

The Embeddability of $\mathfrak{sl}(n, \mathbb{C})$ Modules

Dongseok Kim¹⁾

Abstract

In present article, we consider the embeddability problems for finite dimensional irreducible modules over a complex simple Lie algebra L . For $\mathfrak{sl}(n, \mathbb{C})$ modules, we determine when one can be embedded into the other if $\mathfrak{sl}(n, \mathbb{C})$ modules are tensor products of fundamental modules.

Keywords : Embeddability, Fundamental Modules, Tensor Products

1. Introduction

Let A, B be two algebraic objects. One can consider an embeddability of A into B . A embeds into B if there is a faithful map $\varphi: A \hookrightarrow B$ such that φ preserves the algebraic structure, denoted by $A \hookrightarrow B$. For some algebraic objects, the answer is very simple, such as sets and vector spaces : if A, B are sets, then the cardinalities determine, and if A, B are vector spaces, the dimensions determine. But for algebras and modules, it becomes much more interesting.

In the present paper, we consider the embeddability problem for finite dimensional L -modules where L is a complex simple Lie algebra. Due to the Schur's lemma if we know how to decompose the given L -modules into direct sums of irreducible L -modules, it completely determines the embeddability as follows. Let A, B be finite dimensional L -modules. To answer the embeddability problem, we need to decompose A, B into direct sums of irreducible modules,

$$A \cong \bigoplus_{\lambda} a_{\lambda} V_{\lambda}, \quad B \cong \bigoplus_{\mu} b_{\mu} V_{\mu},$$

where a_{λ}, b_{λ} are the numbers of copies of V_{λ} , the irreducible module of highest weight λ in the decomposition of A, B respectively. The Schur's lemma states

¹⁾ Lecturer, Department of Mathematics, Kyungpook National University, Taegu, 702-201, Korea
E-mail: dongseok@knu.ac.kr

that

$$\dim(\text{Hom}_L(V_\lambda, V_\mu)) = \begin{cases} 1 & \text{if } V_\lambda \cong V_\mu \\ 0 & \text{otherwise.} \end{cases}$$

Thus, one can easily see that $A \hookrightarrow B$ if and only if $a_\lambda \leq b_\lambda$ for all λ . However, the decomposition problem itself is one of very difficult problems in the representation theory of Lie algebras, for example the honeycomb model by Knutson and Tao [Knutson, A. and Tao, T. (1999), Knutson, A. and Tao, T. (2001), Knutson, A. and Tao, T. and Woodward, C. (2004)]. In particular, the decomposition problem of a tensor product of two irreducible representations is known as the ‘‘Clebsch–Gordan Problem’’. In fact, the unique decomposition of tensor products into irreducible L -modules was very recently proved by Rajan, C. (2004).

In particular, we consider the embeddability problem for a Lie algebra L which is the set of all complex $n \times n$ trace zero matrices, $\mathfrak{sl}(n, \mathbb{C})$. We determine when an L -module can be embedded into the other L -module if these L -modules are tensor products of fundamental modules over a complex simple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. Suppose A, B are tensor products of fundamental modules of $\mathfrak{sl}(n, \mathbb{C})$, i. e.,

$$A \cong \bigotimes_{k=1}^m V_{\lambda_{i_k}}, \quad B \cong \bigotimes_{k=1}^l V_{\lambda_{j_k}} \quad (1)$$

where λ_i is a fundamental dominant weight of $\mathfrak{sl}(n, \mathbb{C})$. We find a necessary and sufficient condition that A embeds into B as L -modules in Theorem 2. 1.

2. The embeddability of tensor products of fundamental modules of $\mathfrak{sl}(n, \mathbb{C})$

For definitions and notations in complex simple Lie algebras, we refer to [Fulton, W. and Harris, J. (1991), Humphreys, J. E. (1972)].

