

ON ANNIHILATOR IDEALS OF A NEARRING OF SKEW POLYNOMIALS OVER A RING

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ABSTRACT. For a ring endomorphism α and an α -derivation δ of a ring R , we study relation between the set of annihilators in R and the set of annihilators in nearring $R[x; \alpha, \delta]$ and $R_0[[x; \alpha]]$. Also we extend results of Armendariz on the Baer and p.p. conditions in a polynomial ring to certain analogous annihilator conditions in a nearring of skew polynomials. These results are somewhat surprising since, in contrast to the skew polynomial ring and skew power series case, the nearring of skew polynomials and skew power series have substitution for its “multiplication” operation.

0. Introduction

We use R and N to denote a ring and a nearring respectively. Throughout this paper all rings are associative with identity, all nearrings are left nearrings, α is an endomorphism of R and δ is an α -derivation of R , that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote by $(R[x; \alpha, \delta], +, \cdot)$ the Ore extension whose elements are the polynomials $\sum_{i=0}^n r_i x^i$, $r_i \in R$, where addition defined as usual and multiplication by $xb = \alpha(b)x + \delta(b)$ for each $b \in R$. We use $(R[[x; \alpha]], +, \cdot)$ to denote the skew power series ring over a ring R , where addition defined as usual and multiplication by $xb = \alpha(b)x$ for each $b \in R$. Recall a ring or nearring is said to be *reduced* if it has no nonzero nilpotent element. A ring R is called (*quasi-*)*Baer* if R has identity and the right annihilator of every nonempty subset (*ideal*) of R is generated, as a right ideal, by an idempotent. The class of Baer rings includes all right Noetherian Rickart rings and all von Neumann regular rings. In [16], Kaplansky introduced Baer rings to abstract various properties of AW*-algebra and von Neumann algebras. For more on Baer rings see [3]. In 1974, Armendariz obtained the following result: Let R be a reduced ring. Then R is a Baer ring if and only if polynomial ring $(R[x], +, \cdot)$ is a Baer ring. In [10], Clark

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defined quasi-Baer rings and used them to characterized when a finite dimensional algebra with identity over an algebraically closed field is isomorphic to twisted matrix units semigroup algebra. In [8], Birkenmeier et al. showed that the quasi-Baer condition is preserved by polynomial extensions. For more on quasi-Baer rings see [7-8]. As a generalization of quasi-Baer rings, in [4], G. F. Birkenmeier, J. Y. Kim and J. K. Park introduced the concept of principally quasi-Baer rings. A ring R is called *right (resp. left) principally quasi-Baer* (or simply right (resp. left) p.q.-Baer) if the right (resp. left) annihilator of a principal right (resp. left) ideal of R is generated by an idempotent. Observe that every biregular ring and every quasi-Baer ring is right p.q.-Baer. In [9], the authors showed that R is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring.

According to Krempa [17], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α -rigid if there exists a rigid endomorphism α of R . In [15], C. Y. Hong, N. K. Kim and T. K. Kwak studied Ore extensions of quasi-Baer rings over α -rigid rings and proved the following results. Let R be an α -rigid ring. Then R is a Baer ring if and only if Ore extension $(R[x; \alpha, \delta], +, \cdot)$ is a Baer ring if and only if skew power series ring $(R[[x; \alpha]], +, \cdot)$ is a Baer ring. In [12], the authors introduced α -compatible rings (i.e., $ab = 0 \Leftrightarrow a\alpha(b) = 0$ for each $a, b \in R$) which are a generalization of α -rigid rings and study on the relationship between the quasi Baerness and the p.q.-Baer property of a ring R and those of the polynomial extensions (including formal skew power series, skew Laurent polynomials and skew Laurent series).

In [4-5], the authors introduced several Baer-type annihilator conditions on a nearring which are equivalent to the Baer condition when the nearring is a ring with identity. Also, they investigated various annihilator conditions on polynomials and formal power series under addition and substitution. In [6], the authors introduced *quasi-Baer annihilator conditions* in the class of nearrings by defining the following: (For a nonempty subset $S \subseteq N$, let $r_N(S) = \{a \in N \mid Sa = 0\}$ and $\ell_N(S) = \{a \in N \mid aS = 0\}$. If the context is clear, the subscript may be omitted.)

(1) $N \in q\mathcal{B}_{r1}$ if for every ideal $I \subseteq N$ the right annihilator $r_N(I) = eN$ for some idempotent $e \in N$;

(2) $N \in q\mathcal{B}_{r2}$ if for every ideal $I \subseteq N$ the right annihilator $r_N(I) = r_N(e)$ for some idempotent $e \in N$;

(3) $N \in q\mathcal{B}_{\ell1}$ if for every ideal $I \subseteq N$ the left annihilator $\ell_N(I) = Ne$ for some idempotent $e \in N$;

(4) $N \in q\mathcal{B}_{\ell2}$ if for every ideal $I \subseteq N$ the left annihilator $\ell_N(I) = \ell_N(e)$ for some idempotent $e \in N$.

The *quasi-Rickart* annihilator conditions in the class of nearrings are also defined and denoted similarly except replacing $q\mathcal{B}$ by $q\mathcal{R}$, when I is a principally generated ideal. If the ideal I considered in the above definition is replaced with any nonempty subset (singleton subset), we obtain the *Baer-type (Rickart-type)*

annihilator conditions [4], denoted without the prefix “q” in the above notations. In [3, p. 28], the \mathcal{R}_{r_2} condition is considered for rings with involution. If N is a ring with identity, then $N \in q\mathcal{B}_{r_1} \cup q\mathcal{B}_{r_2} \cup q\mathcal{B}_{\ell_1} \cup q\mathcal{B}_{\ell_2}$ is equivalent to N being a quasi-Baer ring. Similarly $N \in q\mathcal{R}_{r_1} \cup q\mathcal{R}_{r_2}$ ($N \in q\mathcal{R}_{\ell_1} \cup q\mathcal{R}_{\ell_2}$) is equivalent to N being a right (left) p.q.-Baer. In [4], the authors proved the following result which is analogue of Armendariz’s result: Let R be a reduced ring. Then R is a Baer (Rickart) ring if and only if nearring $(R[x], +, \circ) \in q\mathcal{B}_{r_2}$ ($R[x] \in \mathcal{R}_{r_2} \cup q\mathcal{R}_{r_2}$).

