqrt{m^2 - 3n})) < 0.$$

We need one more condition: 1 cannot be the root. Otherwise, the eigenvalues will be $e^{\lambda s}$, $e^{-\lambda s}$ and 1 so that the Lie group becomes $\mathrm{Sol}^3 \times \mathbb{R}$. Since $\chi_A(1) = 0$ if and only if $m = n$, we need to exclude the cases $m = n$.

Thus, if the group $\mathrm{Sol}^4_\lambda = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ has a lattice, then there exists a pair of positive integers (m, n) for which $x^3 - mx^2 + nx - 1 = 0$ has 3 distinct positive real roots $e^{\lambda s_0}$, e^{s_0} and $e^{-(1+\lambda)s_0}$. [Then (m, n) lies in the region].



Conversely, suppose (m, n) lies in the shaded region minus the line $n = m$. Then the equation $x^3 - mx^2 + nx - 1 = 0$ has 3 distinct positive real roots, say $\alpha_1 > \alpha_2 > \alpha_3$. Then the equations (2–1) yields

$$\lambda = \frac{\ln \alpha_1}{\ln \alpha_2}; \quad s_0 = \ln \alpha_2.$$

For example, for the point $(m, n) = (8, 11)$, the equation $\chi_A(x) = 0$ has 3 positive real roots. [All the integer points in the shaded region containing $(8, 11)$ in the picture give rise to the same results]. The region contains only countably infinite pairs (m, n) of integers. Consequently, for only countably infinite values of λ's, the matrix $\varphi(s)$ can be conjugated to an integral matrix for some s. This proves that Sol^4_λ has generically no lattice except for countably many values of λ's. □

Next, we look at the groups Sol^4_0 and Sol'^4_0.

Proposition 2.2. *The groups* Sol^4_0 *and* Sol'^4_0 *do not admit any lattice.*

Proof. Recall Sol^4_0 and Sol'^4_0 are of the form $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$,

$$\varphi(s) = \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e^s & se^s & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix}.$$

Suppose Γ is a lattice. As before, $\Gamma \cap \mathbb{R}^3 \cong \mathbb{Z}^3$ must be a lattice in $\mathbb{R}^3$. Thus Γ is of the form $\mathbb{Z}^3 \rtimes \mathbb{Z}$, where a generator of $\mathbb{Z}$ acts on $\mathbb{Z}^3$ via $A \in \mathrm{GL}(3, \mathbb{Z})$. Let P be a matrix diagonalizing A. Then

$$\varphi(s_0) = PAP^{-1}.$$

Let

$$\chi_A(x) = x^3 - mx^2 + nx - 1$$

be the characteristic polynomial of A (so $m, n \in \mathbb{Z}$). Since A and $\varphi(s_0)$ have the same characteristic polynomial, we have

$$m = 2e^s + e^{-2s}$$
$$n = 2e^{-s} + e^{2s}.$$

The function $\chi_A(x)$ has two critical points

$$\tfrac{1}{3}(m \pm \sqrt{m^2 - 3n}).$$

Then $\chi_A(x) = 0$ has 2 positive real roots (one of them is a double root) if and only if

$$\chi_A(\tfrac{1}{3}(m - \sqrt{m^2 - 3n})) = 0 \text{ and } \chi_A(\tfrac{1}{3}(m + \sqrt{m^2 - 3n})) < 0$$

or

$$\chi_A(\tfrac{1}{3}(m - \sqrt{m^2 - 3n})) > 0 \text{ and } \chi_A(\tfrac{1}{3}(m + \sqrt{m^2 - 3n})) = 0.$$

For the first, the equation

$$-27 - 2m^3 + 9mn + 2(m^2 - 3n)^{3/2} = 0$$

must have integer solutions m, n. Clearly, $\sqrt{m^2 - 3n}$ must be an integer. Let

$$m^2 - 3n = r^2$$

with $r > 0$. Then the above equation yields the polynomial

$$\begin{aligned} g(m,r) &= -27 + m^3 - 3mr^2 + 2r^3 \\ &= (m-r)^2(m+2r) - 27. \end{aligned}$$

Suppose $g(m,r) = 0$. Then, for $m > 26$, we have $0 < |r - m| < 1$, which is impossible since m and r are both integers. (In fact, this is true for $m \geq 9$). By checking for $1 \leq m \leq 26$, we conclude that $g(m,r) = 0$ has no integer solutions. Consequently, the group does not admit a lattice.

