J. Korean Math. Soc. **44** (2007), No. 6, pp. 1243–1254

ON THE MEAN VALUES OF THE HOMOGENEOUS DEDEKIND SUMS AND COCHRANE SUMS IN SHORT INTERVALS

Huaning Liu

Abstract. In this paper, we study the mean values of the homogeneous Dedekind sums and Cochrane sums in short intervals $[1, \frac{p}{3}]$ and $[1, \frac{p}{4}]$, and give some asymptotic formulae by using the mean values of the Dirichlet L-functions.

§ 1. Introduction

For integers a and $q > 0$, the classical Dedekind sum is defined by

$$S(a,q) = \sum_{r=1}^{q} \left(\left(\frac{r}{q}\right)\right)\left(\left(\frac{ar}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $S(a,q)$ plays an important role in the transformation theory of the Dedekind η function (see [7] and Chapter 3 of [1] for details).

J. B. Conrey, E. Fransen, R. Klein and C. Scott [3] studied the mean values of Dedekind sums and proved the following:

Proposition 1.1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then*

$$\sideset{}{'}\sum_{h=1}^{k} S^{2m}(h,k) = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{9/5} + k^{2m-1+1/(m+1)}\right)\log^3 k\right),$$

Received November 4, 2005; Revised September 10, 2007.

2000 *Mathematics Subject Classification.* 11F20.

Key words and phrases. Dedekind sums, Cochrane sums, homogeneous, mean values.

Research partially supported by the National Natural Science Foundation of China under Grant No.60472068 and No.10671155; Natural Science Foundation of Shaanxi province of China under Grant No.2006A04; and the Natural Science Foundation of the Education Department of Shaanxi Province of China under Grant No.06JK168.

where $\sum_h{}'$ *denotes the summation over all* h *such that* $(h,k) = 1$, $f_m(k)$ *is defined by the Dirichlet series*

$$\sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2 \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s),$$

and $\zeta(s)$ *is the Riemann zeta function.*

In [5], C. Jia improved the error terms in Proposition 1.1 for $m > 1$. H. Walum [8] showed that for a prime p,

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{\pi^4(p-1)}{p^2} \sum_{h=1}^{p} |S(h,p)|^2 .$$

In the spirit of [3] and [8], W. Zhang [9] used the estimate for character sums to prove the following:

Proposition 1.2. *Suppose that* p *is any sufficiently large prime number and* n *is any positive integer. Then for* $k = p^n$, *we have*

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{5}{144} \cdot \frac{(p^2-1)^2}{p(p^3-1)} \cdot k^2 + O\left(k \exp\left(\frac{3\log k}{\log\log k}\right)\right),$$

where $\exp(y) = e^y$.

For integers a, b and $q > 0$, the homogeneous Dedekind sum is defined by (see [4])

$$S(a,b,q) = \sum_{r=1}^{q} \left(\left(\frac{ar}{q}\right)\right)\left(\left(\frac{br}{q}\right)\right).$$

Z. Zheng [11] gave a generalized Knopp's identity for $S(a,b,q)$. However, it seems that no one has studied the mean values of $S(a,b,q)$. In Section 3, we shall study the mean values of the homogeneous Dedekind sums in short intervals $\left[1, \frac{p}{3}\right]$ and $\left[1, \frac{p}{4}\right]$, and give some asymptotic formulae.

For integers a and $q > 0$, the Cochrane sum, which is analogous to the Dedekind sum, is defined by

$$C(a,q) = \sum_{r=1}^{q}{}' \left(\left(\frac{r}{q}\right)\right)\left(\left(\frac{a\bar{r}}{q}\right)\right),$$

where $\sum_r{}'$ denotes the summation over all r such that $(r,q) = 1$, and $\bar{r}$ is defined by the equation $r\bar{r} \equiv 1 \bmod q$. W. Zhang and Y. Yi [10] gave the following upper bound estimate

$$|C(a,q)| \ll \sqrt{q}\, d(q) \ln^2 q$$

and proved that

$$\sum_{a=1}^{p-1} C^2(a,p) = \frac{5}{144}p^2 + O\left(p \exp\left(\frac{4\ln p}{\ln\ln p}\right)\right),$$

where $d(q)$ is the divisor function.