The binary operation of substitution, denoted by \circ , of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for $(x)f, (x)g \in R[x; \alpha, \delta]$ with

$$(x)f = \sum_{i=0}^m f_i x^i, \quad (x)g \circ (x)f = \sum_{i=0}^m f_i ((x)g)^i.$$

Observe that the system $(R[x; \alpha, \delta], +, \circ)$ is a left nearring. We use $R[x; \alpha, \delta]$ to denote the left nearring of skew polynomials $(R[x; \alpha, \delta], +, \circ)$ with coefficients from R and $R_0[x; \alpha, \delta] = \{f \in R[x; \alpha, \delta] \mid f \text{ has zero constant term}\}$. Clearly, $R_0[x; \alpha, \delta]$ is a 0-symmetric left nearring. For instance, let $(x)g = a_0 + a_1 x$ and $(x)f = b_0 + b_1 x + b_2 x^2 \in R[x; \alpha, \delta]$. Through a simple calculation, we have

$$\begin{aligned} (x)g \circ (x)f &= ((x)g)f = b_0 + b_1((x)g) + b_2((x)g)^2 \\ &= (b_0 + b_1 a_0 + b_2 a_0^2 + b_2 a_1 \delta(a_0)) + (b_1 a_1 + b_2 a_0 a_1 + b_2 a_1 \alpha(a_0) \\ &\quad + b_2 a_1 \delta(a_0))x + b_2 a_1 \alpha(a_1)x^2. \end{aligned}$$

Also we denote the collection of all skew power series with positive orders using the operations of addition and substitution by $R_0[[x; \alpha]]$ unless specifically indicated otherwise (i.e., $R_0[[x; \alpha]]$ denotes $(R_0[[x; \alpha]], +, \circ)$). Observe that the system $(R_0[[x; \alpha]], +, \circ)$ is a 0-symmetric left nearring. However, the operation “ \circ ”, left distributes but does not right distribute over addition. Thus $(R_0[[x; \alpha]], +, \circ)$ forms a left nearring but not a ring. For instance, let $(x)f = \sum_{i=1}^{\infty} f_i x^i$ and $(x)g = \sum_{j=1}^{\infty} g_j x^j \in R_0[[x; \alpha]]$. Then

$$\begin{aligned} (x)f \circ (x)g &= ((x)f)g = b_1(x)f + b_2((x)f)^2 + b_3((x)f)^3 + \dots \\ &= b_1 a_1 x + (b_1 a_2 + b_2 a_1 \alpha(a_1))x^2 \\ &\quad + (b_1 a_3 + b_2 a_1 \alpha(a_2) + b_2 a_2 \alpha^2(a_1) + b_3 a_1 \alpha(a_1) \alpha^2(a_1))x^3 + \dots \end{aligned}$$

In this paper for an α -rigid ring R , we show that: (1) R is quasi-Baer if and only if $R[x; \alpha, \delta] \in q\mathcal{B}_{r_2}$ (resp. $R_0[[x; \alpha]] \in q\mathcal{B}_{r_2}$) if and only if $R_0[x; \alpha, \delta] \in q\mathcal{B}_{r_1}$ (resp. $R_0[[x; \alpha]] \in q\mathcal{B}_{r_1}$). (2) R is Rickart if and only if $R[x; \alpha, \delta] \in \mathcal{R}_{r_2}$ if and only if $R[x; \alpha, \delta] \in q\mathcal{R}_{r_2}$. Examples to show that α -rigid condition on R is not superfluous, are provided.

1. Narrings of Ore extensions

Definition 1.1 (Krempa [17]). Let α be an endomorphism of R . α is called a *rigid* endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called to be α -*rigid* if there exists a rigid endomorphism α of R .

Clearly, any rigid endomorphism is a monomorphism. Note that α -rigid rings are reduced rings. In fact, if R is an α -rigid ring and $a^2 = 0$ for $a \in R$, then $a\alpha(a)\alpha(a\alpha(a)) = 0$. Thus $a\alpha(a) = 0$ and so $a = 0$. Therefore R is reduced. But there exists an endomorphism of a reduced ring which is not a rigid endomorphism (see [15, Example 9]). However, if α is an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for any $r \in R$) of a reduced ring R , then R is α -rigid.

Definition 1.2. Let δ be an α -derivation of a ring R . Let I be an ideal of R .

I is said to be α -ideal if $\alpha(I) \subseteq I$;

I is said to be δ -ideal if $\delta(I) \subseteq I$;

I is said to be (α, δ) -ideal if it is both α -ideal and δ -ideal.

Lemma 1.3 (Hong et al. [15]). *Let R be an α -rigid ring and $a, b \in R$. Then we have the following:*

- (i) *If $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for each positive integer n ,*
- (ii) *If $a\alpha^k(b) = 0 = \alpha^k(a)b$ for some positive integer k , then $ab = 0$,*
- (iii) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any positive integers m, n ,*
- (iv) *If $e^2 = e \in R$, then $\alpha(e) = e$ and $\delta(e) = 0$.*

A nerring N is said to has the *insertion of factors property* (or simply, IFP) if for all $a, b, n \in N$, $ab = 0$ implies $anb = 0$. Clearly, each reduced nerring has the IFP. The following is the key lemma for narrings of skewed polynomials over α -rigid rings satisfying Baer-type annihilator conditions.

Let $f(x) = a_0 + \dots + a_n x^n \in R[x; \alpha, \delta]$. We say $\deg(f(x)) = n$, when $a_n \neq 0$. If $f(x) = 0$, then $\deg(f(x)) = \infty$.

Proposition 1.4. *Suppose that R is an α -rigid ring. Let $(x)f = a_0 + a_1 x + \dots + a_n x^n$, $(x)g = b_0 + b_1 x + \dots + b_m x^m \in R[x; \alpha, \delta]$. Then $(x)f \circ (x)g = 0$ if and only if $b_j a_i = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $b_0 + b_1 a_0 + b_2 a_0^2 + \dots + b_m a_0^m = 0$.*