For the second, the equation

$$-27 - 2m^3 + 9mn - 2(m^2 - 3n)^{3/2} = 0$$

must have integer solutions m, n. Clearly, $\sqrt{m^2 - 3n}$ must be an integer. Let

$$m^2 - 3n = r^2$$

with $r > 0$. Then the above equation yields the polynomial

$$\begin{aligned} g(m,r) &= -27 + m^3 - 3mr^2 - 2r^3 \\ &= (m+r)^2(m-2r) - 27. \end{aligned}$$

Suppose $g(m,r) = 0$. Then, for $m > 5$, we have $0 < |m - 2r| < 1$, which is impossible since m and r are both integers. By checking for $1 \leq m \leq 5$, we conclude that $g(m,r) = 0$ has no integer solutions. Consequently, the group does not admit a lattice. □

3. Infra-homogeneous spaces

We shall need Gordon-Wilson's result:

Theorem 3.1 ([7]). *Let G be a solvable Lie group which is of type* (R) *and is unimodular. Then, with respect to any left-invariant Riemannian metric, the group of left-translations $\ell(G)$ is normal in* $\mathrm{Isom}_0(G)$*, the connected component of the group of isometries of G.*

Consequently, with respect to any left-invariant Riemannian metric on a unimodular Lie group G of type (R),

$$\mathrm{Isom}(G) \subset \ell(G) \rtimes K,$$

where K is a maximal compact subgroup of $\mathrm{Aut}(G)$. Conversely, for any maximal compact subgroup K of $\mathrm{Aut}(G)$, there exists a left-invariant Riemannian metric on G for which $\mathrm{Isom}(G) = \ell(G) \rtimes K$. Therefore, in order to understand the isometry group $\mathrm{Isom}(G)$, it is enough to calculate $\mathrm{Aut}(G)$.

For the case when $\varphi(s)$ has all eigenvalues not equal to 1 (that is, Sol_0^4, $\mathrm{Sol'}_0^4$, Sol_λ^4), $\mathbb{R}^3$ is the nil-radical in $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$, and hence is a characteristic subgroup of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$; every automorphism of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ restricts to an automorphism of

$\mathbb{R}^3$. Consequently an automorphism of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ induces an automorphism on the quotient group $\mathbb{R}$. Thus there is a natural homomorphism

$$\begin{array}{ccc}\mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R}) & \longrightarrow & \mathrm{GL}(3,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R}) \\ \theta & \mapsto & (\hat{\theta}, \bar{\theta}).\end{array}$$

By Gram-Schmidt, $\widehat{\theta}$ is conjugate to a blocked upper triangular matrix. We need to look into the eigenvalues of the matrix $\widehat{\theta}$. Except for the case when $\widehat{\theta}$ has complex eigenvalues, G is always of type (R). But in general, the trace of $\widehat{\theta}$ will not be zero. Such G will not have any lattice, and the group of isometries is hard to calculate.

Proposition 3.2. *Let*

$$C = \{(\widehat{\theta}, \overline{\theta}) \in \mathrm{GL}(3,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R}) : \varphi(\overline{\theta}(s)) = \widehat{\theta} \circ \varphi(s) \circ \widehat{\theta}^{-1}\}.$$

Then

$$\mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R}) = \mathbb{R}^3 \rtimes C \subset \mathbb{R}^3 \rtimes \Big(\mathrm{GL}(3,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})\Big).$$

Proof. Let $\theta \in \mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$. Then $(\hat{\theta}, \bar{\theta}) \in \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R})$ and $\theta(\mathbf{x}, 0) = (\hat{\theta}(\mathbf{x}), 0)$. Define $\eta : \mathbb{R} \to \mathbb{R}^3$ by $\theta(\mathbf{0}, s) = (-\eta(\overline{\theta}(s)), \bar{\theta}(s))$. Thus,

$$\theta(\mathbf{x}, s) = \theta((\mathbf{x}, 0)(\mathbf{0}, s)) = (\hat{\theta}(\mathbf{x}), 0)(-\eta(\overline{\theta}(s)), \bar{\theta}(s)) = (\hat{\theta}(\mathbf{x}) - \eta(\overline{\theta}(s)), \bar{\theta}(s)).$$

We write this θ as $(\eta, \hat{\theta}, \overline{\theta})$. Thus,

$$(\eta, \hat{\theta}, \overline{\theta})(\mathbf{x}, s) = (\hat{\theta}(\mathbf{x}) - \eta(\overline{\theta}(s)), \bar{\theta}(s)).$$

For this to be an automorphism of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$, it should satisfy

$$\varphi(\overline{\theta}(s)) = \widehat{\theta} \circ \varphi(s) \circ \widehat{\theta}^{-1},$$
$$\eta(s+t) = \eta(s) + \varphi(\bar{\theta}(s))\eta(t)$$

for all $s, t \in \mathbb{R}$, or equivalently $(\hat{\theta}, \bar{\theta}) \in C$ and $\eta : \mathbb{R} \to \mathbb{R}^3$ is a crossed homomorphism with respect to the action homomorphism $\varphi \circ \bar{\theta}$. Therefore, we have a homomorphism $\mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R}) \to C$.

