

ENUMERATION OF WEIGHTED COMPLETE GRAPHS

HYEONG-KWAN JU

ABSTRACT. We enumerate the number of weighted complete graphs and compute its generating function.

1. Introduction

For a given nonnegative integer n and a simple graph $G = (V, E)$ (no loops or no multi-edges allowed) with $V = (v_1, v_2, \dots, v_k)$, we give an $\alpha = (n_1, n_2, \dots, n_k) \in [n]^k$ in such a way that

$$(0) \quad n_i + n_j \leq n \quad \text{if } v_i v_j \in E,$$

where $[n] = \{0, 1, 2, \dots, n\}$. Then we call such a triplet $WG_\alpha = (V, E, \alpha)$ a *weighted graph* of G with a distribution α . Let $WG(n)$ be the number of all weighted graphs of G with a fixed upper bound n . In other words, $WG(n)$ is the number of all WG_α such that α satisfies the condition (0).

The cases that G is a linear graph (that is, a tree with k vertices having two vertices of degree 1) and G is a circular graph (that is, a connected graph with k vertices of degree 2) were studied by Bona and Ju [3].

In this paper, we enumerate $WG(n)$ for the complete graph $G = K_t$ of order t , its generating function $U_t(x) = \sum_{n=0}^{\infty} WK_t(n)x^n$ and

$$F(x, y) = \sum_{t=0}^{\infty} U_t(x)y^t/t!.$$

Maple was used for all computations in this paper.

2. Reduction of the generating functions

The number $WK_t(n)$ of weighted complete graphs for a given complete graph K_t of order t is the same as the number of solutions to the following system of homogeneous linear inequalities:

$$(1) \quad n_i + n_j \leq n \quad \text{for } 1 \leq i, j \leq t$$

by the equivalent condition (0).

Received March 21, 2006; Revised October 17, 2006.

2000 *Mathematics Subject Classification.* 05A15.

Key words and phrases. weighted graph, generating function.

This research was financially supported by Chonnam National University.

Case 1 : when n is even (i.e., $n = 2m$).

If $r = \max\{n_1, n_2, \dots, n_t\} (= n_1, \text{ say})$ is greater than m , then rest of them can be any integer in $[n - r] := \{0, 1, \dots, n - r\}$. So the number of solutions of the system (1) in this case is $(n + 1 - r)^{t-1}$. We can switch n_1 with any other n_i , one of the rests of them. Therefore, the number of solutions of the system (1) when the maximum number in n_i 's is r is $t(n + 1 - r)^{t-1}$.

If $r = \max\{n_1, n_2, \dots, n_t\}$ is at most m , then n_i can be an arbitrary number in $[m]$. The number of solutions of the system (1) in this case is $(m + 1)^t = (\frac{n}{2} + 1)^t$. The total number $\omega e_t(n)$ of solutions in the case 1 is :

$$(2e) \quad \omega e_t(n) = t \sum_{r=1}^{\frac{n}{2}} r^{t-1} + (\frac{n}{2} + 1)^t.$$

Case 2 : when n is odd (i.e., $n = 2m + 1$).

In the same way as in the case 1, if $r = \max\{n_1, n_2, \dots, n_t\} (= n_1, \text{ say})$ is greater than m , then rest of them can be any integer in $[n - r]$. So the number of solutions of the system (1) in this case is $(n + 1 - r)^{t-1}$. We can switch n_1 with any other n_i , one of the rest of them. Therefore, the number of solutions of the system (1) when the maximum number in n_i 's is r is $t(n + 1 - r)^{t-1}$.

