LEONHARD EULER (1707-1783) AND THE COMPUTATIONAL ASPECTS OF SOME ZETA-FUNCTION SERIES

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ABSTRACT. In this presentation dedicated to the tricentennial birth anniversary of the great eighteenth-century Swiss mathematician, Leonhard Euler (1707-1783), we begin by remarking about the so-called *Basler problem* of evaluating the Zeta function $\zeta(s)$ [in the much later notation of Georg Friedrich Bernhard Riemann (1826-1866)] when $s = 2$, which was then of vital importance to Euler and to many other contemporary mathematicians including especially the Bernoulli brothers [Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748)], and for which a fascinatingly large number of seemingly independent solutions have appeared in the mathematical literature ever since Euler first solved this problem in the year 1736. We then investigate various recent developments on the evaluations and representations of $\zeta(s)$ when $s \in \mathbb{N} \setminus \{1\}$, $\mathbb{N}$ being the set of natural numbers. We emphasize upon several interesting classes of rapidly convergent series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ which have been developed in recent years. In two of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of series which converge much more rapidly than that in Euler's celebrated formula as well as the series used recently by Roger Apéry (1916-1994) in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of one of these series are capable of producing an accuracy of seven decimal places.

1. Introduction and motivation

Some of the important functions in *Analytic Number Theory* include (for example) the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized)

Zeta function $\zeta(s, a)$, which are defined (for $\Re(s) > 1$) by

$$(1.1) \quad \zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\[4mm] \displaystyle\frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1) \end{cases}$$

and
(1.2)

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right),$$

and (for $\Re(s) \leqq 1;\ s \neq 1$) by their *meromorphic* continuations (see, for details, Titchmarsh [39]; see also Whittaker and Watson [42]), so that (obviously)

$$(1.3) \quad \zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1}\, \zeta\left(s, \frac{1}{2}\right) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1.$$

More generally, we have the following relationships:

$$(1.4) \qquad\qquad \zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right)$$

$$(m \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1, 2, 3, \ldots\})$$

and

$$(1.5) \qquad\qquad \zeta(s, ma) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s, a + \frac{j}{m}\right) \qquad (m \in \mathbb{N}).$$

A fascinatingly large number of seemingly independent solutions of the so-called *Basler problem* of evaluating the Riemann Zeta function $\zeta(s)$ when $s = 2$, which was of vital importance to Leonhard Euler (1707-1783) and the Bernoulli brothers [Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748)], have appeared in the mathematical literature ever since Euler first solved this problem in the year 1736. Another remarkable *classical* result involving Riemann's ζ-function is the following elegant series representation for $\zeta(3)$:

$$(1.6) \qquad\qquad \zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\, 2^{2k}},$$

which was actually contained in Euler's 1772 paper entitled *"Exercitationes Analyticae"* (*cf.*, *e.g.*, Ayoub [3, pp. 1084-1085]). In fact, this result of Euler was rediscovered (among others) by Ramaswami [27] (see also Srivastava [28, p. 7, Equation (2.23)]) and (more recently) by Ewell [13]. And, as pointed out by (for example) Chen and Srivastava [5, pp. 180-181], another series

representation:

$$(1.7) \qquad \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}},$$

which played a key rôle in the *celebrated* proof [2] of the irrationality of $\zeta(3)$ by Roger Apéry (1916-1994), was derived *independently* by (among others) Hjortnaes [19], Gosper [17], and Apéry [2].

Clearly, Euler's series in (1.6) converges faster than the defining series for $\zeta(3)$, but obviously not as fast as the series in (1.7). Such Zeta values as $\zeta(3)$, $\zeta(5)$, *et cetera* are known to arise naturally in a wide variety of applications such as those in Elastostatics, Quantum Field Theory, *et cetera* (see, for example, Tricomi [40], Witten [44], and Nash and O'Connor [25], [26]). On the other hand, in the case of *even* integer arguments, we already have the following computationally useful relationship:

$$(1.8) \qquad \zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} \, B_{2n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$$

with the *well-tabulated* Bernoulli numbers defined by the generating function:

$$(1.9) \qquad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \, \frac{z^n}{n!} \qquad (|z| < 2\pi),$$

as well as the familiar recursion formula:

$$(1.10) \qquad \zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{k=1}^{n-1} \zeta(2k) \, \zeta(2n - 2k) \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Motivated essentially by a genuine need, for computational purposes, for expressing $\zeta(2n+1)$ as a rapidly converging series for all $n \in \mathbb{N}$, we propose to present here a rather systematic investigation of the various interesting families of rapidly convergent series representations for the Riemann $\zeta(2n+1)$ ($n \in \mathbb{N}$). We also consider relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$). In two of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of series which converge much more rapidly than that in Euler's celebrated formula (1.6) as well as the series (1.7) used recently by Apéry [2] in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of one of these series are capable of producing an accuracy of seven decimal places.