First, we assign an array of integers, (i_1, i_2, \dots, i_m) for A and (j_1, j_2, \dots, j_l) for B . Without loss of generality, we assume i_k, j_k are non-increasing. For a fixed n , let KI_k be the set of all k -tuples whose entries are non-increasing positive integers between 1 and $n-1$. Let $KI = \bigcup_{k=0}^{\infty} KI_k$. Then we define a partial order $<$ on KI as follows.

From a given tensor product A of fundamental modules of $\mathfrak{sl}(n, \mathbb{C})$ we can find $I = (i_1, i_2, \dots, i_m)$. From I , we define

$$S(I) = V_{\sum_{k=1}^m \lambda_{i_k}}, \quad T(I) = \bigotimes_{k=1}^m V_{\lambda_{i_k}} \cong A$$

where $V_{\sum_{k=1}^m \lambda_{i_k}}$ is an irreducible module of $\mathfrak{sl}(n, \mathbb{C})$ of highest weight $\sum_{k=1}^m \lambda_{i_k}$ and we use a convention $S(\emptyset) = T(\emptyset) = V_0$, the one dimensional module of $\mathfrak{sl}(n, \mathbb{C})$.

We define a partial order on KI as follows: $I_1 < I_2$ if there is a finite sequence $(I_{j_0}, I_{j_1}, \dots, I_{j_c})$ where $I_{j_0} = I_1, I_{j_c} = I_2$ and $I_{j_{d+1}}$ can be obtained from I_{j_d} by one of the following moves.

Type I: if $i_{l-1} \geq i_{l+1}$, $i_{j-1} \geq i_{j+1}$ and $l < j$, we change $(i_1, i_2, \dots, i_l, \dots, i_j, \dots, i_k)$ to $(i_1, i_2, \dots, i_l - 1, \dots, i_j + 1, \dots, i_k)$.

Type II: if $l < k$ and $i_{l-1} \geq i_{l+1}$ or $l = k$ and $i_l > 1$, we change $(i_1, i_2, \dots, i_j, \dots, i_k)$ to $(i_1, i_2, \dots, i_l - 1, \dots, i_k, 1)$.

Type III: if $j > 1$ and $i_{j-1} \geq i_{j+1}$ or $j = 1$ and $i_1 < n - 1$, we change $(i_1, i_2, \dots, i_l, \dots, i_k)$ to $(n - 1, i_1, i_2, \dots, i_j + 1, \dots, i_k)$.

Type IV: we change (i_1, i_2, \dots, i_k) to $(n - 1, i_1, i_2, \dots, i_k, 1)$.

Then we find the following theorem.

Theorem 2. 1. Let A, B be $\mathfrak{sl}(n, \mathbb{C})$ -modules of the form in (1). Then $A \hookrightarrow B$ if and only if $(i_1, i_2, \dots, i_m) < (j_1, j_2, \dots, j_l)$ on KI .

Proof. For sufficiency, we need to show that every move from I to I' induces an inclusion from $T(I)$ to $T(I')$ as $\mathfrak{sl}(n, \mathbb{C})$ modules. For $i \leq j \leq \lceil \frac{n}{2} \rceil$, one can easily see that

$$V_{\lambda_i} \otimes V_{\lambda_j} = V_{\lambda_i + \lambda_j} \oplus V_{\lambda_{i-1} + \lambda_{j+1}} \oplus \dots \oplus V_{\lambda_1 + \lambda_{j+i-1}} \oplus V_{\lambda_{j+i}},$$

so we have $V_{\lambda_i} \otimes V_{\lambda_j} \hookrightarrow V_{\lambda_{i+1}} \otimes V_{\lambda_{j-1}}$. Thus type I and II moves induce inclusions. Also we can take care of all possible cases because of $V_{\lambda_{n-i}} \cong (V_{\lambda_i})^*$. For last two moves, if we use the convention $\lambda_n = 0$, then the same formula works.

For the necessity, we start with introducing some notations.

$$\Omega(I) = \{J \mid S(J) \hookrightarrow T(I)\}, \quad \Lambda(I) = \{J \mid J < I\}.$$

Then we prove a couple of lemmas which play a main role in the proof of the necessity.