Proof. Let $(x)f, (x)g \in R[x; \alpha, \delta]$ such that $(x)f \circ (x)g = 0$. We proceed by induction on $\deg(f) + \deg(g)$. It is clear for $\deg(f) + \deg(g) = 2$. Now suppose that our claim is true for each $(x)f, (x)g \in R[x; \alpha, \delta]$, with $\deg(f) \geq 1$, $\deg(g) \geq 1$, $\deg(f) + \deg(g) < k$. Let $(x)f = a_0 + a_1 x + \dots + a_n x^n$, $(x)g = b_0 + b_1 x + \dots + b_m x^m \in R[x; \alpha, \delta]$ such that $n, m \geq 1$ and $m + n = k$. Then $\sum_{j=0}^m b_j ((x)f)^j = 0$ and that $b_m a_n \alpha^n(a_n) \dots \alpha^{(m-1)n}(a_n) = 0$, since it is the leading coefficient of $(x)f \circ (x)g$. Hence $b_m a_n = a_n b_m = 0$, by Lemma 1.3. Thus $\sum_{j=0}^{m-1} a_n b_j ((x)f)^j = 0$ and that $(x)f \circ (a_n b_0 + a_n b_1 x + \dots + a_n b_{m-1} x^{m-1}) = 0$. By induction hypothesis, we have $a_n b_j a_n = 0$ for $1 \leq j \leq m - 1$. Hence

$a_n b_j = 0$ for $1 \leq j \leq m$, since R is reduced. Therefore $(x)f \circ (x)g = (a_0 + a_1x + \dots + a_{n-1}x^{n-1}) \circ (b_0 + b_1x + \dots + b_mx^m) = 0$, by Lemma 1.3. Then our assertion follows from induction hypothesis.

Since R has the IFP, the converse follows from Lemma 1.3. □

Lemma 1.5. *Let δ be an α -derivation of a ring R and $R[x; \alpha, \delta]$ the nerring of skew polynomials over R . Let R be an α -rigid ring. If $(x)\varepsilon \in R[x; \alpha, \delta]$ is an idempotent, then $(x)\varepsilon = e_0 + e_1x$, where e_1 is an idempotent in R with $e_1e_0 = 0$.*

Proof. Since $(x)\varepsilon \circ (x)\varepsilon = (e)\varepsilon$ and R is α -rigid, so $e_n = 0$ for all $n \geq 2$. Thus we have $e_0 + e_1(e_0 + e_1x) = e_0 + e_1x$ and that $e_1^2 = e_1, e_1e_0 = 0$. □

For a nerring N , put $rAnn_N(id(N)) = \{r_N(U) \mid U \text{ is an ideal of } N\}$ and $\ell Ann_N(id(N)) = \{\ell_N(U) \mid U \text{ is an ideal of } N\}$.

Proposition 1.6. *Let R be an α -rigid ring and $S = R[x; \alpha, \delta]$. Then*

$$\psi : rAnn_R(id(R)) \rightarrow rAnn_S(id(S)); I \rightarrow I_0[x; \alpha, \delta]$$

is bijective, where $I_0[x; \alpha, \delta]$ is the 0-symmetric left nerring of skew polynomials with coefficients from I .

Proof. Let $A \in rAnn_R(id(R))$. Then $A = r_R(I)$ for some ideal I of R . Let $I^{(\alpha, \delta)}$ be the (α, δ) -ideal of R generated by I . Then $J = I^{(\alpha, \delta)}[x; \alpha, \delta]$ is a left nerring of skew polynomials with coefficients from $I^{(\alpha, \delta)}$. We first show that J is an ideal of S . Let $(x)a = \sum_{i=0}^n a_i x^i \in J$ and $(x)f, (x)g = \sum_{j=0}^m g_j x^j \in S$. Observe that $(x)f \circ (x)a = \sum_{i=0}^n a_i ((x)f)^i \in I^{(\alpha, \delta)}[x; \alpha, \delta]$ and

$$\begin{aligned} & ((x)a + (x)f) \circ (x)g - (x)f \circ (x)g \\ &= \sum_{j=1}^m g_j ((x)a + (x)f)^j - \sum_{j=1}^m g_j ((x)f)^j \\ &= \sum_{j=1}^m g_j [((x)a + (x)f)^j - ((x)f)^j] \in I^{(\alpha, \delta)}[x; \alpha, \delta], \end{aligned}$$

since coefficients of $[((x)a + (x)f)^j - ((x)f)^j]$ and $a_j((x)f)^j$ belong to $I^{(\alpha, \delta)}$ for each j . Therefore J is an ideal of S . Since $A = r_R(I)$, hence A is an (α, δ) -ideal of R , by Lemma 1.3. Thus $A_0[x; \alpha, \delta]$ is a 0-symmetric left nerring of skew polynomials with coefficients from A . Now we show that $r_S(J) = A_0[x; \alpha, \delta]$. Since $IA = 0$, hence by Lemma 1.3, $AI^{(\alpha, \delta)} = I^{(\alpha, \delta)}A = 0$, which implies $I^{(\alpha, \delta)}[x; \alpha, \delta] \circ A_0[x; \alpha, \delta] = 0$. Thus $A_0[x; \alpha, \delta] \subseteq r_S(J)$. Let $(x)g = \sum_{j=0}^m g_j x^j \in r_S(J)$. Since $0 \in J$, hence $g_0 = 0$. Also $ax \circ (x)g = 0$, for each $a \in I$. Hence $g_j a = ag_j = 0$ for each $j = 1, \dots, m$, by Proposition 1.4. Thus $g_j \in r_R(I) = A$ for each j and that $(x)g \in A_0[x; \alpha, \delta]$. Consequently $r_S(J) = A_0[x; \alpha, \delta]$. Therefore ψ is a well defined map.

Suppose that $B \in rAnn_S(id(S))$. Then $B = r_S(J)$ for some ideal J of S . Let J^1 and B^1 denote the set of all coefficients of elements of J and B

respectively. Let $J^{1(\alpha,\delta)}$ and $B^{1(\alpha,\delta)}$ be the (α,δ) -ideals of R generated by J^1 and B^1 respectively. Hence $r_R(J^{1(\alpha,\delta)}) = r_R(J^1)$, by Lemma 1.3. We claim that $r_R(J^{1(\alpha,\delta)}) = B^{1(\alpha,\delta)}$ and $r_S(J) = B_0^{1(\alpha,\delta)}[x; \alpha, \delta]$. Since $0 \in J$, we have $B \subseteq R_0[x; \alpha, \delta]$. Let $\sum_{i=1}^n b_i x^i \in B$ and $(x)g = \sum_{j=0}^m g_j x^j \in J$. Then $(\sum_{j=0}^m g_j x^j) \circ (\sum_{i=1}^n b_i x^i) = 0$ and that $b_i g_j = g_j b_i = 0$ for each $1 \leq i \leq n, 1 \leq j \leq m$, by Proposition 1.4. Hence