2. A class of series representations for $\zeta\left(2n+1\right)\left(n \in \mathbb{N}\right)$

We begin by recalling the following simple consequence of the binomial theorem *and* the definition (1.1):

$$(2.1) \qquad \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \, \zeta\left(s+k,a\right) t^k = \zeta\left(s,a-t\right) \quad \left(|t| < |a|\right),$$

which, for $a = 1$ and $t = \pm 1/m$, readily yields the series identity:

$$(2.2) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \, \frac{\zeta\left(s+2k\right)}{m^{2k}}$$
$$= \begin{cases} \left(2^s - 1\right) \, \zeta\left(s\right) - 2^{s-1} & (m = 2) \\[2ex] \dfrac{1}{2} \left[\left(m^s - 1\right) \, \zeta\left(s\right) - m^s - \displaystyle\sum_{j=2}^{m-2} \zeta\left(s, \frac{j}{m}\right)\right] & \left(m \in \mathbb{N} \setminus \{1,2\}\right), \end{cases}$$

$(\lambda)_n := \Gamma\left(\lambda + n\right)/\Gamma\left(\lambda\right)$ being the Pochhammer symbol (or the *shifted factorial*, since $(1)_n = n!$ for $n \in \mathbb{N}_0$). (See, for details, [29] and [34]).

In terms of the familiar harmonic numbers

$$(2.3) \qquad\qquad H_n := \sum_{j=1}^{n} \frac{1}{j} \qquad \left(n \in \mathbb{N}\right),$$

the following set of series representations for $\zeta\left(2n+1\right)\left(n \in \mathbb{N}\right)$ were proven recently by Srivastava [32] by appealing appropriately to the series identity (2.2) in its special cases when $m = 2, 3, 4,$ and 6, and also to many other properties and characteristics of the Riemann Zeta function such as the familiar functional equation:

$$(2.4) \qquad \zeta\left(s\right) = 2 \cdot \left(2\pi\right)^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma\left(1-s\right) \zeta\left(1-s\right)$$

or, equivalently,

$$(2.5) \qquad \zeta\left(1-s\right) = 2 \cdot \left(2\pi\right)^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma\left(s\right) \zeta\left(s\right),$$

the familiar derivative formula:

$$(2.6) \qquad \zeta'\left(-2n\right) = \lim_{\varepsilon \to 0} \left\{\frac{\zeta\left(-2n+\varepsilon\right)}{\varepsilon}\right\}$$
$$= \frac{\left(-1\right)^n}{2 \cdot \left(2\pi\right)^{2n}} \left(2n\right)! \, \zeta\left(2n+1\right) \qquad \left(n \in \mathbb{N}\right)$$

with, of course,

$$(2.7) \qquad \zeta\left(0\right) = -\frac{1}{2}; \quad \zeta\left(-2n\right) = 0 \quad \left(n \in \mathbb{N}\right); \quad \zeta'\left(0\right) = -\frac{1}{2}\log\left(2\pi\right),$$

and each of the following limit relationships:

$$(2.8) \qquad \lim_{s \to -2n} \left\{ \frac{\sin\left(\frac{1}{2}\pi s\right)}{s + 2n} \right\} = (-1)^n \frac{\pi}{2} \qquad (n \in \mathbb{N})$$

and

$$(2.9) \qquad \lim_{s \to -2n} \left\{ \frac{\zeta\left(s + 2k\right)}{s + 2n} \right\} = \frac{(-1)^{n-k}}{2 \cdot (2\pi)^{2(n-k)}} (2n - 2k)! \, \zeta\left(2n - 2k + 1\right)$$

$$(k = 1, \ldots, n-1; \ n \in \mathbb{N} \setminus \{1\}).$$

Series Representation 1:

$$\zeta\left(2n + 1\right)$$

$$(2.10) \qquad = (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1} - 1} \left[\frac{H_{2n} - \log \pi}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \, \frac{\zeta\left(2k + 1\right)}{\pi^{2k}} \right.$$

$$\left. + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \, \frac{\zeta\left(2k\right)}{2^{2k}} \right] \qquad (n \in \mathbb{N});$$

Series Representation 2:

$$\zeta\left(2n + 1\right)$$

$$(2.11) \qquad = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{3^{2n+1} - 1} \left[\frac{H_{2n} - \log\left(\frac{2}{3}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \, \frac{\zeta\left(2k + 1\right)}{\left(\frac{2}{3}\pi\right)^{2k}} \right.$$

$$\left. + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \, \frac{\zeta\left(2k\right)}{3^{2k}} \right] \qquad (n \in \mathbb{N});$$

Series Representation 3:

$$(2.12)$$

$$\zeta\left(2n + 1\right)$$

$$= (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{1}{2}\pi\right)}{(2n)!} \right.$$

$$\left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \, \frac{\zeta\left(2k + 1\right)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \, \frac{\zeta\left(2k\right)}{4^{2k}} \right] \qquad (n \in \mathbb{N});$$

Series Representation 4:

$$(2.13)$$

$$\zeta\left(2n + 1\right)$$

$$= (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{3^{2n}\left(2^{2n} + 1\right) + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{1}{3}\pi\right)}{(2n)!} \right.$$

$$\left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \, \frac{\zeta\left(2k + 1\right)}{\left(\frac{1}{3}\pi\right)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \, \frac{\zeta\left(2k\right)}{6^{2k}} \right] \qquad (n \in \mathbb{N}).$$

Here (*and elsewhere in this presentation*) an empty sum is to be interpreted (as usual) to be zero.