Lemma 2. 2. Let $I_1, I_2 \in KI$.

- a) If $T(I_1) \hookrightarrow T(I_2)$, then $\Omega(I_1) \subset \Omega(I_2)$.
- b) $\Lambda(I_1) \subset \Lambda(I_2)$ if and only if $I_1 < I_2$.

Proof. First part follows from the definition. For second part, if $I_1 < I_2$, then $\Lambda(I_1) \subset \Lambda(I_2)$ because of the transitivity of the partial order $<$. For converse, we get $I_1 < I_2$ because of $I_1 \in \Lambda(I_1)$ and $\Lambda(I_1) \subset \Lambda(I_2)$.

Lemma 2. 3. For $I \in KI$, $\Omega(I) = \Lambda(I)$.

Proof. If $J \in \Lambda(I)$, we get $J < I$. But we already proved that $J < I$ implies $T(I) \hookrightarrow T(J)$ and so $S(J) \hookrightarrow T(J) \hookrightarrow T(I)$. Thus we show that $\Lambda(I) \subset \Omega(I)$. If $I \in KI_m$, we say I has a length m , denoted by $|I|$. To show $\Omega(I) \subset \Lambda(I)$ we induct on $|I|$. If $|I| = 1$, $I = (i)$ for some i . Then $\Omega((i)) = (i) = \Lambda((i))$.

To proceed the induction we assume $\Omega(I) = \Lambda(I)$ for $|I| \leq m$. For $I \in KI_{m+1}$, we can write I as $I' \wedge (i_{m+1})$ where $|I'| = m$ and the wedge (\wedge) of two tuples is defined by reordering elements to make all entries non-increasing in the juxtaposition of two.

A main ingredient of the proof is to show the following equalities and inclusions.

$$\begin{aligned} \Omega(I) &= \{J \mid S(J) \hookrightarrow T(I) = T(I') \otimes T((i_{m+1}))\} \\ &= \left\{J \mid S(J) \hookrightarrow \bigoplus_{I'' \in \Omega(I')} S(I'') \otimes S((i_{m+1}))\right\} \\ &= \bigcup_{I'' \in \Omega(I')} \{J \mid S(J) \hookrightarrow S(I'') \otimes S((i_{m+1}))\} \\ &\subset \bigcup_{I'' \in \Lambda(I')} \{J \mid J < I'' \wedge (i_{m+1})\} \subset \Lambda(I' \wedge (i_{m+1})) = \Lambda(I). \end{aligned}$$

Now we will look at each step. One can prove the first three equalities by the definitions of S, T and the irreducibility of $S(J)$. Also, the last two steps can be proven by $I'' < I'$ and $I = I' \wedge (i_{m+1})$. Thus, we need to show the fourth inclusion, i. e., let $I' = (k_1, k_2, \dots, k_m)$ then we have to show that if there is a $J = (l_1, l_2, \dots, l_j) \in KI_j$ such that $S(J) \in \Omega((k_1, k_2, \dots, k_m) \wedge (i_{m+1}))$, then $J < I' \wedge (i_{m+1})$. To find all such J , we look at $S(I') \otimes S((i_{m+1})) = V_{\sum_{s=1}^m \lambda_{k_s}} \otimes V_{\lambda_{i_{m+1}}}$.

The weight spaces of $V_{\lambda_{i_{m+1}}}$ can be obtained by the following way: let $B(n, i_{m+1})$ be the set of all binary codes of length n with i_{m+1} many 1's or precisely,

$$\begin{aligned} B(n, i_{m+1}) &= \{B = (b_1, b_2, \dots, b_n) \mid b_i \in \{0, 1\}, |\{i \mid b_i \neq 0\}| = i_{m+1}\} \\ &= \bigcup_{k=0}^{i_{m+1}} \{B \in B(n, i_{m+1}) \mid b_{k+1} = 0, b_j = 1 \text{ for } 1 \leq j \leq k\}. \end{aligned}$$

We will denote $\{B \in B(n, i_{m+1}) \mid b_{k+1} = 0, b_j = 1 \text{ for } 1 \leq j \leq k\}$ by $B(n, i_{m+1}, k)$.