Similarly, we define the homogeneous Cochrane sum as follows:

$$C(a,b,q) = \sum_{r=1}^{q}{}' \left(\left(\frac{ar}{q}\right)\right)\left(\left(\frac{b\overline{r}}{q}\right)\right).$$

In Section 4, we shall study the mean values of the homogeneous Cochrane sums in short intervals $\left[1, \frac{p}{3}\right]$ and $\left[1, \frac{p}{4}\right]$, and give some asymptotic formulae.

Now we state the following results, which will be useful.

Lemma 1.1. *Let $p \geq 5$ be a prime. For any non-principal character χ modulo p, we have*

$$\sum_{a\leq \frac{p}{3}} \chi(a) = \frac{3\tau(\chi)}{2\pi i} L(1, \overline{\chi}\chi_3^0)$$

and

$$\sum_{a\leq \frac{p}{4}} \chi(a) = \frac{\tau(\chi)}{\pi i}\left[1 + \frac{\overline{\chi}(2)}{2} - \frac{\overline{\chi}(4)}{2}\right] L(1, \overline{\chi}),$$

where χ_3^0 is the principal character modulo 3, and $\tau(\chi) = \sum_{a=1}^{p} \chi(a)e^{2\pi i a/p}$ is the Gauss sum.

Proof. These identities can be easily deduced from the Fourier expansion for primitive character sums (see [6])

$$\sum_{a\leq \lambda p} \chi(a) = \begin{cases} \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{+\infty} \frac{\overline{\chi}(n)\sin(2\pi n\lambda)}{n}, & \text{if } \chi(-1) = 1; \\ \dfrac{\tau(\chi)}{\pi i} \displaystyle\sum_{n=1}^{+\infty} \frac{\overline{\chi}(n)(1-\cos(2\pi n\lambda))}{n}, & \text{if } \chi(-1) = -1. \end{cases}$$

See also reference [2]. □

§ 2. Mean values of the Dirichlet L-functions

In this section, we shall prove some mean values of the Dirichlet L-functions, which will be used in Section 3 and Section 4. First we have

Lemma 2.1. *Let $p \geq 5$ be a prime. We have*

(I)
$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 \left|L(1,\chi\chi_3^0)\right|^2 = \frac{2\pi^4}{81}p + O\left(p^\epsilon\right);$$

$$\text{(II)} \qquad \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} L^2(1,\overline{\chi})L^2(1,\chi\chi_3^0) = \frac{16\pi^4}{729}p + O\left(p^{\epsilon}\right).$$

Proof. We only prove (I), since similarly we can deduce (II). For any non-principal character χ modulo p, and parameter $N \geq p$, by Abel's identity we get

$$L(1,\chi) = \sum_{n=1}^{+\infty} \frac{\overline{\chi}(n)}{n} = \sum_{1\leq n\leq N} \frac{\chi(n)}{n} + \int_N^{+\infty} \frac{\sum_{N<n\leq y} \chi(n)}{y^2} dy$$

$$= \sum_{1\leq n\leq N} \frac{\chi(n)}{n} + O\left(\frac{\sqrt{p}\log p}{N}\right). \tag{2.1}$$

Let $\tau(n) = \sum_{d|n} \chi_3^0(d)$. Then using the methods in Lemma 3 of [9] we have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 \left|L(1,\chi\chi_3^0)\right|^2$$

$$= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \sum_{1\leq n_1\leq N} \frac{\chi(n_1)}{n_1} \sum_{1\leq m_1\leq N} \frac{\overline{\chi}(m_1)}{m_1}$$

$$\times \sum_{1\leq n_2\leq N} \frac{\chi\chi_3^0(n_2)}{n_2} \sum_{1\leq m_2\leq N} \frac{\overline{\chi}\chi_3^0(m_2)}{m_2} + O\left(\frac{p^{\frac{3}{2}}\log p \log^3 N}{N}\right)$$