If $r = \max\{n_1, n_2, \dots, n_t\}$ is at most m , then n_i can be arbitrary in $[m]$. The number of solutions of the system (1) in this case $(m + 1)^t = (\frac{n+1}{2})^t$. The total number $\omega o_t(n)$ of solutions in the case 2 is :

$$(2o) \quad \omega o_t(n) = t \sum_{r=1}^{\frac{n+1}{2}} r^{t-1} + (\frac{n+1}{2})^t.$$

Hence, by the equations(2e) and (2o),

$$(2) \quad WK_t(n) = t \sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor} r^{t-1} + (\lfloor \frac{n+2}{2} \rfloor)^t.$$

Let $W(n, y) = \sum_{t=0}^{\infty} WK_t(n) \frac{y^t}{t!}$. Then

$$(3e) \quad \begin{aligned} W(2m, y) &= \sum_{t=0}^{\infty} \omega e_t(2m) \frac{y^t}{t!} \\ &= \sum_{t=0}^{\infty} (t \sum_{r=1}^m r^{t-1} + (m + 1)^t) \frac{y^t}{t!} \quad (\text{from the formula (2e)}) \\ &= y \sum_{t=1}^{\infty} \sum_{r=1}^m r^{t-1} \frac{y^{(t-1)}}{(t-1)!} + \sum_{t=0}^{\infty} (m + 1)^t \frac{y^t}{t!} \end{aligned}$$

$$\begin{aligned}
 (3e) \quad &= y \sum_{r=1}^m \exp(ry) + \exp((m+1)y) \\
 &= y \frac{\exp((m+1)y) - 1}{\exp(y) - 1} + \exp((m+1)y) - y.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (3o) \quad W(2m+1, y) &= \sum_{t=0}^{\infty} \omega_{ot}(2m+1) \frac{y^t}{t!} \\
 &= y \frac{\exp((m+2)y) - 1}{\exp(y) - 1} + \exp((m+1)y) - y.
 \end{aligned}$$

Let $F(x, y)$ be the generating function for the double sequence $\{WK_t(n)\}_{n,t=0}^{\infty}$, ordinary in n and exponential in t . Then

$$\begin{aligned}
 F(x, y) &= \sum_{n=0}^{\infty} \sum_{t=0}^{\infty} WK_t(n) \frac{y^t}{t!} x^n = \sum_{n=0}^{\infty} W(n, y) x^n \\
 &= \sum_{m=0}^{\infty} (W(2m, y) + W(2m+1, y)) x^{2m} \\
 &= \left[\frac{ye^y(1+xe^y)}{e^y-1} + (1+x)e^y \right] \frac{1}{1-x^2e^y} - \frac{(1+x)ye^y}{(1-x^2)(e^y-1)}
 \end{aligned}$$

by using (3e) and (3o).

Theorem 1. *The generating function of the double sequence*

$$\{WK_t(n)\}_{n,t=0}^{\infty}$$

enumerating the solutions of the system (1) is as follows:

$$\begin{aligned}
 (4) \quad F(x, y) &= \sum_{t=0}^{\infty} \sum_{n=0}^{\infty} WK_t(n) x^n \frac{y^t}{t!} \\
 &= \left[\frac{ye^y(1+xe^y)}{e^y-1} + (1+x)e^y \right] \frac{1}{1-x^2e^y} - \frac{(1+x)ye^y}{(1-x^2)(e^y-1)}.
 \end{aligned}$$

In order to change the formula (4) into simpler form, let

$$f(y) = \frac{1}{e^y - 1} \quad \text{and} \quad g(x, y) = \frac{1}{1 - x^2 e^y}.$$

Then

$$\begin{aligned}
 F(x, y) &= \frac{y(e^y - 1 + 1)}{e^y - 1} \frac{1}{1 - x^2 e^y} + \frac{y(e^y - 1 + 1)}{e^y - 1} \frac{1 - (1 - x^2 e^y)}{x(1 - x^2 e^y)} \\
 &\quad + \frac{(1+x)(1 - (1 - x^2 e^y))}{x^2(1 - x^2 e^y)} + \frac{y}{1-x} \frac{(e^y - 1) + 1}{e^y - 1}
 \end{aligned}$$

$$\begin{aligned}
 &= y(1+f)g + y(1+f)(g-1)\frac{1}{x} + \frac{1+x}{x^2}(g-1) + \frac{y}{1-x}(1+f) \\
 (5) \quad &= \frac{y+xy}{x}fg - \left(\frac{y}{x} + \frac{y}{1-x}\right)f \\
 &\quad + \left(\frac{y+xy}{x} + \frac{1+x}{x^2}\right)g - \left(\frac{y}{x} + \frac{1+x}{x^2} + \frac{y}{1-x}\right).
 \end{aligned}$$