Then the weight space of $V_{\lambda_{i_{m+1}}}$ is $W(V_{\lambda_{i_{m+1}}}) = \{(b_1 - b_2, b_2 - b_3, \dots, b_{n-1} - b_n) \mid B \in B(n, i_{m+1})\}$. If we rewrite I' as $W(I'') = (a_1, a_2, \dots, a_{n-1})$ where a_i is the number of j such that $k_j = i$, where $I'' = (k_1, k_2, \dots, k_m)$, we can find

$$V_{\sum_{s=1}^m \lambda_{k_s}} \otimes V_{\lambda_{i_{m+1}}} = \bigoplus_{B \in B(n, i_{m+1})} V_{\sum_{i=1}^{n-1} (a_i + b_i - b_{i+1}) \lambda_i} = \bigoplus_{k=0}^{i_{m+1}} \bigoplus_{B \in B(n, i_{m+1}, k)} V_{\sum_{i=1}^{n-1} (a_i + b_i - b_{i+1}) \lambda_i},$$

where $V_{\sum_{i=1}^{n-1} (a_i + b_i - b_{i+1}) \lambda_i} = \{1\}$ if $(a_i + b_i - b_{i+1}) = -1$ for an i .

Then one can see that every nontrivial term in $(\bigoplus_{B \in B(n, i_{m+1}, k)} V_{\sum_{i=1}^{n-1} (a_i + b_i - b_{i+1}) \lambda_i})$ can be obtained from $(\bigoplus_{B \in B(n, i_{m+1}, k+1)} V_{\sum_{i=1}^{n-1} (a_i + b_i - b_{i+1}) \lambda_i})$ by one of moves we defined for the partial order $<$ (one may have to take S^{-1} to apply the moves but these processes should be clear) and so all possible J must be obtained by these moves from $S^{-1}(V_{\sum_{s=1}^m \lambda_{k_s} + \lambda_{i_{m+1}}}) = I'' \wedge (i_{m+1})$. It completes the proof of Lemma 2.3.

Now we continue to prove the theorem. If we assume $T(I_1) \hookrightarrow T(I_2)$, then

$$\begin{aligned} \Omega(I_1) &\subset \Omega(I_2) && \text{by Lemma 2.2.} \\ \Rightarrow \Lambda(I_1) &\subset \Lambda(I_2) && \text{by Lemma 2.3.} \\ \Rightarrow I_1 &< I_2 && \text{by Lemma 2.2.} \end{aligned}$$

Therefore, it completes the proof of Theorem 2.1.

References

1. Fulton, W. and Harris, J. (1991), *Representation theory*, Graduate Texts in Mathematics, 129, Springer-Verlag, New York-Heidelberg-Berlin,.
2. Humphreys, J. E. (1972), *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Heidelberg-Berlin.
3. Kim, D. and Lee, J. (2007), On Stable embeddability of partitions, *European J. Comb.* 28, 848-857.
4. Knutson, A. and Tao, T. (1999), The honeycomb model of $GL(n)$ tensor products I: proof of the saturation conjecture, *Journal of the AMS*, 12(4), 1055-1090.
5. Knutson, A. and Tao, T. (2001), Honeycombs and sums of Hermitian matrices, *Notices of the AMS*, February.
6. Knutson, A. and Tao, T. and Woodward, C. (2004), The honeycomb

- model of $GL(n)$ tensor products II: Puzzles determine facets of the Littlewood Richardson cone, *Journal of the AMS*, 17, 19-48.
7. Rajan, C. (2004), Unique decomposition of tensor products of irreducible representations of simple algebraic groups, *Annals of Math.*, 160, 683-704.

[received date : June 2007, accepted date : July 2007]