The series representation (2.10) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [45, p. 1590, Equation (3.13)] and (subsequently) by Cvijović and Klinowski [10, p. 1265, Theorem A]. Since $\zeta(2k) \to 1$ as $k \to \infty$, the general term in the series representation (2.10) has the following *order estimate*:

$$O\left(2^{-2k} \cdot k^{-2n-1}\right) \qquad (k \to \infty; \ n \in \mathbb{N}),$$

whereas the general term in each of these *earlier* series representations has the order estimate given below:

$$O\left(2^{-2k} \cdot k^{-2n}\right) \qquad (k \to \infty; \ n \in \mathbb{N}).$$

By suitably combining (2.10) and (2.12), it is fairly straightforward to obtain the series representation:

$$
\begin{aligned}
\zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} & \left[\frac{\log 2}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\
& \left. -2 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \qquad (n \in \mathbb{N}).
\end{aligned}
$$

(2.14)

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions (1.9) and

$$(2.15) \qquad \frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \qquad (|z| < \pi),$$

respectively, it is known that (*cf.*, *e.g.*, Magnus *et al.* [24, p. 29])

$$(2.16) \qquad E_n(0) = (-1)^n E_n(1) = \frac{2\left(1 - 2^{n+1}\right)}{n+1} B_{n+1} \qquad (n \in \mathbb{N}),$$

which, together with the identity (1.8), implies that

$$(2.17) \qquad E_{2n-1}(0) = \frac{4 \cdot (-1)^n}{(2\pi)^{2n}} (2n-1)! \left(2^{2n}-1\right) \zeta(2n) \qquad (n \in \mathbb{N}).$$

By appealing to the relationship (2.17), the series representation (2.14) can immediately be put in the alternative form:

$$\zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} \right.$$

$$+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}}$$

$$(2.18) \qquad \left. + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n+2k)!} \left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0) \right] \qquad (n \in \mathbb{N}),$$

which is a slightly modified (and corrected) version of a result proven, using a significantly different technique, by Tsumura [41, p. 383, Theorem B].

Another interesting combination of our series representations (2.10) and (2.12) leads us to the following variant of Tsumura's result (2.14) or (2.18):

$$\zeta(2n+1) = (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1}-1} \left[\frac{H_{2n} - \log\left(\frac{1}{4}\pi\right)}{(2n)!} \right.$$

$$+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}}$$

$$(2.19) \qquad \left. - 4 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k-1}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \qquad (n \in \mathbb{N}),$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived recently by Ewell [14, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Many other similar combinations of the series representations (2.10) to (2.13) would yield some interesting companions of Ewell's result (2.19).

Next, by setting $t = 1/m$ and differentiating both sides with respect to s, we find from the following obvious consequence of the series identity (2.1):

$$\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s+2k+1, a) \ t^{2k+1}$$

$$(2.20) \qquad = \frac{1}{2} \left[\zeta(s, a-t) - \zeta(s, a+t) \right] \qquad (|t| < |a|)$$

that

$$\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! \ m^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right]$$

$$(2.21) \quad = \frac{m}{2} \frac{\partial}{\partial s} \left\{ \zeta\left(s, a-\frac{1}{m}\right) - \zeta\left(s, a+\frac{1}{m}\right) \right\} \qquad (m \in \mathbb{N} \setminus \{1\}).$$

In particular, when $m = 2$, (2.21) immediately yields

$$\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! \, 2^{2k}} \left[\zeta'(s + 2k + 1, a) + \zeta(s + 2k + 1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right]$$

$$(2.22) \quad = -\left(a - \frac{1}{2}\right)^{-s} \log\left(a - \frac{1}{2}\right).$$

By letting $s \to -2n - 1$ $(n \in \mathbb{N})$ in the *further* special of this last identity (2.22) when $a = 1$, Wilton [34, p. 92] obtained the following series representation for $\zeta(2n+1)$ (see also Hansen [18, p. 357, Entry (54.6.9)]):

$$\zeta(2n+1) = (-1)^{n-1} \, \pi^{2n} \left[\frac{H_{2n+1} - \log \pi}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$

$$(2.23) \qquad \left. +2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}),$$

which, in view of the identity:

$$(2.24) \qquad \frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n \frac{(2k-1)!}{(2n+2k)!} \qquad (n \in \mathbb{N}),$$

would combine with the result (2.10) to yield the series representation:

$$\zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1} - 1)} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$

$$(2.25) \qquad \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}).$$

The series representation (2.25) is precisely the aforementioned *main* result of Cvijović and Klinowski [10, p. 1265, Theorem A]. In fact, by virtue of a known derivative formula [32, p. 389, Equation (2.8)], the series representation (2.25) is essentially the same as a result given earlier by Zhang and Williams [45, p. 1590, Equation (3.13)] (see also Zhang and Williams [45, p. 1591, Equation (3.16)] where an obviously more complicated (*asymptotic*) version of (2.25) was proven similarly).