Here, note that

$$(6) \quad \frac{1}{f} + \frac{1}{g} = (e^y - 1) + (1 - x^2e^y) = (1 - x^2)e^y.$$

From the equation(6),

$$fg = \frac{f+g}{(1-x^2)e^y}.$$

Since

$$\frac{f}{e^y} = \frac{1}{e^y(e^y - 1)} = f - \frac{1}{e^y} \quad \text{and} \quad \frac{g}{e^y} = \frac{1}{e^y(1 - x^2e^y)} = x^2g + \frac{1}{e^y},$$

we have

$$(7) \quad fg = \frac{1}{1-x^2}(f+x^2g).$$

Substituting the equation (7) into the last formula of the equation (5), finally we have a next formula of nicer and simpler form given in the Theorem 2 below:

Theorem 2. *The generating function of the double sequence*

$$\{WK_t(n)\}_{n,t=0}^\infty$$

enumerating the solutions of the system (1) is as follows:

$$(8) \quad F(x, y) = \sum_{t=0}^\infty \sum_{n=0}^\infty WK_t(n)x^n \frac{y^t}{t!} = \left(1+x + \frac{xy}{1-x}\right) \frac{e^y}{1-x^2e^y}.$$

We define

$$\alpha(y; x) = 1+x + \frac{xy}{1-x}, \quad \beta(y; x) = \frac{e^y}{1-x^2e^y}.$$

Let $p = p_1p_2 \cdots p_t$ be a permutation. We say that i is a descent of p if $p_i > p_{i+1}$. Let $A(t, k)$ be the number of t -permutation with k descents. The numbers $A(t, k)$ are called the *Eulerian numbers*, and $A_t(x) = \sum_{k=0}^{t-1} A(t, k)x^k$ is called the *Eulerian polynomial*. We list below several known facts about Eulerian numbers and Eulerian polynomials. (See Bona [2] for details about Eulerian numbers)

Facts

- (1) $A(t, k+1) = (k+2)A(t-1, k+1) + (t-k-1)A(t-1, k)$.
- (2) $A(t, k) = A(t, t-k-1)$ and $\sum_{k=0}^{t-1} A(t, k) = k!$.
- (3) $x^t = \sum_{k=0}^{t-1} A(t, k)C(x+t-k-1, t)$, where $C(t, k) = \frac{t!}{k!(t-k)!}$.

- (4) $g(x, y) = \sum_{t=0}^{\infty} \sum_{k=0}^{t-1} A(t, k)x^k \frac{y^t}{t!} = \sum_{t=0}^{\infty} A_t(x) \frac{y^t}{t!} = \frac{1-x}{x(1-xe^{y(1-x)})}$.
- (5) $\frac{A_t(x)}{(1-x)^{t+1}} = \frac{d}{dx} \left\{ \frac{x A_{t-1}(x)}{(1-x)^t} \right\}$.
- (6) $A_t(x) = (1 + (t-1)x)A_{t-1}(x) + x(1-x)A'_{t-1}(x)$.
- (7) $A_t(x) = (1-x)^t + \sum_{k=1}^t C(t, k)(1-x)^{t-k} A_k(x)$.