$$((x)g + b_i x^{2m+1}) \circ x^2 - b_i x^{2m+1} \circ x^2 = g_0^2 + \dots + g_0 b_i x^{2m+1} \in J$$

for each $1 \leq i \leq n$. Therefore $b_i g_0 b_i = 0$ for each $1 \leq i \leq n$, by Proposition 1.4. Hence $gb = bg = 0$ for each $g \in J^1$ and $b \in B^1$. Consequently $g\alpha^{i_1} \delta^{j_1} \alpha^{i_2} \delta^{j_2} \dots \alpha^{i_t} \delta^{j_t} (b) = b\alpha^{i_1} \delta^{j_1} \alpha^{i_2} \delta^{j_2} \dots \alpha^{i_t} \delta^{j_t} (g) = 0$ for each nonnegative integers $i_1, \dots, i_t, j_1, \dots, j_t$ and $b \in B^1, g \in J^1$, by Lemma 1.3. Therefore $B^{1(\alpha,\delta)} \subseteq r_R(J^1) = r_R(J^{1(\alpha,\delta)})$ and $B_0^{1(\alpha,\delta)}[x; \alpha, \delta] \subseteq r_S(J)$. But $r_S(J) = B \subseteq B_0^{1(\alpha,\delta)}[x; \alpha, \delta]$, so $r_S(J) = B_0^{1(\alpha,\delta)}[x; \alpha, \delta]$. Let $t \in r_R(J^{1(\alpha,\delta)})$. Then $tJ^1 = J^1 t = 0$ and that $\sum_{j=0}^m g_j x^j \circ tx = 0$ for each $\sum_{j=0}^m g_j x^j \in J$. Hence $tx \in B$ and that $t \in B^1$. Therefore, $r_R(J^{1(\alpha,\delta)}) = B^{1(\alpha,\delta)}$. Consequently, ψ is surjective. □

Now we can prove the following.

Theorem 1.7. *Let R be an α -rigid ring and $S = R[x; \alpha, \delta]$. Then the following are equivalent:*

- (1) R is quasi-Baer;
- (2) $S \in q\mathcal{B}_{r2}$;
- (3) $(R[x; \alpha, \delta], +, \cdot)$ is quasi-Baer;
- (4) $S_0 = R_0[x; \alpha, \delta] \in q\mathcal{B}_{r1}$.

Proof. (1) \Rightarrow (2). Let $J \in rAnn_S(id(S))$. Then $J = I_0[x; \alpha, \delta]$ for some $I \in rAnn_R(id(R))$, by Proposition 1.6. Since R is quasi-Baer and every idempotent of R is central, there exists an idempotent $e \in R$ such that $I = eR = Re$. Then $J = I_0[x; \alpha, \delta] = eR_0[x; \alpha, \delta] = ex \circ R_0[x; \alpha, \delta] = r_S((1 - e)x)$, since $\alpha(e) = e$ and $\delta(e) = 0$. Therefore $S \in q\mathcal{B}_{r2}$.

(2) \Rightarrow (1). Let $I \in rAnn_R(id(R))$. Then $I_0[x; \alpha, \delta] \in rAnn_S(id(S))$, by Proposition 1.6. Hence $I_0[x; \alpha, \delta] = r_S((x)\varepsilon)$ for some idempotent $(x)\varepsilon \in S$, since $S \in q\mathcal{B}_{r2}$. By Lemma 1.5, $(x)\varepsilon = e_0 + e_1 x$ where e_1 is an idempotent of R with $e_1 e_0 = 0$. Since $-e_0 + (1 - e_1)x \in r_S((x)\varepsilon) = I_0[x; \alpha, \delta]$, we have $e_0 = 0$. On the other hand $r_S(e_1 x) = (1 - e_1)x \circ R_0[x; \alpha, \delta] = (1 - e_1)R_0[x; \alpha, \delta]$, hence $I = (1 - e_1)R$. Therefore R is quasi-Baer.

The equivalence of (1) and (3) follows from Hong et al. [15].

(4) \Rightarrow (1). Let I be an ideal of R . Assume that $I^{(\alpha,\delta)}$ be the (α,δ) -ideal of R generated by I . Hence $I_0^{(\alpha,\delta)}[x; \alpha, \delta]$, the 0-symmetric left nearring of skew polynomials with coefficients from $I^{(\alpha,\delta)}$, is an ideal of $R_0[x; \alpha, \delta]$. Since $R_0[x; \alpha, \delta] \in q\mathcal{B}_{r1}$, there exists an idempotent $(x)\varepsilon \in R_0[x; \alpha, \delta]$ such that $r_{S_0}(I_0^{(\alpha,\delta)}[x; \alpha, \delta]) = (x)\varepsilon \circ R_0[x; \alpha, \delta]$. By Lemma 1.5, $(x)\varepsilon = ex$ for some

idempotent $e \in R$. Hence $r_{S_0}(I_0^{(\alpha, \delta)}[x; \alpha, \delta]) = (x)\varepsilon \circ R_0[x; \alpha, \delta] = eR_0[x; \alpha, \delta]$, since $\alpha(e) = e$, $\delta(e) = 0$ and e is a central idempotent of R . Since $I \subseteq I^{(\alpha, \delta)}$, hence $rea x = ax \circ (ex \circ rx) = 0$ for each $a \in I$ and $r \in R$. Consequently $eRI = IeR = 0$, since R is reduced and e is a central idempotent of R . Hence $eR \subseteq r_R(I)$. Now, let $t \in r_R(I)$. Then $It = tI = 0$ and that $tI^{(\alpha, \delta)} = 0$, by Lemma 1.3. Hence $I_0^{(\alpha, \delta)}[x; \alpha, \delta] \circ tx = 0$. Thus $tx \in r_{S_0}(I_0^{(\alpha, \delta)}[x; \alpha, \delta]) = ex \circ R_0[x; \alpha, \delta]$. Therefore $tx = ex \circ tx = tex$ and that $t = et \in eR$. Consequently $r_R(I) = eR$. Therefore R is a quasi-Baer ring.