$$= \sum_{1\leq n\leq N^2} \frac{\tau(n)}{n} \sum_{1\leq m\leq N^2} \frac{\tau(m)}{m} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(n)\overline{\chi}(m) + O\left(\frac{p^{\frac{3}{2}}\log p \log^3 N}{N}\right)$$

$$= \frac{1}{2}(p-1) \sum_{\substack{1\leq n\leq N^2\\ (n,p)=1}} \frac{\tau^2(n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}}\log p \log^3 N}{N}\right)$$

$$= \frac{1}{2}(p-1) \sum_{\substack{n=1\\ (n,p)=1}}^{+\infty} \frac{\tau^2(n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}}\log p \log^3 N}{N}\right).$$

By Euler products we have

$$\sum_{\substack{n=1\\ (n,p)=1}}^{+\infty} \frac{\tau^2(n)}{n^2} = \prod_{p_1\neq p} \left(1 + \frac{\tau^2(p_1)}{p_1^2} + \frac{\tau^2(p_1^2)}{p_1^4} + \cdots\right)$$

$$= \left(1+\frac{1}{3^2}+\frac{1}{3^4}+\cdots\right)\prod_{\substack{p_1\neq 3\\ p_1\neq p}}\left(1+\frac{4}{p_1^2}+\frac{9}{p_1^4}+\cdots+\frac{(\alpha+1)^2}{p_1^{2\alpha}}+\cdots\right)$$

$$= \frac{1}{1-\frac{1}{3^2}}\prod_{\substack{p_1\neq 3\\ p_1\neq p}}\frac{1+\frac{1}{p_1^2}}{\left(1-\frac{1}{p_1^2}\right)^3} = \frac{\left(1-\frac{1}{3^2}\right)^2}{1+\frac{1}{3^2}}\prod_{p_1}\frac{1+\frac{1}{p_1^2}}{\left(1-\frac{1}{p_1^2}\right)^3}+O\left(\frac{1}{p}\right)$$

$$= \frac{\left(1-\frac{1}{3^2}\right)^2}{1+\frac{1}{3^2}}\cdot\frac{\zeta^4(2)}{\zeta(4)}+O\left(\frac{1}{p}\right) = \frac{4}{81}\pi^4+O\left(\frac{1}{p}\right).$$

Now taking $N = p^{\frac{3}{2}}$ in the above, we immediately get

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2\left|L(1,\chi\chi_3^0)\right|^2 = \frac{2\pi^4}{81}p+O\left(p^{\epsilon}\right).$$

This proves Lemma 2.1.

Lemma 2.2. *Let $p\geq 5$ be a prime. For any integer $j\geq 0$, we have*
(I)

$$\Psi_1 = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^j)L^2(1,\chi)L(1,\overline{\chi})L(1,\overline{\chi}\chi_3^0) = \frac{\left[\frac{3}{4}(j+1)+\frac{1}{2}\right]\pi^4}{45\cdot 2^j}p+O\left(p^{\epsilon}\right);$$

(II)

$$\Psi_2 = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(2^j)L^2(1,\chi)L(1,\overline{\chi})L(1,\overline{\chi}\chi_3^0) = \frac{\left[\frac{3}{4}(j+1)+\frac{1}{2}\right]\pi^4}{45\cdot 2^j}p+O\left(p^{\epsilon}\right).$$

Proof. We only prove (I), since similarly we can deduce (II). For parameter $N\geq p$, from (2.1) and the methods in Lemma 3 of [9] we have

$$\begin{aligned}&\Psi_1\\ &= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^j)\sum_{1\leq n_1\leq N}\frac{\chi(n_1)}{n_1}\sum_{1\leq n_2\leq N}\frac{\chi(n_2)}{n_2}\sum_{1\leq m_1\leq N}\frac{\overline{\chi}(m_1)}{m_1}\sum_{1\leq m_2\leq N}\frac{\overline{\chi}\chi_3^0(m_2)}{m_2}\\ &\qquad +O\left(\frac{p^{\frac{3}{2}}\log p\log^3 N}{N}\right)\\ &= \sum_{1\leq n\leq N^2}\frac{d(n)}{n}\sum_{1\leq m\leq N^2}\frac{\tau(m)}{m}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^j)\chi(n)\overline{\chi}(m)+O\left(\frac{p^{\frac{3}{2}}\log p\log^3 N}{N}\right)\end{aligned}$$