Using the Facts above we can show that the following holds:

$$\begin{aligned} \beta(y; x) &= e^y \frac{1}{1-x^2 e^y} = \left(\sum_{t=0}^{\infty} \frac{y^t}{t!} \right) \left(\frac{1}{1-x^2} + \sum_{t=1}^{\infty} \frac{x^2 A_t(x^2)}{(1-x^2)^{t+1}} \frac{y^t}{t!} \right) \\ &= \frac{1}{1-x^2} + \sum_{t=1}^{\infty} \frac{A_t(x^2)}{(1-x^2)^{t+1}} \frac{y^t}{t!}, \end{aligned}$$

and

$$(9) \quad F(x, y) = \left(1 + x + \frac{xy}{1-x} \right) \beta(y; x) = \sum_{t=0}^{\infty} \frac{A_t(x^2) + tx A_{t-1}(x^2)}{(1-x)(1-x^2)^t} \frac{y^t}{t!}.$$

The numerator of the summand in the equation (9) is

$$A_t(x^2) + tx A_{t-1}(x^2) = (1+x) \{ (1 + (t-1)x)A_{t-1}(x^2) + x^2(1-x)A'_{t-1}(x^2) \}.$$

Let

$$r_t(x) = (1 + (t-1)x)A_{t-1}(x^2) + x^2(1-x)A'_{t-1}(x^2),$$

for $t = 2, 3, \dots$. By the Fact 2 (symmetric condition),

$$r_t(-1) = \sum_{k=0}^{t-2} (2-t+2k)A(t-1, k) = 0,$$

for $t = 2, 3, \dots$. This implies that

$$A_t(x^2) + tx A_{t-1}(x^2) = (1+x)^2 p_{2t-4}(x)$$

for some polynomial $p_{2t-4}(x)$ in x of degree $2t-4$.

Theorem 3. For $U_t(x) = \sum_{n=0}^{\infty} WK_t(n)x^n$,

$$F(x, y) = \sum_{t=0}^{\infty} U_t(x) \frac{y^t}{t!} = \frac{1}{1-x} + \frac{y}{(1-x)^2} + \sum_{t=2}^{\infty} \frac{p_{2t-4}(x)}{(1-x)^3(1-x^2)^{t-2}} \frac{y^t}{t!},$$

where

$$p_{2t-4}(x) = \frac{A_t(x^2) + tx A_{t-1}(x^2)}{(1+x)^2}$$

is a polynomial of degree $2t-4$ for $t = 2, 3, 4, \dots$.

We denote $\beta^{(t)}$ the t th partial derivative of β with respect to the variable y .

Theorem 4. Let $S(t, r)$ be the Stirling number of the second kind. Then for $t = 0, 1, 2, \dots$,

$$(10) \quad \beta^{(t)} = \sum_{r=1}^{t+1} S(t+1, r)(r-1)! \beta^r x^{2(r-1)}.$$

Proof. Use an induction on t . $\beta(y; x)^{(0)} = \beta(y; x) = S(1, 1)0! \beta(y; x)^1 x^{2(1-1)}$, and (for reference, we write the next one too)

$$(11) \quad \begin{aligned} \beta^{(1)}(y; x) &= \frac{\partial \beta(y; x)}{\partial y} = \beta(y; x) - e^y (1 - x^2 e^y)^{-2} (-x^2) e^y \\ &= \beta(y; x) + e^{2y} (1 - x^2 e^y)^{-2} x^2 \\ &= \beta(y; x) + x^2 \beta(y; x)^2 \\ &= S(2, 1)0! \beta + S(2, 2)1! \beta^2 x^2 \\ &\quad (\text{note that } S(2, 1) = S(2, 2) = 1). \end{aligned}$$

This says that the formula (10) is satisfied for $t = 0$ and 1 .