(1) \Rightarrow (4). Assume that R is a quasi-Baer ring. Let J be an ideal of S_0 . Assume that $J^{1(\alpha, \delta)}$ be the (α, δ) -ideal of R generated by the set of all coefficients of elements of J . Then $J_0^{1(\alpha, \delta)}[x; \alpha, \delta]$, the 0-symmetric left nearring of skew polynomials with coefficients from $J^{1(\alpha, \delta)}$, is an ideal of $R_0[x; \alpha, \delta]$. By using Lemma 1.3 and Proposition 1.4, one can show that $r_{S_0}(J) = r_{S_0}(J_0^{1(\alpha, \delta)}[x; \alpha, \delta])$. Since R is quasi-Baer, hence $\ell_R(J^{1(\alpha, \delta)}) = r_R(J^{1(\alpha, \delta)}) = eR$ for some idempotent $e \in R$. We show that $r_{S_0}(J) = ex \circ R_0[x; \alpha, \delta]$. Since $e \in r_R(J^{1(\alpha, \delta)})$, we have $ex \circ R_0[x; \alpha, \delta] \subseteq r_{S_0}(J)$. Now, let $(x)g = g_1x + \dots + g_mx^m \in r_{S_0}(J) = r_{S_0}(J_0^{1(\alpha, \delta)}[x; \alpha, \delta])$. Then $J^{1(\alpha, \delta)}g_i = g_iJ^{1(\alpha, \delta)} = 0$ for each $i = 1, \dots, m$, by Proposition 1.4. Therefore $g_i \in r_R(J^{1(\alpha, \delta)}) = eR$ and that $g_i = eg_i = g_ie$ for each $i = 1, \dots, m$. Hence $(x)g = ex \circ (x)g$, since $\alpha(e) = e$ and $\delta(e) = 0$. Consequently $r_{S_0}(J) = r_{S_0}(J_0^{1(\alpha, \delta)}[x; \alpha, \delta]) = ex \circ R_0[x; \alpha, \delta]$, which implies $R_0[x; \alpha, \delta] \in q\mathcal{B}_{r_1}$. \square

The following example shows that there exists a Baer ring R such that $S_0 = R_0[x; \alpha, \delta] \notin q\mathcal{B}_{r_1}$, where $\delta = 0$. So “ α -rigid condition on R ” in Theorem 1.7 is not superfluous.

Example 1.8. Let F be a field and consider the polynomial ring $R = F[y]$ over F . Then R is a commutative domain and so R is Baer. Let $\alpha : R \rightarrow R$ be an endomorphism defined by $\alpha(f(y)) = f(0)$. Then

(i) R is not α -rigid:

Since $y\alpha(y) = 0$ but $y \neq 0$.

(ii) The only idempotents of 0-symmetric left nearring $R_0[x; \alpha]$ are 0 and x :
Let $(x)\varepsilon = f_1(y)x + \dots + f_n(y)x^n$ be a nonzero idempotent of $R_0[x; \alpha]$. Then

$$\begin{aligned} &(x)\varepsilon \circ (x)\varepsilon \\ &= f_1(y)(f_1(y)x + \dots + f_n(y)x^n) + \dots + f_n(y)(f_1(y)x + \dots + f_n(y)x^n)^n \\ &= f_1(y)x + \dots + f_n(y)x^n. \end{aligned}$$

Hence $f_1(y)^2 = f_1(y)$ and that $f_1(y) = 0$ or $f_1(y) = 1$, since R is a domain. If $f_1(y) = 0$, then by a simple calculation we can show that $(x)\varepsilon = 0$, which is a contradiction. Hence $f_1(y) = 1$. Since $f_1(y)f_2(y) + f_2(y)f_1(y)\alpha(f_1(y)) = f_2(y)$ and $\alpha(f_1(y)) = 1$, hence $f_2(y) = 0$. Continuing this process, we have $f_i(y) = 0$ for each $i \geq 2$, which implies $(x)\varepsilon = x$.

(iii) $S_0 = R_0[x; \alpha] \notin q\mathcal{B}_{r_1}$:

Let $I = \{a_1(y)x + \cdots + a_n(y)x^n \in S_0 \mid n > 0, a_i(0) = 0 \text{ for } i = 1, \dots, n\}$. Then I is an ideal of S_0 . Assume to the contrary that $S_0 \in q\mathcal{B}_{r_1}$. Then there exists an idempotent $(x)\varepsilon = e_1(y)x + \cdots + e_n(y)x^n \in S_0$ such that $r_{S_0}(I) = (x)\varepsilon \circ R_0[x; \alpha]$. By (ii), $(x)\varepsilon = 0$ or $(x)\varepsilon = 1$. Since $x^2 \in r_{S_0}(I)$, so $(x)\varepsilon \neq 0$. If $(x)\varepsilon = 1$, then $r_{S_0}(I) = x \circ S_0 = S_0$ and so $x \in r_{S_0}(I)$, which is a contradiction.

Corollary 1.9 (Birkenmeier and Huang, [6]). *Let R be a reduced ring and $S = R[x]$. Then the following are equivalent:*

- (1) R is Baer;
- (2) $S \in q\mathcal{B}_{r_2}$;
- (3) $(R[x], +, \cdot)$ is quasi-Baer.

By Birkenmeier and Huang [6, Example 2.5], there is a Baer ring R such that the nerring $R[x] \notin q\mathcal{B}_{r_2}$. Hence “reduced condition on R ” in Corollary 1.9, is not superfluous. Here we give another example of commutative ring R such that $R[x; \alpha] \in q\mathcal{B}_{r_2}$ but R is not quasi-Baer. So “ α -rigid condition on R ” in Theorem 1.7 is not superfluous.

Example 1.10. Let \mathbb{Z} be the ring of integers and consider the ring $\mathbb{Z} \oplus \mathbb{Z}$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{2}\}$ of $\mathbb{Z} \oplus \mathbb{Z}$ is a commutative reduced ring. Note that the only idempotents of R are $(0, 0)$ and $(1, 1)$. One can show that $r_R((2, 0)) = \{(0, 2n) \mid n \in \mathbb{Z}\}$. So we can see that $r_R((2, 0))$ does not contain a nonzero idempotent of R , hence R is not quasi-Baer. Now, let $\alpha : R \rightarrow R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R . Let I be an ideal of nerring $S = R[x; \alpha]$. Let $(x)f = (a_0, b_0) + (a_1, b_1)x + \cdots + (a_n, b_n)x^n \in I$, with $(a_n, b_n) \neq (0, 0)$, $n \geq 1$ and $(x)g = (u_0, v_0) + (u_1, v_1)x + \cdots + (u_m, v_m)x^m \in r_S(I)$. Then $0 \circ (x)f = (a_0, b_0) \in I$ and that $(x)f_1 = (a_1, b_1)x + \cdots + (a_n, b_n)x^n \in I$. Since $0 \in I$ and $(x)g \in r_S(I)$, we have $0 \circ (x)g = (u_0, v_0) = (0, 0)$. Suppose that i be the smallest positive integer number such that $(a_i, b_i) \neq (0, 0)$. Then $(x)f_1 \circ (x)g = (u_1, v_1)((x)f_1) + \cdots + (u_n, v_n)((x)f_1)^n = 0$ and that $(u_1, v_1)(a_i, b_i) = (0, 0)$. Hence $(u_1, v_1)x^2 \circ (x)f_1 = (a_{i+1}, b_{i+1})(u_1^{2(i+1)}, v_1^{2(i+1)})x^4 + \cdots + (a_n, b_n)(u_1^{2n}, v_1^{2n})x^{2n} \in I$. Since $(x)g \in r_S(I)$, hence by Proposition 1.4 we have $(u_1, v_1)(a_{i+1}, b_{i+1})(u_1^{2(i+1)}, v_1^{2(i+1)}) = (0, 0)$. Thus $(a_{i+1}, b_{i+1})(u_1, v_1) = (0, 0)$. Similarly, we obtain $(a_k, b_k)(u_1, v_1) = (0, 0)$ for each $1 \leq k \leq n$. Continuing this process, we can show that