Conversely, suppose that $(\hat{\theta}, \bar{\theta}) \in C$. For any crossed homomorphism $\eta : \mathbb{R} \to \mathbb{R}^3$ with respect to the action homomorphism $\varphi \circ \bar{\theta}$, we define $\theta \in \mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$ by

$$\theta(\mathbf{x}, s) = (\hat{\theta}(\mathbf{x}) - \eta(\overline{\theta}(s)), \bar{\theta}(s)).$$

Then it is easy to check that θ is an automorphism of $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$. Moreover, this with $\eta = 0$ defines a split homomorphism $C \to \mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$.

In particular, we have observed that $\theta \in \mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$ with $(\hat{\theta}, \bar{\theta}) = (\mathrm{id}_{\mathbb{R}^3}, \mathrm{id}_{\mathbb{R}})$ induces a crossed homomorphism with action homomorphism exactly $\varphi \circ \bar{\theta} = \varphi$.

Observe that a crossed homomorphism η is completely determined by the value $\eta(1)$, and hence the subgroup of all crossed homomorphisms is isomorphic to $\mathbb{R}^3$. □

A finite quotient of a homogeneous space $\Gamma\backslash G$ (where Γ is a lattice of G) is an infra-homogeneous space. We consider the infra-homogeneous spaces for each of the groups.

(1) $\mathbb{R}^4$: There are 75 flat manifolds in dimension 4. See [3], for example.
(2) $\mathrm{Nil}^3 \times \mathbb{R}$: There are 74 families of infra-nilmanifolds. For Nil^4, roughly speaking, there are 7 families some of which split into 2 or 3 subfamilies. The only holonomy groups are 1, $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. See [4] and [5].
(3) $\mathrm{Sol}^3 \times \mathbb{R}$: See Proposition 3.4 below.
(4) Sol_0^4 and $\mathrm{Sol'}_0^4$: No lattices, see Proposition 2.2.
(5) Sol_λ^4: Only $\mathrm{Sol}_{m,n}^4$ has a lattice. There are no other infra-solvmanifolds, see Proposition 2.1.

Recall the embedding of $G = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ into $\mathrm{Aff}(4)$:

$$\begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$$

where $\varphi(s) \in \mathrm{GL}(3,\mathbb{R})$, $\mathbf{x} \in \mathbb{R}^3$ is a column vector, and $s \in \mathbb{R}$.

In general, the normalizer of G in $\mathrm{Aff}(4)$ is not enough to get all of automorphisms of G. But for our $G = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$, the natural map $N_{\mathrm{Aff}(4)}(G) \to \mathrm{Aut}(G)$ is surjective. The normalizer of G is

$$\alpha = \begin{bmatrix} \widehat{\theta} & \mathbf{m} & \mathbf{u} \\ 0 & \overline{\theta} & v \\ 0 & 0 & 1 \end{bmatrix}$$

with the conditions

$$\widehat{\theta} \circ \varphi(s) \circ \widehat{\theta}^{-1} = \varphi(\overline{\theta}(s)) \tag{3–1}$$

$$(I - \varphi(\overline{\theta}(s)))\mathbf{m} = 0 \tag{3–2}$$

for all $s \in \mathbb{R}$. For such α,

$$\begin{bmatrix} \widehat{\theta} & \mathbf{m} & \mathbf{u} \\ 0 & \overline{\theta} & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\theta} & \mathbf{m} & \mathbf{u} \\ 0 & \overline{\theta} & v \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \varphi(\overline{\theta}(s)) & 0 & \mathbf{x}' \\ 0 & 1 & \overline{\theta}(s) \\ 0 & 0 & 1 \end{bmatrix} \tag{3–3}$$

where $\mathbf{x}' = \widehat{\theta}(\mathbf{x}) + s\mathbf{m} + (I - \varphi(\overline{\theta}(s)))\mathbf{u}$. This shows conjugation by $(\widehat{\theta}, \overline{\theta})$ (i.e., with $\mathbf{m} = \mathbf{u} = 0$ and $v = 0$) is an automorphism. Conversely, for any $\eta(1) \in \mathbb{R}^3$, write it as

$$\eta(1) = \mathbf{m} + (I - \varphi(1))\mathbf{u},$$

where $\mathbf{m} \in \ker(I - \varphi(1))$. Then conjugation by $(\mathbf{m}, \mathbf{u}, v)$ (with $\widehat{\theta} = \mathrm{id}, \overline{\theta} = \mathrm{id}$) is exactly the automorphism induced by η.