Observing also that

$$(2.26) \quad \frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1) \frac{(2k-1)!}{(2n+2k+1)!} \qquad (n, k \in \mathbb{N}),$$

we obtain yet another series representation for $\zeta(2n+1)$ by applying (2.10) and (2.23):

$$\zeta(2n+1) = (-1)^n \frac{2 \cdot (2\pi)^{2n}}{(2n-1)\, 2^{2n}+1} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}\, k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$

$$\text{(2.27)} \qquad \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}),$$

which provides a *significantly* simpler (and *much more* rapidly convergent) version of the other *main* result of Cvijović and Klinowski [10, p. 1265, Theorem B]:

$$\text{(2.28)} \qquad \zeta(2n+1) = (-1)^n \frac{2 \cdot (2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \qquad (n \in \mathbb{N}),$$

where the coefficients $\Omega_{n,k}$ are given explicitly as a finite sum of Bernoulli numbers [10, p. 1265, Theorem B(i)] (see, for details, Srivastava [32, pp. 393-394]):

$$\text{(2.29)} \qquad \Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j+2k+1)(j+1)\, 2^j} \qquad (n \in \mathbb{N};\ k \in \mathbb{N}_0).$$

3. Further classes of series representations

By starting once again from the identity (2.1) with (of course) $a = 1$, $t = \pm 1/m$, and s replaced by $s+1$, and applying (2.2), we find yet another class of series identities including, for example,

$$\text{(3.1)} \qquad \sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\, \zeta(s)$$

and

$$\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}}$$

$$= \frac{1}{2m} \left[m(m^s - 3)\, \zeta(s) + \left(m^{s+1} - 1\right) \zeta(s+1) - 2\zeta\left(s+1, \frac{1}{m}\right) \right.$$

$$\text{(3.2)} \qquad \left. - \sum_{j=2}^{m-2} \left\{ m\zeta\left(s, \frac{j}{m}\right) + \zeta\left(s+1, \frac{j}{m}\right) \right\} \right] \qquad (m \in \mathbb{N} \setminus \{1,2\}).$$

It is the series identity (3.1) which was first applied by Zhang and Williams [45] (and, subsequently, by Cvijović and Klinowski [10]) in order to prove two (only seemingly different) versions of the series representation (2.25). Indeed, by appealing to (3.2) *with $m = 4$*, we can derive the following much more

rapidly convergent series representation for $\zeta(2n+1)$ (see Srivastava [31, p. 9, Equation (41)]):

$$
\begin{aligned}
\zeta(2n+1) \\
= (-1)^n \frac{2 \cdot (2\pi)^{2n}}{n(2^{4n+1} + 2^{2n} - 1)} & \left[\frac{4^{n-1} - 1}{(2n)!} \, B_{2n} \log 2 \right. \\
\text{(3.3)} \qquad - \frac{2^{2n-1} - 1}{2(2n-1)!} \, \zeta'(1 - 2n) & - \frac{4^{2n-1}}{(2n-1)!} \, \zeta'\left(1 - 2n, \frac{1}{4}\right) \\
+ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \, \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} & \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \, \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N}),
\end{aligned}
$$

where (and in what follows) a prime denotes the derivative of $\zeta(s)$ or $\zeta(s,a)$ with respect to s.

In view of the identities (2.24) and (2.26), the results (2.12) and (3.3) would lead us eventually to the following *additional* series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) (see Srivastava [31, p. 10, Equations (42) and (43)]):

(3.4)
$$
\begin{aligned}
\zeta(2n+1) \\
= (-1)^{n-1} \left(\frac{\pi}{2}\right)^{2n} & \left[\frac{H_{2n+1} - \log\left(\frac{1}{2}\pi\right)}{(2n+1)!} + \frac{2(4^n - 1)}{(2n+2)!} B_{2n+2} \log 2 \right. \\
- \frac{2^{2n+1} - 1}{(2n+1)!} \, \zeta'(-2n-1) & - \frac{2^{4n+3}}{(2n+1)!} \, \zeta'\left(-2n-1, \frac{1}{4}\right) \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \, \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} & \left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \, \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N});
\end{aligned}
$$

(3.5)
$$
\begin{aligned}
\zeta(2n+1) \\
= (-1)^n \frac{4 \cdot (2\pi)^{2n}}{n \cdot 4^{2n+1} - 2^{2n} + 1} & \left[\frac{2^{2n+1} - 1}{2 \cdot (2n)!} \, \zeta'(-2n-1) \right. \\
+ \frac{4^{2n+1}}{(2n)!} \, \zeta'\left(-2n-1, \frac{1}{4}\right) & - \frac{(2n+1)(4^n - 1)}{(2n+2)!} \, B_{2n+2} \log 2 \\
+ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \, \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} & \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \, \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N}).
\end{aligned}
$$

Explicit expressions for the derivatives $\zeta'(-2n \pm 1)$ and $\zeta'\left(-2n \pm 1, \frac{1}{4}\right)$, occurring in the series representations (3.3), (3.4), and (3.5), can be found and substituted into these results in order to represent $\zeta(2n+1)$ in terms of Bernoulli numbers and polynomials and various rapidly convergent series of ζ-functions (see, for details, Srivastava [31, Section 3]).

Of the four seemingly analogous results (2.12), (3.3), (3.4), and (3.5), the infinite series in (3.4) would obviously converge most rapidly, with its general term having the order estimate:

$$O\left(k^{-2n-2} \cdot 4^{-2k}\right) \qquad (k \to \infty;\ n \in \mathbb{N}).$$

Srivastava and Tsumura [36] derived the following three *new* members of the class of the series representations (2.12) and (3.4):

(3.6)
$$\zeta(2n+1)$$
$$= (-1)^{n-1}\left(\frac{2\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log\left(\frac{2}{3}\pi\right)}{(2n+1)!}+\frac{\left(3^{2n+2}-1\right)\pi}{2\sqrt{3}\,(2n+2)!}\,B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{\sqrt{3}\,(2\pi)^{2n+1}}\,\zeta\left(2n+2,\frac{1}{3}\right)$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{3^{2k}}\right]\quad (n\in\mathbb{N}),$$