$$= \frac{p-1}{2^{j+1}} \sum_{\substack{1\le n\le N^2\\(n,p)=1}} \frac{d(n)\tau(2^j n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}}\log p\log^3 N}{N}\right)$$

$$= \frac{p-1}{2^{j+1}} \sum_{\substack{n=1\\(n,p)=1}}^{+\infty} \frac{d(n)\tau(2^j n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}}\log p\log^3 N}{N}\right).$$

From the properties of multiplicative functions we have

$$\sum_{\substack{n=1\\(n,p)=1}}^{+\infty} \frac{d(n)\tau(2^j n)}{n^2}$$

$$= \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(n)\tau(2^j n)}{n^2} + \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(2n)\tau(2^{j+1} n)}{(2n)^2} + \cdots$$

$$+ \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(2^k n)\tau(2^{j+k} n)}{(2^k n)^2} + \cdots$$

$$= \left(\sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(n)\tau(n)}{n^2}\right)\left(\tau(2^j) + \frac{d(2)\tau(2^{j+1})}{2^2} + \cdots + \frac{d(2^k)\tau(2^{j+k})}{2^{2k}} + \cdots\right)$$

$$= S_1 \cdot S_2.$$

By Euler products we have

$$S_1 = \prod_{\substack{p_1\neq 2\\p_1\neq p}} \left(1 + \frac{d(p_1)\tau(p_1)}{p_1^2} + \frac{d(p_1^2)\tau(p_1^2)}{p_1^4} + \cdots\right)$$

$$= \left(1 + \frac{2}{3^2} + \frac{3}{3^4} + \cdots\right) \prod_{\substack{p_1\neq 2\\p_1\neq 3\\p_1\neq p}} \left(1 + \frac{4}{p_1^2} + \frac{9}{p_1^4} + \cdots + \frac{(\alpha+1)^2}{p_1^{2\alpha}} + \cdots\right)$$

$$= \frac{1}{\left(1-\frac{1}{3^2}\right)^2} \prod_{\substack{p_1\neq 2\\p_1\neq 3\\p_1\neq p}} \frac{1+\frac{1}{p_1^2}}{\left(1-\frac{1}{p_1^2}\right)^3} = \frac{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{2^2}\right)^3}{\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)} \cdot \frac{\zeta^4(2)}{\zeta(4)} + O\left(\frac{1}{p}\right)$$

$$= \frac{3}{160}\pi^4 + O\left(\frac{1}{p}\right).$$

On the other hand, we easily get

$$S_2 = (j+1) + \frac{2(j+2)}{4} + \frac{3(j+3)}{4^2} + \cdots = \frac{\frac{3}{4}(j+1) + \frac{1}{2}}{\left(1-\frac{1}{4}\right)^3}.$$

Now taking $N = p^{\frac{3}{2}}$ in the above, we immediately have

$$\Psi_1 = \frac{\left[\frac{3}{4}(j+1) + \frac{1}{2}\right]\pi^4}{45 \cdot 2^j} p + O\left(p^{\epsilon}\right).$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $p \geq 5$ be a prime. For any integer $j \geq 0$, we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^j)\, |L(1,\chi)|^4 = \frac{\left[\frac{3}{4}(j+1) + \frac{1}{2}\right]\pi^4}{9 \cdot 2^{j+2}} p + O\left(p^{\epsilon}\right).$$