$$(\dagger) \quad (a_k, b_k)(u_j, v_j) = (0, 0) \text{ for each } 1 \leq k \leq n, 1 \leq j \leq m.$$

We claim that $r_S(I) = 0$.

(i) If $a_n \neq 0$ and $b_n \neq 0$, then from Eq.(\dagger) we have $(x)g = 0$.

(ii) If $a_n \neq 0$ and $b_n = 0$, then from Eq.(\dagger), $u_j = 0$ for each $j = 1, \dots, m$. Since $x^2 \circ (x)f = (a_0, b_0) + (a_1, b_1)x^2 + \cdots + (a_n, 0)x^{2n} \in I$, we have $(x^2 \circ (x)f + (0, a_n)x^{2n+1}) \circ x^2 - (0, a_n)x^{2n+1} \circ x^2 = (a_0, b_0)^2 + \cdots + (0, a_n^2)x^{4n+1} \in I$. Hence from Eq.(\dagger), $v_j = 0$ for each $j = 1, \dots, m$. Consequently, $(x)g = 0$.

(iii) If $a_n = 0$ and $b_n \neq 0$, then from Eq.(†), $v_j = 0$ for each $j = 1, \dots, m$. Since $x^2 \circ (x)f = (a_0, b_0) + (a_1, b_1)x^2 + \dots + (0, b_n)x^{2n} \in I$, we have $(x^2 \circ (x)f + (b_n, 0)x^{2n+1}) \circ x^2 - (b_n, 0)x^{2n+1} \circ x^2 = (a_0, b_0)^2 + \dots + (b_n^2, 0)x^{4n+1} \in I$. Hence from Eq.(†), $u_j = 0$ for each $j = 1, \dots, m$ and that $(x)g = 0$.

Consequently, $r_S(I) = 0 = r_S((1, 1)x)$. Therefore $S \in q\mathcal{B}_{r_2}$.

In the following we investigate the transfer of Rickart-type annihilator conditions between R and $R[x; \alpha, \delta]$.

Theorem 1.11. *Let R be an α -rigid ring. Then the following conditions are equivalent:*

- (1) R is Rickart;
- (2) $S = R[x; \alpha, \delta] \in \mathcal{R}_{r_2}$;
- (3) $S = R[x; \alpha, \delta] \in q\mathcal{R}_{r_2}$;
- (4) $(R[x; \alpha, \delta], +, \cdot)$ is Rickart.

Proof. The equivalence of (1) and (4) follows from [15, Corollary 15]. We first show the equivalence of (1) and (3). Assume R is Rickart. Let $(x)f = \sum_{i=0}^m f_i x^i \in S$ and I the ideal of S generated by $(x)f$. Since R is a reduced Rickart ring, there exists idempotents $e_i \in R, i = 0, \dots, m$ such that $\cap_{i=0}^m \ell_R(f_i) = \cap_{i=0}^m \ell_R(e_i) = \cap_{i=0}^m (1 - e_i)R$. Since every idempotent of R is central, hence $\cap_{i=0}^m (1 - e_i)R = (1 - e_1)(1 - e_2) \dots (1 - e_m)R = dR = Rd$ for some idempotent $d \in R$. Then $I \circ dx = 0$. Let $e = 1 - d$. Since $\alpha(e) = e$ and $\delta(e) = 0$, hence it is not difficult to show that $r_S(ex) = dx \circ R_0[x; \alpha, \delta]$. Hence $r_S(ex) \subseteq r_S(I)$. Let $(x)g = \sum_{j=0}^n g_j x^j \in r_S(I)$. Since $0 \in I$, hence $g_0 = 0$ and $f_0 \in I$. By Proposition 1.4, $g_j f_i = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that

$$\begin{aligned} (x)h &= x^2 \circ [(f_0 + x) \circ x^2 - x \circ x^2] = x^2 \circ [(f_0 + x)^2 - x^2] \\ &= x^2 \circ [f_0^2 + \delta(f_0) + (f_0 + \alpha(f_0))x] = f_0^2 + \delta(f_0) + (f_0 + \alpha(f_0))x^2 \in I \end{aligned}$$

and

$$\begin{aligned} (x)k &= (f_0 + x) \circ x^3 - x \circ x^3 = (f_0 + x)^3 - x^3 \\ &= (f_0^3 + f_0\delta(f_0) + \delta(f_0^2) + \delta^2(f_0)) \\ &\quad + (f_0^2 + f_0\alpha(f_0) + \alpha(f_0^2) + \alpha(\delta(f_0)) + \delta(f_0) + \delta(\alpha(f_0)))x \\ &\quad + (f_0 + \alpha(f_0) + \alpha^2(f_0))x^2 \in I. \end{aligned}$$

Then $(x)k - (x)h = (f_0^3 + f_0\delta(f_0) + \delta(f_0^2) + \delta^2(f_0) - f_0^2 - \delta(f_0)) + (f_0^2 + f_0\alpha(f_0) + \alpha(f_0^2) + \alpha(\delta(f_0)) + \delta(f_0) + \delta(\alpha(f_0)))x + \alpha^2(f_0)x^2 \in I$. Again, by Proposition 1.4, $g_j \alpha^2(f_0) = 0$ for all $1 \leq j \leq n$. Hence $g_j f_0 = 0$ for all $1 \leq j \leq n$, by Lemma 1.3. So each $g_j \in Rd$. Thus $(x)g \in dx \circ R_0[x; \alpha, \delta]$. Consequently, $r_S(I) = r_S(ex)$. Hence $R[x; \alpha, \delta] \in q\mathcal{R}_{r_2}$.