The equation (3–3) shows also the centralizer of G in Aff(4). It consists of

$$\begin{bmatrix} I & 0 & \mathbf{u}_0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$$

where $(I - \varphi(1))\mathbf{u}_0 = 0$. In case 1 is an eigenvalue of $\varphi(1)$, we denote the "complementary eigenspace" (so that V is $\varphi(1)$-invariant) by

$$\ker(I - \varphi(1))^{\perp}.$$

Proposition 3.3. *There is a one-one correspondence*

$$\mathrm{Aut}(\mathbb{R}^3 \rtimes_{\varphi} \mathbb{R}) \cong \left\{ \begin{bmatrix} \widehat{\theta} & \mathbf{m} & \mathbf{u} \\ 0 & \overline{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

where $\widehat{\theta}$, $\overline{\theta}$, $\mathbf{m}$ *and* $\mathbf{u}$ *satisfy*

$$\begin{aligned} &\widehat{\theta} \circ \varphi(s) \circ \widehat{\theta}^{-1} = \varphi(\overline{\theta}(s)) \\ &\mathbf{m} \in \ker(I - \varphi(\overline{\theta}(s))) \\ &\mathbf{u} \in \ker(I - \varphi(1))^{\perp}. \end{aligned}$$

Note that conjugations by the two matrices

$$\begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(where the latter is used in the Proposition 3.3) result in the same automorphism.

The isometry group of Sol^3 is not always $\mathrm{Sol}^3 \rtimes D_8$. Sometimes, it is $\mathrm{Sol}^3 \rtimes (\mathbb{Z}_2)^2$, depending on the left-invariant Riemannian metric. See Ha-Lee [8] and [9, Theorem 3.3]. With the best left-invariant Riemannian metric, Sol^3 has isometry group $\mathrm{Sol}^3 \rtimes D_8$ (see for example, [11]). None of these finite subgroups of the isometry group can act freely. We denote by $\mathbb{R}^*$ the multiplicative group $\mathbb{R} - \{0\}$. On $\mathrm{Sol}^3 \times \mathbb{R}$, it is not too hard to see the following:

Theorem 3.4. *The group of automorphisms of* $\mathrm{Sol}^3 \times \mathbb{R}$ *is* $\mathbb{R}^3 \rtimes ((\mathbb{R}^*)^3 \rtimes \mathbb{Z}_2)$, *and the maximal group of isometries is*

$$\begin{aligned} \mathrm{Isom}(\mathrm{Sol}^3 \times \mathbb{R}) &= (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes ((\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_2) \\ &= (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_8 \times \mathbb{Z}_2). \end{aligned}$$

Every infra-solvmanifold is the quotient by torsion free extension π *of a lattice* Γ $(= \Delta \times \mathbb{Z}$, *where* Δ *is a lattice of* $\mathrm{Sol}^3)$ *by a cyclic subgroup of* D_8. *The possible holonomies are* 1, $\mathbb{Z}_2$ *and* $\mathbb{Z}_4$. *The group* π *is again an extension of* Δ *by* $\mathbb{Z}$.

Proof. Recall

$$\varphi(s) = \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The conditions on $\mathbf{m}$ and $\mathbf{u}$ in Proposition 3.3 yield

$$\mathbf{m} = \begin{bmatrix} 0 \\ 0 \\ m \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}.$$

Examining the first condition there, we find

$$\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R}) = \mathbb{R}^3 \rtimes ((\mathbb{R}^*)^3 \rtimes \mathbb{Z}_2)$$

corresponds to the following matrices:

$$\begin{bmatrix} p_{11} & 0 & 0 & 0 & u_1 \\ 0 & p_{22} & 0 & 0 & u_2 \\ 0 & 0 & p_{33} & m & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $(u_1, u_2, m) \in \mathbb{R}^3$, $(p_{11}, p_{22}, p_{33}) \in (\mathbb{R}^*)^3$, respectively. The maximal compact subgroup is $D_8 \times \mathbb{Z}_2$ generated by