(3.7)
$$\zeta(2n+1)$$
$$= (-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n}\left[\frac{H_{2n+1}-\log\left(\frac{1}{2}\pi\right)}{(2n+1)!}+\frac{2^{2n}\left(2^{2n+2}-1\right)\pi}{(2n+2)!}\,B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{2\cdot(2\pi)^{2n+1}}\,\zeta\left(2n+2,\frac{1}{4}\right)$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{4^{2k}}\right]\quad (n\in\mathbb{N}),$$

and
(3.8)
$$\zeta(2n+1)$$
$$= (-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log\left(\frac{1}{3}\pi\right)}{(2n+1)!}+\frac{2^{2n}\left(3^{2n+2}-1\right)\pi}{\sqrt{3}\,(2n+2)!}\,B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{2\sqrt{3}\,(2\pi)^{2n+1}}\left\{\zeta\left(2n+2,\frac{1}{3}\right)+\zeta\left(2n+2,\frac{1}{6}\right)\right\}$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{6^{2k}}\right]\quad (n\in\mathbb{N}).$$

Indeed the general terms of the infinite series occurring in these three members [(3.6), (3.7), and (3.8)] have the order estimates:

(3.9) $O\left(k^{-2n-2}\cdot m^{-2k}\right)$ $(k\to\infty;\ n\in\mathbb{N};\ m=3,4,6),$

which exhibit the fact that *each* of the three series representations (3.6), (3.7), and (3.8) converges more rapidly than Wilton's result (2.23) and two of them [*cf.* Equations (3.7) and (3.8)] at least as rapidly as Srivastava's result (3.4).

Next we recall that, in their aforecited work on the Ray-Singer torsion and topological field theories, Nash and O'Connor ([25] and [26]) obtained a number of remarkable integral expressions for $\zeta(3)$, including (for example) the following result [26, p. 1489 *et seq.*]:

$$(3.10) \qquad \zeta(3) = \frac{2\pi^2}{7} \log 2 - \frac{8}{7} \int_0^{\pi/2} z^2 \cot z \, dz.$$

Since [12, p. 51, Equation 1.20(3)]

$$(3.11) \qquad z \cot z = -2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{z}{\pi}\right)^{2k} \qquad (|z| < \pi),$$

the result (3.10) is, in fact, equivalent to the series representation (*cf.* Dąbrowski [11, p. 202]; see also Chen and Srivastava [5, p. 191, Equation (3.19)]):

$$(3.12) \qquad \zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1) 2^{2k}} \right).$$

Moreover, upon integrating by parts, it is easily seen that

$$(3.13) \qquad \int_0^{\pi/2} z^2 \cot z \, dz = -2 \int_0^{\pi/2} z \log \sin z \, dz,$$

so that the result (3.10) is equivalent *also* to the integral representation:

$$(3.14) \qquad \zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7} \int_0^{\pi/2} z \log \sin z \, dz,$$

which was proven in the aforementioned 1772 paper by Euler (*cf.*, *e.g.*, Ayoub [3, p. 1084]).

Furthermore, since

$$(3.15) \qquad i \cot iz = \coth z = \frac{2}{e^{2z} - 1} + 1 \qquad \left(i := \sqrt{-1}\right),$$

by replacing z in the known expansion (3.11) by $\frac{1}{2} i\pi z$, it is easily seen that (*cf.*, *e.g.*, Koblitz [14, p. 25]; see also Erdélyi *et al.* [12, p. 51, Equation 1.20(1)])

$$(3.16) \qquad \frac{\pi z}{e^{\pi z} - 1} + \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \zeta(2k)}{2^{2k-1}} z^{2k} \qquad (|z| < 2).$$

By setting $z = it$ in (3.16), multiplying both sides by t^{m-1} ($m \in \mathbb{N}$), and then integrating the resulting equation from $t = 0$ to $t = \tau$ ($0 < \tau < 2$), Srivastava [24] derived the following series representations for $\zeta(2n+1)$ (see also

Srivastava *et al.* [35]):

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\,(2^{2n+1}-1)}$$
$$\cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j} \frac{(2j)!\,(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)\,2^{2k}} \right] \qquad (n \in \mathbb{N})$$

(3.17)

and

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!\,(2^{2n}-1)}$$
$$\cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n+1}{2j} \frac{(2j)!\,(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right)2^{2k}} \right] \qquad (n \in \mathbb{N}).$$

(3.18)

In its special case when $n = 1$, (3.18) immediately reduces to the following series representation for $\zeta(3)$:

(3.19)
$$\zeta(3) = \frac{2\pi^2}{9}\left(\log 2 + 2\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)\,2^{2k}}\right),$$

which was proven *independently* by (among others) Glasser [16, p. 446, Equation (12)], Zhang and Williams [45, p. 1585, Equation (2.13)], and Dąbrowski [11, p. 206] (see also Chen and Srivastava [5, p. 183, Equation (2.15)]). And a special case of (3.17) when $n = 1$ yields (*cf.* Dąbrowski [11, p. 202]; see also Chen and Srivastava [5, p. 191, Equation (3.19)])

(3.20)
$$\zeta(3) = \frac{2\pi^2}{7}\left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)\,2^{2k}}\right).$$

In view of the familiar sum:

(3.21)
$$\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)\,2^{2k}} = -\frac{1}{2}\log 2,$$

Euler's formula (1.6) is indeed a *simple* consequence of (3.20).