Proof. For parameter $N \geq p$, from (2.1) we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^j)\, |L(1,\chi)|^4 \\
&= \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^j) \sum_{1\leq n_1\leq N} \frac{\chi(n_1)}{n_1} \sum_{1\leq n_2\leq N} \frac{\chi(n_2)}{n_2} \\
&\qquad \times \sum_{1\leq m_1\leq N} \frac{\overline{\chi}(m_1)}{m_1} \sum_{1\leq m_2\leq N} \frac{\overline{\chi}(m_2)}{m_2} + O\left(\frac{p^{\frac{3}{2}} \log p \log^3 N}{N}\right) \\
&= \sum_{1\leq n\leq N^2} \frac{d(n)}{n} \sum_{1\leq m\leq N^2} \frac{d(m)}{m} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^j)\chi(n)\overline{\chi}(m) + O\left(\frac{p^{\frac{3}{2}} \log p \log^3 N}{N}\right) \\
&= \frac{p-1}{2^{j+1}} \sum_{\substack{1\leq n\leq N^2 \\ (n,p)=1}} \frac{d(n)d(2^j n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}} \log p \log^3 N}{N}\right) \\
&= \frac{p-1}{2^{j+1}} \sum_{\substack{n=1 \\ (n,p)=1}}^{+\infty} \frac{d(n)d(2^j n)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{p^{\frac{3}{2}} \log p \log^3 N}{N}\right).
\end{aligned}$$

From the properties of multiplicative functions we have

$$\sum_{\substack{n=1 \\ (n,p)=1}}^{+\infty} \frac{d(n)d(2^j n)}{n^2}$$

$$\begin{aligned}
&= \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(n)d(2^j n)}{n^2} + \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(2n)d(2^{j+1}n)}{(2n)^2} + \cdots \\
&\quad + \sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d(2^k n)d(2^{j+k}n)}{(2^k n)^2} + \cdots \\
&= \left(\sum_{\substack{n=1\\(n,2p)=1}}^{+\infty} \frac{d^2(n)}{n^2}\right)\left(d(2^j) + \frac{d(2)d(2^{j+1})}{2^2} + \cdots + \frac{d(2^k)d(2^{j+k})}{2^{2k}} + \cdots\right) \\
&= T_1 \cdot T_2.
\end{aligned}$$

By Euler products we have

$$\begin{aligned}
T_1 &= \prod_{\substack{p_1 \neq 2\\ p_1 \neq p}} \left(1 + \frac{d^2(p_1)}{p_1^2} + \frac{d^2(p_1^2)}{p_1^4} + \cdots\right) \\
&= \prod_{\substack{p_1 \neq 2\\ p_1 \neq p}} \left(1 + \frac{4}{p_1^2} + \frac{9}{p_1^4} + \cdots + \frac{(\alpha+1)^2}{p_1^{2\alpha}} + \cdots\right) \\
&= \prod_{\substack{p_1 \neq 2\\ p_1 \neq p}} \frac{1 + \frac{1}{p_1^2}}{\left(1 - \frac{1}{p_1^2}\right)^3} = \frac{\left(1 - \frac{1}{2^2}\right)^3}{\left(1 + \frac{1}{2^2}\right)} \cdot \frac{\zeta^4(2)}{\zeta(4)} + O\left(\frac{1}{p}\right) = \frac{3}{128}\pi^4 + O\left(\frac{1}{p}\right).
\end{aligned}$$

On the other hand, we easily get

$$T_2 = (j+1) + \frac{2(j+2)}{4} + \frac{3(j+3)}{4^2} + \cdots = \frac{\frac{3}{4}(j+1) + \frac{1}{2}}{\left(1 - \frac{1}{4}\right)^3}.$$

Now taking $N = p^{\frac{3}{2}}$ in the above, we immediately have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^j)\,|L(1,\chi)|^4 = \frac{\left[\frac{3}{4}(j+1) + \frac{1}{2}\right]\pi^4}{9 \cdot 2^{j+2}} p + O\left(p^{\epsilon}\right).$$

This proves Lemma 2.3. □

§ 3. Mean values of the homogeneous Dedekind sums

In this section, we shall prove the following:

Theorem 3.1. *Let $p \geq 5$ be a prime. We have*

(I) $$\sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{3}} S(a,b,p) = \frac{1}{18}p^2 + O\left(p^{1+\epsilon}\right);$$

(II)
$$\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{4}} S(a,b,p)=\frac{3}{64}p^2+O\left(p^{1+\epsilon}\right);$$