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The factor $\mathbb{Z}_2$ is generated by -1 on $(3,3)$-slot. Every element of the dihedral group has fixed point on Sol^3. For a cyclic subgroup, one can resolve the fixed point by advancing to the $\mathbb{R}$-direction ($(3,5)$-slot) so that finite power generates a lattice of $\mathbb{R}$. The cyclic subgroup Φ is either trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_4$. The extended group π fits the short exact sequence

$$1 \to \Delta \rtimes \mathbb{Z} \to \pi \to \Phi \to 1.$$

Note that $\pi/\Delta \cong \mathbb{Z}$ also. □

Theorem 3.5. (1) *The group of isometries of* Sol^4_λ *is* $\mathrm{Sol}^4_\lambda \rtimes (\mathbb{Z}_2)^3$. (2) *The only infra-solvmanifolds modeled on* $\mathrm{Sol}^4_{m,n}$ *are solvmanifolds* $\Gamma\backslash\mathrm{Sol}^4_{m,n}$ *for some lattice* Γ.

Proof. Recall

$$\varphi(s) = \begin{bmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^s 0 & \\ 0 & 0 & e^{-(1+\lambda)s} \end{bmatrix}.$$

The conditions on $\mathbf{m}$ and $\mathbf{u}$ in Proposition 3.3 yield

$$\mathbf{m} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Examining the first condition there, we find

$$\mathrm{Aut}(\mathrm{Sol}^4) = \mathbb{R}^3 \rtimes (\mathbb{R}^*)^3$$

corresponds to the following matrices:

$$\begin{bmatrix} p_{11} & 0 & 0 & 0 & u_1 \\ 0 & p_{22} & 0 & 0 & u_2 \\ 0 & 0 & p_{33} & 0 & u_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $(u_1, u_2, u_3) \in \mathbb{R}^3$, $(p_{11}, p_{22}, p_{33}) \in (\mathbb{R}^*)^3$, respectively. The maximal compact subgroup is the diagonal matrices $(\mathbb{Z}_2)^3$ in $(\mathbb{R}^*)^3$.

Clearly, then an element of $\mathrm{Aut}(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$ is torsion if and only if it is of the form

$$\alpha = \begin{bmatrix} \epsilon & 0 & \mathbf{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \epsilon = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

satisfying

$$(I + \epsilon)\mathbf{x} = 0.$$

Consequently, such a torsion element lies in $(\mathbb{R}^3 \rtimes_\varphi \mathbb{R}) \rtimes (\mathbb{Z}_2)^3$.

Now we specialize to the case where $\mathrm{Sol}^4_\lambda = \mathrm{Sol}^4_{m,n}$. We claim that α cannot leave any lattice Γ invariant. [Therefore, there cannot exist a finite extension of Γ]. Since $\mathbb{R}^3$ is a nil-radical of our group,

$$Z = \Gamma \cap \mathbb{R}^3$$

is a lattice of $\mathbb{R}^3$, and is a characteristic subgroup of Γ. See, [10, Corollary 3.5]. Thus, α should leave Z invariant as well. Let

$$\mathbf{z} = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in Z.$$

Then

$$(\alpha \mathbf{z} \alpha^{-1})\mathbf{z}^{-1} = (\epsilon - I)\mathbf{z}.$$

Therefore, for $\mathbf{z} \in Z$, we must have

$$(\epsilon - I)\mathbf{z}, \ (\epsilon + I)\mathbf{z} \in Z.$$

Recall that ϵ was a diagonal matrix with entries ± 1's. Unless $\epsilon = I$, at least one axis contains a subgroup $\mathbb{Z}$ of Z. This is not possible because each of the 3 axes is an eigenspace of $\varphi(s)$; ϵ should conjugate this $\mathbb{Z}$ onto itself, but $\varphi(s)$

does not have eigenvalue 1. This completes the proof that any lattice Γ does not have an extension by a finite group. □

4. The first Bieberbach Theorem

Statement 4.1. Let G be a connected, simply connected Lie group and let K be a compact subgroup of $\mathrm{Aut}(G)$. Suppose $\pi \subset G \rtimes K$ is a lattice, then $\Gamma = \pi \cap G$ is a lattice of G (and Γ has finite index in π).