To prove that (3) implies (1), assume $S = R[x; \alpha, \delta] \in q\mathcal{R}_{r_2}$. Let $w \in R$ and A the (α, δ) -ideal of R generated by w . Then $A[x; \alpha, \delta]$ is a left nearring of skew polynomial with coefficients from A . Let $(x)a = \sum_{i=0}^m a_i x^i \in A[x; \alpha, \delta]$ and $(x)f, (x)g = \sum_{j=0}^n g_j x^j \in R[x; \alpha, \delta]$. Observe that $(x)f \circ$

$(x)a = \sum_{i=0}^m a_i((x)f)^i \in A[x; \alpha, \delta]$ and $((x)a + (x)f) \circ (x)g - (x)f \circ (x)g = \sum_{j=0}^n g_j((x)a + (x)f)^j - \sum_{j=0}^n g_j((x)f)^j = \sum_{j=0}^n g_j[((x)a + (x)f)^j - ((x)f)^j] \in A[x; \alpha, \delta]$, since coefficients of $[(x)a + (x)f)^j - ((x)f)^j$ belong to A for all j . Therefore $A[x; \alpha, \delta]$ is an ideal of $R[x; \alpha, \delta]$. It is not difficult to show that $A[x; \alpha, \delta]$ is the ideal of $R[x; \alpha, \delta]$ generated by wx . Since $S = R[x; \alpha, \delta] \in q\mathcal{R}_{r_2}$, there exists an idempotent $(x)\varepsilon \in R[x; \alpha, \delta]$ such that $r_S(A[x; \alpha, \delta]) = r_S((x)\varepsilon)$. Note that $(x)\varepsilon = e_0 + e_1x$ for some $e_0, e_1 \in R$ where $e_1^2 = e_1$ and $e_1e_0 = 0$, by Lemma 1.5. Since $0 \in A[x; \alpha, \delta]$, we have $r_S(A[x; \alpha, \delta]) \subseteq R_0[x; \alpha, \delta]$. Since $-e_0 + (1 - e_1)x \in r_S(e_0 + e_1x) = r_S(A[x; \alpha, \delta]) \subseteq R_0[x; \alpha, \delta]$, we see that e_0 must be 0 and so $(x)\varepsilon = e_1x$. We will show that the left annihilator $\ell_R(A) = \ell_R(e_1) = R(1 - e_1)$. Let $a \in A$ and $t \in R$ be arbitrary. Since $t(1 - e_1)x \in r_R((x)\varepsilon) = r_S(A[x; \alpha, \delta])$, we have $0 = ax \circ t(1 - e_1)x = t(1 - e_1)ax$. Therefore $t(1 - e_1)a = 0$ and so $R(1 - e_1) \subseteq \ell_R(A)$. On the other hand, if $t \in \ell_R(A)$, then $ta = 0$ for all $a \in A$. Hence $tx \in r_S(A[x; \alpha, \delta]) = r_S((x)\varepsilon)$ and that $0 = e_1x \circ tx = te_1x$. Therefore $t \in \ell_R(e_1) = R(1 - e_1)$. Consequently, $\ell_R(A) = R(1 - e_1)$. Therefore R is Rickart.

We now show the equivalence of (1) and (2). Assume R is Rickart. Let $(x)f = \sum_{i=0}^m f_i x^i \in S$. Since R is reduced Rickart, there exists idempotents $e_i \in R, i = 0, \dots, m$ such that $\cap_{i=0}^m \ell_R(f_i) = \cap_{i=0}^m \ell_R(e_i) = \cap_{i=0}^m (1 - e_i)R$. Since every idempotent of R is central, hence $\cap_{i=0}^m (1 - e_i)R = (1 - e_1)(1 - e_2) \cdots (1 - e_m)R = (1 - e)R = \ell_R(e)$ for some idempotent $e \in R$. Let $(x)\varepsilon = e_0 + ex$ where $e_0 = -ef_0 + f_0$. Clearly $(x)\varepsilon$ is an idempotent in S . We will show that $r_S((x)f) = r_S((x)\varepsilon)$. Let $(x)g = \sum_{j=0}^n g_j x^j \in r_S((x)f)$. Then $g_j f_i = 0$ for all $1 \leq j \leq n, 1 \leq i \leq m$ and $g_0 + g_1 f_0 + \cdots + g_n f_0^n = 0$, by Proposition 1.4. Since $ee_0 = e\alpha(e_0) = e\delta(e_0) = 0$, we have $((x)\varepsilon)^k = e_0^k + ex^k$ for all $k \geq 1$. Hence $(x)\varepsilon \circ (x)g = \sum_{j=0}^n g_j ((x)\varepsilon)^j = g_0 + \sum_{j=1}^n g_j (e_0^j + ex^j) = g_0 + \sum_{j=1}^n g_j e_0^j + \sum_{j=1}^n g_j ex^j = g_0 + g_1 f_0 + \cdots + g_n f_0^n + 0 = 0$. Thus $(x)g \in r_S((x)\varepsilon)$. Therefore, $r_S((x)f) \subseteq r_S((x)\varepsilon)$. Now, assume $(x)g \in r_S((x)\varepsilon)$. Then $g_i \in \ell_R(e) = \cap_{i=1}^m \ell_R(f_i)$ and $g_0 + g_1 e_0 + \cdots + g_n e_0^n = 0$, by Proposition 1.4. Hence $g_0 + g_1 f_0 + \cdots + g_n f_0^n = 0$ and that $(x)g \in r_S((x)f)$, by Proposition 1.4. Therefore $r_S((x)\varepsilon) \subseteq r_S((x)f)$. Consequently $R[x; \alpha, \delta] \in \mathcal{R}_{r_2}$.