We remark in passing that an integral representation for $\zeta(2n+1)$, which is easily seen to be equivalent to the series representation (3.17), was given by Dąbrowski [11, p. 203, Equation (16)], who [11, p. 206] mentioned the existence of (but did not fully state) the series representation (3.18) as well.

The series representation (3.17) is derived also in a recent paper by Borwein *et al.* (*cf.* [4, p. 269, Equation (57)]).

By suitably combining the series occurring in (3.12), (3.19), and (3.21), it is not difficult to derive several other series representations for $\zeta(3)$, which are analogous to Euler's formula (1.6). More generally, since

$$\frac{\lambda k^2 + \mu k + \nu}{(2k + 2n - 1)(2k + 2n)(2k + 2n + 1)}$$

(3.22)
$$= \frac{\mathcal{A}}{2k + 2n - 1} + \frac{\mathcal{B}}{2k + 2n} + \frac{\mathcal{C}}{2k + 2n + 1},$$

where, for convenience,

$$(3.23) \qquad \mathcal{A} = \mathcal{A}_n(\lambda, \mu, \nu) := \frac{1}{2}\left[\lambda n^2 - (\lambda + \mu)n + \frac{1}{4}(\lambda + 2\mu + 4\nu)\right],$$

$$(3.24) \qquad \mathcal{B} = \mathcal{B}_n(\lambda, \mu, \nu) := -\left(\lambda n^2 - \mu n + \nu\right),$$

and

$$(3.25) \qquad \mathcal{C} = \mathcal{C}_n(\lambda, \mu, \nu) := \frac{1}{2}\left[\lambda n^2 + (\lambda - \mu)n + \frac{1}{4}(\lambda - 2\mu + 4\nu)\right],$$

by applying (3.17), (3.18), and another result (proven by Srivastava [33, p. 341, Equation (3.17)]):

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{2n+1}{2j} \frac{(2j)!\left(2^{2j} - 1\right)}{(2\pi)^{2j}} \zeta(2j+1)$$

(3.26)
$$= \log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{\left(k + n + \frac{1}{2}\right)2^{2k}} \qquad (n \in \mathbb{N}_0),$$

with n replaced by $n - 1$, Srivastava [33] derived the following unification of a large number of known (or new) series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$, including (for example) Euler's formula (1.6):

(3.23)
$$\zeta(2n+1)$$

$$= \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\left\{\left(2^{2n+1} - 1\right)\mathcal{B} + (2n+1)\left(2^{2n} - 1\right)\mathcal{C}\right\}}$$

$$\cdot \left[\frac{1}{4}\lambda \log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2}\left\{2j(2j-1)\mathcal{A} + [\lambda(4n-1) - 2\mu]nj\right.\right.$$

$$\left. + \lambda n\left(n + \frac{1}{2}\right)\right\} \frac{(2j-2)!\left(2^{2j} - 1\right)}{(2\pi)^{2j}}\zeta(2j+1)$$

$$\left. + \sum_{k=0}^{\infty} \frac{\left(\lambda k^2 + \mu k + \nu\right)\zeta(2k)}{(2k + 2n - 1)(k + n)(2k + 2n + 1)2^{2k}}\right] \qquad (n \in \mathbb{N}; \ \lambda, \mu, \nu \in \mathbb{C}),$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by (3.23), (3.24), and (3.25), respectively.

Numerous other interesting series representations for $\zeta(2n+1)$, which are analogous to (3.17) and (3.18), were also given by Srivastava *et al.* [35].

4. Some useful deductions and consequences

By suitably specializing the parameter λ, μ, and ν in (3.27), and then applying a rather elaborate scheme, the following rapidly convergent series representation for $\zeta(2n+1)$ $(n \in \mathbb{N})$ was derived by Srivastava [33, pp. 348–349, Equation (3.50)]):

(4.1)

$$\zeta(2n+1)$$

$$= (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\Delta_n} \left[\sum_{j=1}^{n-1} (-1)^j \right.$$

$$\cdot \left(\left\{ (2n-3)\, 2^{2n+2} - 2n \right\} \left\{ \binom{2n-1}{2j} - \binom{2n+2}{2j} + 6n \binom{2n-1}{2j-2} \right\} \right.$$

$$\left. - \left(2^{2n+3} - 1 \right) \left\{ \binom{2n}{2j} - \binom{2n+3}{2j} + 3 \binom{2n+1}{2j-1} \right\} \right) \frac{(2j)!\, \left(2^{2j} - 1 \right)}{(2\pi)^{2j}}\, \zeta(2j+1)$$

$$\left. + 12 \sum_{k=0}^{\infty} \frac{(\xi_n k + \eta_n)\, \zeta(2k)}{(2k+2n-1)\, (2k+2n)\, (2k+2n+1)\, (2k+2n+2)\, (2k+2n+3)\, 2^{2k}} \right]$$

$$(n \in \mathbb{N}),$$

where, for convenience,

(4.2)
$$\Delta_n := \left(2^{2n+3} - 1 \right) \left\{ \frac{1}{3} (2n+1)\, \left(2n^2 - 4n + 3 \right) \left(2^{2n} - 1 \right) - 2^{2n+1} + 1 \right\}$$
$$- \left\{ (2n-3)\, 2^{2n+2} - 2n \right\} \left\{ 2^{2n+2} + n\, (2n-3)\, \left(2^{2n} - 1 \right) - 1 \right\},$$