The first Bieberbach Theorem states that Statement 4.1 holds for $G = \mathbb{R}^n$. This was generalized to nilpotent Lie groups by L. Auslander, [1] and [2]. Statement 4.1 was further generalized to some solvable Lie groups, see [6]. Using the results in [6], we can see easily that the first Bieberbach Theorem holds for all the Lie groups $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ except for Sol_0^4.

Theorem 4.2. [6, Theorem B] *Let G be a connected, simply connected solvable Lie group of type* (E) *with nil-radical N, and let $G/N = \mathbb{R}^n$. Let $\rho : \mathbb{R}^n \to Out(N)$ be the canonical representation. Assume*:

The centralizer of $\rho(\mathbb{R}^n)$ in $Out(N)$ has trivial maximal torus.

Then Statement 4.1 holds for this G.

The condition (3–1) indicates that, unless $\varphi(s)$ has two dimensional eigenspace, $\widehat{\theta}$ cannot contain a circle. Thus, except for $G = \mathrm{Sol}_0^4$, Statement 4.1 holds for all other solvable G's. (Thus, if there is no lattice in G, then there is no compact form in $\mathrm{Isom}(G)$ either).

Theorem 4.3. *For the group $G = \mathrm{Sol}_0^4$,*

(1) *The group of isometries is* $\mathrm{Isom}_0(G) = G \rtimes (O(2) \times O(1))$.
(2) *There is no lattice in G.*
(3) *There are countably infinite distinct lattices in* $\mathrm{Isom}_0(G)$. *Consequently, the first Bieberbach theorem does not hold for G.*
(4) *For any lattice Π of* $\mathrm{Isom}_0(G)$, *there is no extension $\pi \subset$* $\mathrm{Isom}(G)$ *such that the image of π under the natural map*

$$\mathrm{Isom}(G) \longrightarrow \mathrm{Isom}(G)/\mathrm{Isom}_0(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

is non-trivial.

Proof. With the conditions in Proposition 3.3, we get

$$\mathrm{Aut}(\mathrm{Sol}_0^4) = \mathrm{Sol}_0^4 \rtimes (\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R}))$$

where $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})$ is generated by

$$\begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & 0 \\ 0 & 0 & p_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A maximal compact subgroup is $O(2) \times O(1)$ which is of the form

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \in O(2), \qquad p_{33} = \pm 1 \in O(1)$$

in the above matrix representation. Then with the best left-invariant Riemannian metric on G, we have

$$\mathrm{Isom}(G) = G \rtimes (O(2) \times O(1)) \subset \mathrm{Aut}(G).$$

We shall find a lattice Π of $\mathrm{Isom}_0(G) = G \rtimes \mathrm{SO}(2)$. As noted before, $\mathbb{R}^3$ is the nil-radical of G, so $\Pi \cap \mathbb{R}^3 = \mathbb{Z}^3$ must be a lattice in $\mathbb{R}^3$. Thus Π is of the form $\mathbb{Z}^3 \rtimes_A \mathbb{Z}$, where the generator $1 \in \mathbb{Z}$ acts on $\mathbb{Z}^3$ by $A \in \mathrm{GL}(3, \mathbb{Z})$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 \rtimes_A \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \rtimes_\varphi (\mathbb{R} \times \mathrm{SO}(2)) & \longrightarrow & \mathbb{R} \times \mathrm{SO}(2) & \longrightarrow & 1 \end{array}$$

Since $\mathbb{Z}^3 \rtimes_A \mathbb{Z} \hookrightarrow \mathbb{R}^3 \rtimes_\varphi (\mathbb{R} \times \mathrm{SO}(2))$, there exists a matrix $P \in \mathrm{GL}(3, \mathbb{R})$ so that

$$\begin{aligned} A' \equiv PAP^{-1} &= \begin{bmatrix} e^\alpha & 0 & 0 \\ 0 & e^\alpha & 0 \\ 0 & 0 & e^{-2\alpha} \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^\alpha \cos\beta & e^\alpha \sin\beta & 0 \\ -e^\alpha \sin\beta & e^\alpha \cos\beta & 0 \\ 0 & 0 & e^{-2\alpha} \end{bmatrix} \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$. The vertical maps are

$$\begin{array}{ccc} \mathbf{z} & (\mathbf{z}, n) & n \\ \downarrow & \downarrow & \downarrow \\ P\mathbf{z} & (P\mathbf{z}, (\alpha n, e^{i\beta n})) & (\alpha n, e^{i\beta n}) \end{array}$$

and $(\alpha n, e^{i\beta n})$ acts on $P\mathbf{z}$ by PA^nP^{-1}. Let

$$\chi_A(x) = x^3 - mx^2 + nx - 1$$

be the characteristic polynomial of A (so $m, n \in \mathbb{Z}$). Since A is conjugate to A', they have the same characteristic polynomial. Thus, $\chi_A(x) = 0$ must have only *one* (positive) real root.