(III)
$$\sum_{a\le\frac{p}{4}}\sum_{b\le\frac{p}{4}} S(a,b,p)=\frac{131}{2304}p^2+O\left(p^{1+\epsilon}\right).$$

Proof. From [9] we know that
$$S(a,p)=\frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)\left|L(1,\chi)\right|^2.$$

Therefore
$$S(a,b,p)=S(a\bar{b},p)=\frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)\overline{\chi}(b)\left|L(1,\chi)\right|^2.$$

Then from Lemma 1.1 and Lemma 2.1 we have
$$\begin{aligned}
&\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{3}} S(a,b,p)\\
=\ & \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\sum_{a\le\frac{p}{3}}\chi(a)\sum_{b\le\frac{p}{3}}\overline{\chi}(b)\\
=\ & \frac{9p^2}{4\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\left|L(1,\chi\chi_3^0)\right|^2\\
=\ & \frac{1}{18}p^2+O\left(p^{1+\epsilon}\right).
\end{aligned}$$

Similarly, from Lemma 1.1, Lemma 2.2 and Lemma 2.3 we get
$$\begin{aligned}
&\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{4}} S(a,b,p)\\
=\ & \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\sum_{a\le\frac{p}{3}}\chi(a)\sum_{b\le\frac{p}{4}}\overline{\chi}(b)\\
=\ & \frac{3p^2}{2\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(1+\frac{\chi(2)}{2}-\frac{\chi(4)}{2}\right)\left|L(1,\chi)\right|^2 L(1,\overline{\chi}\chi_3^0)L(1,\chi)\\
=\ & \frac{3p^2}{2\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(1+\frac{\chi(2)}{2}-\frac{\chi(4)}{2}\right)L^2(1,\chi)L(1,\overline{\chi})L(1,\overline{\chi}\chi_3^0)\\
=\ & \frac{3}{64}p^2+O\left(p^{1+\epsilon}\right)
\end{aligned}$$

and

$$\sum_{a\leq\frac{p}{4}}\sum_{b\leq\frac{p}{4}} S(a,b,p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 \sum_{a\leq\frac{p}{4}} \chi(a) \sum_{b\leq\frac{p}{4}} \overline{\chi}(b)$$

$$= \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left|1+\frac{\overline{\chi}(2)}{2}-\frac{\overline{\chi}(4)}{2}\right|^2 |L(1,\chi)|^4$$

$$= \frac{3p^2}{2\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^4 + \frac{p^2}{2\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)\,|L(1,\chi)|^4$$

$$-\frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(4)\,|L(1,\chi)|^4$$

$$= \frac{131}{2304}p^2 + O\left(p^{1+\epsilon}\right).$$

This proves Theorem 3.1. □

§ 4. Mean values of the homogeneous Cochrane sums

In this section, we shall prove the following:

Theorem 4.1. *Let $p \geq 5$ be a prime. We have*

(I)
$$\sum_{a\leq\frac{p}{3}}\sum_{b\leq\frac{p}{3}} C(a,b,p) = \frac{4}{81}p^2 + O\left(p^{1+\epsilon}\right);$$

(II)
$$\sum_{a\leq\frac{p}{3}}\sum_{b\leq\frac{p}{4}} C(a,b,p) = \frac{3}{64}p^2 + O\left(p^{1+\epsilon}\right);$$

(III)
$$\sum_{a\leq\frac{p}{4}}\sum_{b\leq\frac{p}{4}} C(a,b,p) = \frac{79}{1536}p^2 + O\left(p^{1+\epsilon}\right).$$

Proof. From [10] we know that

$$C(a,p) = \frac{-1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(a)\tau^2(\chi)L^2(1,\overline{\chi}).$$

Therefore

$$C(a,b,p) = C(ab,p) = \frac{-1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(a)\overline{\chi}(b)\tau^2(\chi)L^2(1,\overline{\chi}).$$