(4.3)
$$\xi_n := 2 \left\{ (2n-5)\, 2^{2n+2} - 2n + 1 \right\},$$

and

(4.4)
$$\eta_n := \left(4n^2 - 4n - 7 \right) 2^{2n+2} - (2n+1)^2.$$

In its special case when $n = 1$, (4.1) yields the following (*rather curious*) series representation:

(4.5) $$\zeta(3) = -\frac{6\pi^2}{23} \sum_{k=0}^{\infty} \frac{(98k + 121)\, \zeta(2k)}{(2k+1)\, (2k+2)\, (2k+3)\, (2k+4)\, (2k+5)\, 2^{2k}},$$

where the series obviously converges much more rapidly than that in *each* of the *celebrated* results (1.6) and (1.7).

An interesting companion of (4.5) in the form:

(4.6)
$$\zeta(3)$$

$$= -\frac{120}{1573}\pi^2 \sum_{k=0}^{\infty} \frac{8576k^2 + 24286k + 17283}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)(2k+6)(2k+7)} \frac{\zeta(2k)}{2^{2k}}.$$

was deduced by Srivastava and Tsumura [38], who indeed presented an *inductive* construction of several general series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ (see also [37]).

5. Symbolic and numerical computations based upon *Mathematica* (Version 4.0)

We continue our presentation by first summarizing the results of our symbolic and numerical computations with the series in (4.5) using *Mathematica* (Version 4.0) for Linux:

In[1] := (98k + 121) Zeta[2k] /

$$((2k+1)(2k+2)(2k+3)(2k+4)(2k+5)\,2^{\urcorner}(2k))$$

$$\text{Out}[1] = \frac{(121 + 98k)\,\text{Zeta}\,[2k]}{2^{2k}\,(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)}$$

In[2] := Sum[%, {k, 1, Infinity}] // Simplify

$$\text{Out}[2] = \frac{121}{240} - \frac{23\,\text{Zeta}[3]}{6\text{Pi}^2}$$

In[3] := N[%]

Out[3] = 0.0372903

In[4] := Sum[N [%1] // Evaluate, {k, 1, 50}]

Out[4] = 0.0372903

In[5] := N Sum[%1 // Evaluate, {k, 1, Infinity}]

Out[5] = 0.0372903

Since $\zeta(0) = -\frac{1}{2}$, Out[2] evidently validates the series representation (4.5) *symbolically*. Furthermore, our *numerical* computations in Out[3], Out[4], and Out[5], together, exhibit the fact that only 50 terms ($k = 1$ to $k = 50$) of the series in (4.5) can produce an accuracy of seven decimal places.

Our symbolic and numerical computations with the series in (4.6) using *Mathematica* (Version 4.0) for Linux lead us to the following table:

Number of Terms	Precision of Computation
4	6
10	11
20	18
50	38
98	69

As a matter of fact, since the general term of the series in (4.6) has the following order estimate:

$$O\left(2^{-2k} \cdot k^{-5}\right) \qquad (k \longrightarrow \infty),$$

for getting p *exact* digits, we must have

$$2^{-2k} \cdot k^{-5} < 10^{-p}.$$

Solving this inequality *symbolically*, we find that

$$k \cong \frac{5}{\log 4} \text{ProductLog}\left(\frac{10^{p/5}\log 4}{5}\right),$$

where the function ProductLog (also known as Lambert's function) is the solution of the equation:

$$xe^x = a.$$

We now give below some relevant details about our symbolic and numerical computations with the series in (4.6) using *Mathematica* (Version 4.0) for Linux.

In [1]:= expr $= \left(8576k^{\frown}2 + 24286k + 17283\right)$ Zeta$[2k] /$

$$\left((2k+1)(2k+2)(2k+3)(2k+4)(2k+5)(2k+6)(2k+7)\, 2^{\frown}(2k)\right)$$

Out [1] $= \dfrac{\left(17283 + 24286k + 8576k^2\right)\,\text{Zeta}[2k]}{2^{2k}\,(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)(6+2k)(7+2k)}$

In [2] := Sum[expr, $\{k, 0, \text{infinity}\}$] // Simplify

Out [2] $= -\dfrac{1573}{120\text{Pi}^2}$ Zeta$[3]$

In [3] :=N$\left[-1573/\left(120\text{Pi}^{\frown}2\right)\,\text{Zeta}[3], 50\right]$

$$-\text{Sum}[\text{expr}, \{k, 0, 50\}]$$

Out [3] $= 4.00751120011 \cdot 10^{-38}$

In [4] :=N$\left[-1573/\left(120\text{Pi}^{\frown}2\right)\,\text{Zeta}[3], 100\right]$

$$-\text{Sum}\,[\text{expr}, \{k, 0, 50\}]$$

Out [4] $= 4.0075112001 <\text{skip}> 3481 \cdot 10^{-38}$

Thus the result does not change appreciably when we increase the precision of computation of the symbolic result from 50 to 100. This is expected, because of the following numerical computation of the last term for $k = 50$:

In [5] := N [expr $/.k \to 50, 50$]