Conversely, suppose $x^3 - mx^2 + nx - 1 = 0$ has only one positive real root so that

$$x^3 - mx^2 + nx - 1 = (x - a)((x - b)^2 + c^2)$$

for some real $a > 0$, b and $c \neq 0$. Then

$$a(b^2 + c^2) = 1.$$

Therefore, if we set $a = e^{-2\alpha}$, then

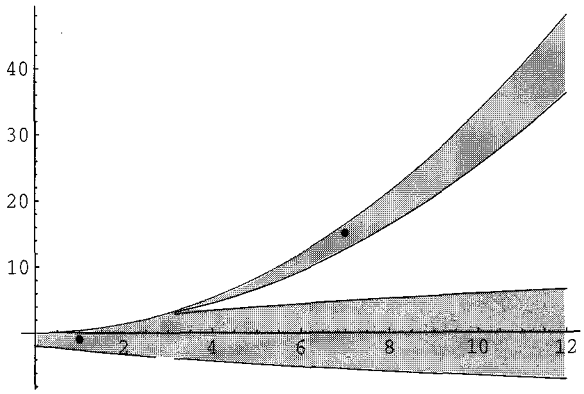
$$b = e^{\alpha}\cos\beta$$
$$c = e^{\alpha}\sin\beta$$

for some β.

Since $\chi_A(0) = -1$ and $\lim_{x\to\infty} = +\infty$, by intermediate value theorem, the condition having a *positive* real root is automatic. That is, there always exists one positive real root. Therefore, the following are equivalent:

(1) $\chi_{A'}(x) = \chi_A(x) \quad (= x^3 - mx^2 + nx - 1)$
(2) $\chi_A(x) = 0$ has only one (positive) real root
(3) (a) $m^2 > 3n$ and $\chi_A(\frac{1}{3}(m + \sqrt{m^2 - 3n})) > 0$, or
(b) $m^2 > 3n$ and $\chi_A(\frac{1}{3}(m - \sqrt{m^2 - 3n})) < 0$.

(Observe that $\frac{1}{3}(m \pm \sqrt{m^2 - 3n})$ are the two critical points of $\chi_A(x)$). All the integer points in the region containing $(7, 15)$ in the picture satisfy the first inequalities (3a). All the integer points in the region containing $(1, -1)$ surrounded by the 3 curves together with $x > 0$ in the picture satisfy the second inequalities (3b). We can easily see that there are infinitely many pairs (m, n)



of integers which satisfy the above inequalities. Then

$$m = e^{-2\alpha} + 2e^{\alpha}\cos\beta$$
$$n = e^{2\alpha} + 2e^{-\alpha}\cos\beta$$

which determines α and β. For example, if $(m, n) = (7, 15)$, we get

$$\alpha = -\frac{1}{2}\ln\left[\frac{1}{3}\left\{4 - \left(\frac{2}{\omega}\right)^{\frac{1}{3}} - \left(\frac{\omega}{2}\right)^{\frac{1}{3}}\right\}\right],$$

where $\omega = 25 - 3\sqrt{69}$. Thus, $\alpha \approx 0.702999$ and $\beta \approx 0.929517$ (thus, the pair (m, n) determines α and β). Since we already know G does not have a lattice, the intersection $(\mathbb{Z}^3 \rtimes_{A'} \mathbb{Z}) \cap G$ cannot be a lattice of G.

Different values of (m,n) yield different lattices of G. There are countably infinite distinct lattices in G. The $\mathbb{Z}$-factor of the lattice $\Pi = \mathbb{Z}^3 \rtimes \mathbb{Z}$ is embedded as $\varphi(n)$, $n \in \mathbb{Z}$, where

$$\varphi(s) = \begin{bmatrix} e^{\alpha s}\cos(\beta s) & e^{\alpha s}\sin(\beta s) & 0 & 0 & 0 \\ -e^{\alpha s}\sin(\beta s) & e^{\alpha s}\cos(\beta s) & 0 & 0 & 0 \\ 0 & 0 & e^{-2\alpha s} & 0 & 0 \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose there exists $\pi \subset \mathrm{Isom}(G)$ so that the commutative diagram of exact rows commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & \pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Isom}_0(G) & \longrightarrow & \mathrm{Isom}(G) & \longrightarrow & \frac{O(2)}{\mathrm{SO}(2)} \times O(1) & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ & & G \rtimes \mathrm{SO}(2) & & G \rtimes (O(2) \times O(1)) & & \mathbb{Z}_2 \times \mathbb{Z}_2 & & \end{array}$$