Then from Lemma 1.1 and Lemma 2.1 we have

$$\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{3}} C(a,b,p) = \frac{-1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \tau^2(\chi)L^2(1,\overline{\chi}) \sum_{a\le\frac{p}{3}} \overline{\chi}(a) \sum_{b\le\frac{p}{3}} \overline{\chi}(b)$$

$$= \frac{9p^2}{4\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} L^2(1,\overline{\chi})L^2(1,\chi\chi_3^0)$$

$$= \frac{4}{81}p^2 + O\left(p^{1+\epsilon}\right).$$

Similarly, from Lemma 1.1, Lemma 2.2 and Lemma 2.3 we get

$$\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{4}} C(a,b,p) = \frac{-1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \tau^2(\chi)L^2(1,\overline{\chi}) \sum_{a\le\frac{p}{3}} \overline{\chi}(a) \sum_{b\le\frac{p}{4}} \overline{\chi}(b)$$

$$= \frac{3p^2}{2\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left(1+\frac{\chi(2)}{2}-\frac{\chi(4)}{2}\right) L^2(1,\overline{\chi})L(1,\chi)L(1,\chi\chi_3^0)$$

$$= \frac{3p^2}{2\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left(1+\frac{\overline{\chi}(2)}{2}-\frac{\overline{\chi}(4)}{2}\right) L^2(1,\chi)L(1,\overline{\chi})L(1,\overline{\chi}\chi_3^0)$$

$$= \frac{3}{64}p^2 + O\left(p^{1+\epsilon}\right)$$

and

$$\sum_{a\le\frac{p}{4}}\sum_{b\le\frac{p}{4}} C(a,b,p) = \frac{-1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \tau^2(\chi)L^2(1,\overline{\chi}) \sum_{a\le\frac{p}{4}} \overline{\chi}(a) \sum_{b\le\frac{p}{4}} \overline{\chi}(b)$$

$$= \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left(1+\frac{\chi(2)}{2}-\frac{\chi(4)}{2}\right)^2 |L(1,\chi)|^4$$

$$= \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left(1+\chi(2)-\frac{3}{4}\chi(4)-\frac{1}{2}\chi(8)+\frac{1}{4}\chi(16)\right) |L(1,\chi)|^4$$

$$= \frac{79}{1536}p^2 + O\left(p^{1+\epsilon}\right).$$

This completes the proof of Theorem 4.1. □

Acknowledgments. The author expresses his gratitude to the referee for his very helpful and detailed comments in improving this paper.

References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Graduate Texts in Mathematics, No. 41. Springer-Verlag, New York-Heidelberg, 1976.

[2] B. C. Berndt, *Classical theorems on quadratic residues*, Enseignement Math. (2) **22** (1976), no. 3-4, 261–304.

[3] J. B. Conrey, E. Fransen, R. Klein, and C. Scott, *Mean values of Dedekind sums*, J. Number Theory **56** (1996), no. 2, 214–226.

[4] R. R. Hall and M. N. Huxley, *Dedekind sums and continued fractions*, Acta Arith. **63** (1993), no. 1, 79–90.

[5] C. Jia, *On the mean value of Dedekind sums*, J. Number Theory **87** (2001), no. 2, 173–188.

[6] G. Polya, *Über die verteilung der quadratischen reste and nicht-reste*, Göttingen Nachrichten (1918), 21–29.

[7] H. Rademacher and E. Grosswald, *Dedekind sums*, The Carus Mathematical Monographs, No. 16. The Mathematical Association of America, Washington, D.C., 1972.

[8] H. Walum, *An exact formula for an average of L-series*, Illinois J. Math. **26** (1982), no. 1, 1–3.

[9] W. Zhang, *On the mean values of Dedekind sums*, J. Theor. Nombres Bordeaux **8** (1996), no. 2, 429–442.

[10] W. Zhang and Y. Yi, *On the upper bound estimate of Cochrane sums*, Soochow J. Math. **28** (2002), no. 3, 297–304.

[11] Z. Zheng, *The Petersson-Knopp identity for the homogeneous Dedekind sums*, J. Number Theory **57** (1996), no. 2, 223–230.

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P. R. China
E-mail address: hnliu@nwu.edu.cn