Conversely, assume $R[x; \alpha, \delta] \in \mathcal{R}_{r_2}$. Let $a \in R$. Then $r_S(ax) = r_S((x)\varepsilon)$ for some idempotent $(x)\varepsilon \in S$. By Lemma 1.5, $(x)\varepsilon = e_0 + e_1x$ where $e_1^2 = e_1$ and $e_1e_0 = 0$. Since $-e_0 + (1 - e_1)x \in r_S((x)\varepsilon) = r_S(ax)$, we have $e_0 = 0$. We show that $\ell_R(a) = \ell_R(e_1)$. Let $r \in \ell_R(a)$. Then $0 = rax = ax \circ rx$, which implies $rx \in r_S(ax) = r_S((x)\varepsilon)$. Therefore $0 = (x)\varepsilon \circ rx = e_1x \circ rx = rex$. Hence $r \in \ell_R(e_1)$, which implies $\ell_R(a) \subseteq \ell_R(e_1)$. Now, let $b \in \ell_R(e_1)$. Then $0 = be_1x = e_1x \circ bx$ and so $bx \in r_S(e_1x) = r_S(ax)$. Hence $0 = ax \circ bx = bax$ and that $ba = 0$. Consequently, $\ell_R(e_1) \subseteq \ell_R(a)$ and that $\ell_R(a) = \ell_R(e_1)$. Therefore R is Rickart. \square

We now study quasi-Baer annihilator conditions on formal skew power series over α -rigid ring. In the sequel, $R_0[[x; \alpha]]$ denotes the 0-symmetric nearring of formal skew power series $(R_0[[x; \alpha]], +, \circ)$ with positive orders.

The following is the key lemma for nearrings of skew power series over α -rigid rings satisfying Baer-type annihilator conditions.

Lemma 1.12. *Let R be an α -rigid ring and $(x)f = \sum_{i=1}^{\infty} a_i x^i$, $(x)g = \sum_{i=1}^{\infty} b_i x^i \in R_0[[x; \alpha]]$. Then $(x)f \circ (x)g = 0$ if and only if $b_i a_j = 0$ for all $i, j \geq 1$.*

Proof. Since $(x)f \circ (x)g = 0$, we have

$$(1) \quad b_1(x)f + b_2((x)f)^2 + b_3((x)f)^3 + \dots = 0.$$

Then $b_1 a_1 = 0$, since it is the coefficient of x . Hence $a_1 b_1 = 0$, since R is reduced. By multiplying a_1 to Eq.(1) from the left-hand side, we obtain

$$(2) \quad a_1 b_2((x)f)^2 + a_1 b_3((x)f)^3 + \dots = 0.$$

Hence $a_1 b_2 a_1 \alpha(a_1) = 0$, since it is the coefficient of x^2 in Eq.(2). Therefore $a_1 b_2 = 0$, by Lemma 1.3. Inductively, we have $a_1 b_i = b_i a_1 = 0$ for all $i \geq 1$. Hence Eq.(1) becomes $(\sum_{i=2}^{\infty} a_i x^i) \circ (\sum_{i=1}^{\infty} b_i x^i) = 0$, since R has the IFP. Continuing this process, we can prove $a_j b_i = 0$ for all $i, j \geq 1$.

Since R has the IFP, the converse follows from Lemma 1.3. □

Lemma 1.13. *Let R be a ring and α be an endomorphism of R . If $(x)\varepsilon = \sum_{i=1}^{\infty} e_i x^i \in R_0[[x; \alpha]]$ is an idempotent, then $e_1^2 = e_1$. If R is α -rigid, then $(x)\varepsilon = e_1 x$.*

Proof. Clearly, $e_1^2 = e_1$. Since $(x)\varepsilon \circ (x)\varepsilon = (x)\varepsilon$, we have

$$(1) \quad e_1(x)\varepsilon + e_2((x)\varepsilon)^2 + \dots = (x)\varepsilon.$$

Multiplying e_1 to Eq. (1) from the left-hand side, we obtain

$$e_1 e_1(x)\varepsilon + e_1 e_2((x)\varepsilon)^2 + \dots = e_1(x)\varepsilon.$$

Hence $(x)\varepsilon \circ (e_1(x)\varepsilon - e_1 x) = (x)\varepsilon \circ (e_1 e_2 x^2 + e_1 e_3 x^3 + \dots) = 0$. Thus $e_1 e_i = e_i e_1 = 0$ for all $i \geq 2$, by Lemma 1.12. Again, multiplying e_2 to Eq.(1) from the left-hand side, we have $(x)\varepsilon \circ (e_2(x)\varepsilon - e_2 x) = (x)\varepsilon \circ (-e_2 x + e_2 e_2 x^2 + \dots) = 0$. Hence $e_2^2 = 0$, by Lemma 1.12 and that $e_2 = 0$. Continuing this process, we can prove $e_i = 0$ for each $i \geq 2$. □

By using Lemma 1.12 and a similar way as in the proof of Proposition 1.6, one can prove the following result.

Proposition 1.14. *Let R be an α -rigid ring and $T = R_0[[x; \alpha]]$. Then*

$$\psi : rAnn_R(id(R)) \rightarrow rAnn_T(id(T)); I \rightarrow I_0[[x; \alpha]]$$

is bijective, where $I_0[[x; \alpha]]$ is the set of all elements of T which coefficients belong to I .

Now we can prove the following result.

Theorem 1.15. *Let R be an α -rigid ring and $T = R_0[[x; \alpha]]$. Then the following are equivalent:*

- (1) R is quasi-Baer;
- (2) $(R[[x; \alpha]], +, \cdot)$ is quasi-Baer;
- (3) $T \in q\mathcal{B}_{r1}$;
- (4) $T \in q\mathcal{B}_{r2}$.

Proof. The equivalence of (1) and (2) follows from [15, Theorem 21]. The equivalence of (1) and (3) follows from Lemmas 1.3, 1.13 and Proposition 1.14.

The equivalence of (3) and (4) follows from the fact that $r_T(ex) = (1 - e) \circ T$ for each idempotent $e \in R$. \square

Example 1.8, also shows that there exists a Baer ring R such that $T = R_0[[x; \alpha]] \notin q\mathcal{B}_{r1}$. In fact, if $I = \{a_1(y)x + a_2(y)x^2 + \cdots \in T \mid a_i(0) = 0 \text{ for each } i\}$, then I is an ideal of T and by a similar way as used in the proof of Example 1.8, we can show that right annihilator $r_T(I)$ do not generated by any idempotent. So “ α -rigid condition on R ” in Theorem 1.15 is not superfluous.

Corollary 1.16. *Let R be a reduced ring and $T = R_0[[x]]$. Then the following are equivalent:*

- (1) R is quasi-Baer;
- (2) $(R[[x]], +, \cdot)$ is quasi-Baer;
- (3) $T \in q\mathcal{B}_{r1}$;
- (4) $T \in q\mathcal{B}_{r2}$.

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