The following are generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The equality

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha s}\cos(\beta s) & e^{\alpha s}\sin(\beta s) & 0 \\ -e^{\alpha s}\sin(\beta s) & e^{\alpha s}\cos(\beta s) & 0 \\ 0 & 0 & e^{-2\alpha s} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} e^{\alpha s}\cos(\beta s) & -e^{\alpha s}\sin(\beta s) & 0 \\ e^{\alpha s}\sin(\beta s) & e^{\alpha s}\cos(\beta s) & 0 \\ 0 & 0 & e^{-2\alpha s} \end{bmatrix}$$

shows that B_1 does not normalize Π. For B_2, suppose it normalized Π. Then it will normalize $\mathbb{Z}^3$ since it is a characteristic subgroup. Let

$$\mathbf{z} = \begin{bmatrix} 1 & 0 & 0 & 0 & z_1 \\ 0 & 1 & 0 & 0 & z_2 \\ 0 & 0 & 1 & 0 & z_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in P\mathbb{Z}^3.$$

We denote $\mathbf{z}$ by (z_1, z_2, z_3). Then

$$B_2(z_1, z_2, z_3)B_2^{-1} = (z_1, z_2, -z_3).$$

This implies

$$(0,0,2z_3) = (z_1,z_2,z_3)\left(B_2(z_1,z_2,z_3)B_2^{-1}\right)^{-1} \in P\mathbb{Z}^3.$$

This is impossible since the 3rd axis is an eigenspace of A' with eigenvalue e^{-2a} with $a > 0$, (a lattice cannot be expanded or shrunk by its automorphism).

Consequently, there is no extension of Π by any subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Remark 4.4. This is the lowest dimensional example of a solvable Lie group of type (R) where the first Bieberbach theorem fails. There was a 5-dimensional example in [6, Example 3.2]. In both cases, the existence of a compact subgroup SO(2) of Aut(G) is essential, as it was a necessary condition for the failure. See Theorem 4.2. There exists a 3-dimensional example which is not of type (E), see below.

Since $\mathbb{R}^2$ is the only 2-dimensional simply connected solvable Lie group, we need to check only 3-dimensional Lie groups. Suppose G is a 3-dimensional simply connected solvable Lie group. Obviously, the nil-radical of G cannot be 1-dimensional. If it is 3-dimensional, G itself is nilpotent, and we know Statement 4.1 holds for nilpotent groups. Now suppose the nil-radical of G is 2-dimensional. Then G is of the form $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ (and its Lie algebra must be of the form $\mathbb{R}^2 \rtimes_{\mathcal{A}} \mathbb{R}$). If G is of type (R), possible A's are

$$\mathcal{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & c \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

For this to have a lattice, the trace must be 0. Then the first case yields G abelian, the second case yields G nilpotent, while the third case ($\lambda_1 + \lambda_2 = 0$) yields G the 3-dimensional Sol. Note that the first Bieberbach theorem holds for all these cases. Thus the group $G = \mathrm{Sol}_0^4$ in Theorem 4.3 is the lowest dimensional example of a solvable Lie group of type (R) where the first Bieberbach theorem fails.

On the other hand, consider the universal covering group G of $E_2(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. So, G is isomorphic to $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$, where $\varphi(t) = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}$. (This is where the $\mathcal{A} = \psi(1)$ above is $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$). Aut($G$) contains SO(2). Now consider the subgroup of $G \rtimes \mathrm{SO}(2) = (\mathbb{R}^2 \rtimes \mathbb{R}) \rtimes \mathrm{SO}(2)$ generated by

$$\left(\begin{bmatrix}1\\0\end{bmatrix}, 0, I\right), \left(\begin{bmatrix}0\\1\end{bmatrix}, 0, I\right), \left(\begin{bmatrix}0\\0\end{bmatrix}, \alpha, \varphi(-\alpha)\right),$$

where α is an irrational number. Clearly, this group Γ is isomorphic to $\mathbb{Z}^3$, but $\Gamma \cap G$ is just $\mathbb{Z}^2$, violating the first Bieberbach theorem. Note that this G is not of type (E).

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