Out [5] $= 1.36085303749223768614438874545151423357570286017 9 \cdot 10^{-37}$

6. Concluding remarks and observations

The foregoing developments (which we have attempted to present here in a rather concise form) have essentially motivated a large number of further investigations, not only involving the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ (and their such relatives as the multiple Zeta functions and the multiple Gamma functions), but indeed also the substantially general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [34, p. 121, et seq.])

$$(6.1) \qquad \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C} \text{ when } |z| < 1; \ \Re(s) > 1 \text{ when } |z| = 1 \right).$$

The Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (6.1) contains, as its *special* cases, not only the Riemann and Hurwitz (or generalized) Zeta functions [cf. Equations (1.1) and (1.2)]:

$$(6.2) \qquad \zeta(s) = \Phi(1, s, 1) \qquad \text{and} \qquad \zeta(s, a) = \Phi(1, s, a)$$

and the Lerch Zeta function:

$$(6.3) \qquad \ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i \xi}}{n^s} = e^{2\pi i \xi} \ \Phi\left(e^{2\pi i \xi}, s, 1 \right)$$

$$(\xi \in \mathbb{R}; \ \Re(s) > 1),$$

but also such other important functions of *Analytic Function Theory* as the Polylogarithmic function:

$$(6.4) \qquad \mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z \ \Phi(z, s, 1)$$

$$\left(s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \ \Re(s) > 1 \quad \text{when} \quad |z| = 1 \right)$$

and the Lipschitz-Lerch Zeta function (cf. [34, p. 122, Eq. 2.5 (11)]):

$$(6.5) \qquad \phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i \xi}}{(n+a)^s} = \Phi\left(e^{2\pi i \xi}, s, a \right) =: L(\xi, s, a)$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z}; \ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z} \right),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions. For details, the interested reader should be referred, in

connection with some of these developments, to the recent works including (among others) [1], [6] to [9], [15], [20], [21], and [23].

Next, in terms of the familiar *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^\mu$ defined by
(6.6)

$$
\mathcal{D}_z^\mu \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \, f(t)\, dt & (\Re(\mu) < 0) \\[4mm] \dfrac{d^m}{dz^m} \, \mathcal{D}_z^{\mu-m} \{f(z)\} & \left(m-1 \leqq \Re(\mu) < m \quad (m \in \mathbb{N})\right), \end{cases}
$$

it is easily observed that

$$
(6.7) \qquad \mathcal{D}_z^\mu \left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} \, z^{\lambda-\mu} \qquad (\Re(\lambda) > -1),
$$

which yields the following fractional derivative formula for the general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ given by (6.1):

$$
(6.8) \qquad \mathcal{D}_z^{\mu-\nu} \left\{z^{\mu-1}\, \Phi(z^\sigma, s, a)\right\} = \frac{\Gamma(\mu)}{\Gamma(\nu)} \, z^{\nu-1} \, \Phi_{\mu,\nu}^{(\sigma,\sigma)}(z^\sigma, s, a)
$$

$$
\left(\Re(\mu) > 0; \ \sigma \in \mathbb{R}^+\right),
$$

where $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ denotes a general family of Lin-Srivastava Zeta functions defined by

$$
(6.9) \qquad \Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^\infty \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \, \frac{z^n}{(n+a)^s}
$$

$$
\left(\mu \in \mathbb{C}; \ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \rho, \sigma \in \mathbb{R}^+; \ \rho < \sigma \text{ when } s, z \in \mathbb{C};\right.
$$

$$
\left. \rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1; \ \rho = \sigma \text{ and } \Re(s - \mu + \nu) > 1 \text{ when } |z| = 1\right),
$$

where $(\lambda)_\kappa$ denotes the Pochhammer symbol or the *shifted* factorial, used already in (for example) (2.1) and (2.2), since

$$
(1)_n = n! \quad \text{when} \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},
$$

given (for $\kappa, \lambda \in \mathbb{C}$ and in terms of the Gamma function) by
(6.10)

$$
(\lambda)_\kappa := \frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)} = \begin{cases} 1 & (\kappa = 0; \ \lambda \in \mathbb{C} \setminus \{0\}) \\[3mm] \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N}; \ \lambda \in \mathbb{C}). \end{cases}
$$

Obviously, we have

$$
(6.11) \qquad \Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \Phi(z,s,a)
$$

and

$$(6.12) \qquad \Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_{\mu}^{*}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s},$$

where $\Phi_{\mu}^{*}(z,s,a)$ is a generalized Hurwitz-Lerch Zeta function introduced and studied earlier (see, for details, the recent investigations by Garg et al. [15] and Lin et al. [23] as well as many of the earlier references cited by them).

In particular, when

$$\nu = \sigma = 1,$$

the fractional-derivative formula (6.8) would reduce at one to the following known form which exhibits the interesting fact that $\Phi_{\mu}^{*}(z,s,a)$ is essentially a Riemann-Liouville fractional derivative of the classical Hurwitz-Lerch function $\Phi(z,s,a)$ defined by (6.1):

$$(6.13) \qquad \Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(\mu)} \mathcal{D}_{z}^{\mu-1} \left\{ z^{\mu-1} \Phi(z,s,a) \right\} \qquad \left(\Re(\mu) > 0 \right).$$

It would be nice and worthwhile to be able to extend the results presented in this lecture to hold true for the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ and for some of its generalizations given by the Lin-Srivastava Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ for special values of the parameters μ, ν, ρ, and σ in the definition (6.9).

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