Suppose that this formula is satisfied for $t \leq k$. Then

$$\begin{aligned} \beta^{(k+1)} &= \frac{\partial}{\partial y} \sum_{r=1}^{k+1} S(k+1, r)(r-1)! \beta^r x^{2(r-1)} \\ &= \sum_{r=1}^{k+1} S(k+1, r)(r-1)! r \beta^{r-1} \beta^{(1)} x^{2(r-1)} \\ &= \sum_{r=1}^{k+1} S(k+1, r)(r-1)! (r \beta^r x^{2(r-1)} + r \beta^{r+1} x^{2r}) \text{ from (11)} \\ &= \beta + \sum_{r=2}^{k+1} r S(k+1, r)(r-1)! \beta^r x^{2(r-1)} \\ &\quad + \sum_{r=2}^{k+1} S(k+1, r)(r-1)! \beta^r x^{2(r-1)} + (k+1)! \beta^{k+2} x^{2(k+1)} \\ &= \beta + \sum_{r=2}^{k+1} [r S(k+1, r) + S(k+1, r-1)] (r-1)! \beta^r x^{2(r-1)} \\ &\quad + (k+1)! \beta^{k+2} x^{2(k+1)} \\ &= \sum_{r=1}^{k+2} S(k+2, r)(r-1)! \beta^r x^{2(r-1)}. \end{aligned}$$

(In the last equality we used the recurrence formula for the sequence on the Stirling number of the second kind, that is, $rS(k+1, r) + S(k+1, r-1) = S(k+2, r)$. See page 92 (Theorem 5.2) in Bona [1], or Graham et. al. [5] for this formula). \square

Note that we can get the same conclusion of Theorem 3 by manipulating certain partial differential equation and formal partial differentiation rather than an induction.

$$\begin{aligned}
 U_t(x) &= \frac{\partial^t F(x, y)}{\partial y^t} \Big|_{y=0} \\
 &= \frac{\partial^t}{\partial y^t} (\alpha(y; x)\beta(y; x)) \Big|_{y=0} \\
 &= \sum_{r=0}^t \binom{t}{r} \alpha^{(r)}(y; x)\beta^{(t-r)}(y; x) \Big|_{y=0} \\
 &= \alpha(0; x)\beta^{(t)}(0; x) + t\alpha^{(1)}(0; x)\beta^{(t-1)}(0; x) \\
 &= (1+x) \sum_{r=1}^{t+1} S(t+1, r)(r-1)! \beta(0; x)^r x^{2(r-1)} \\
 &\quad + \frac{tx}{1-x} \sum_{r=1}^t S(t, r)(r-1)! \beta(0; x)^r x^{2(r-1)} \\
 &\text{by the formula (10) in Theorem 4,} \\
 &= (1+x) \sum_{r=1}^{t+1} S(t+1, r)(r-1)! \frac{x^{2(r-1)}}{(1-x^2)^r} \\
 &\quad + \frac{tx(1+x)}{1-x^2} \sum_{r=1}^t S(t, r)(r-1)! \frac{x^{2(r-1)}}{(1-x^2)^r} \quad (\text{since } \beta(0; x) = \frac{1}{1-x^2}) \\
 &= \frac{1+x}{(1-x^2)^{t+1}} \left[\sum_{r=1}^{t+1} S(t+1, r)(r-1)! x^{2(r-1)} (1-x^2)^{t+1-r} \right. \\
 &\quad \left. + tx \sum_{r=1}^t S(t, r)(r-1)! x^{2(r-1)} (1-x^2)^{t-r} \right] \\
 &= \frac{1+x}{(1-x^2)^{t+1}} \left[\sum_{r=1}^{t+1} \{rS(t, r) + S(t, r-1)\} (r-1)! x^{2(r-1)} (1-x^2)^{t+1-r} \right. \\
 &\quad \left. + tx \sum_{r=1}^t S(t, r)(r-1)! x^{2(r-1)} (1-x^2)^{t-r} \right] \\
 &= \frac{1+x}{(1-x^2)^{t+1}} \sum_{r=1}^t S(t, r)(r-1)! (r+tx) x^{2(r-1)} (1-x^2)^{t-r}.
 \end{aligned}$$

Theorem 5. *An ordinary generating function*

$$U_t(x) = \sum_{n=0}^{\infty} WK_t(n)x^n$$

for the sequence $\{WK_t(n)\}_{n=0}^\infty$ is of the form

$$(12) \quad U_t(x) = \frac{1+x}{(1-x^2)^{t+1}} \sum_{r=1}^t S(t,r)(r-1)!(r+tx)x^{2(r-1)}(1-x^2)^{t-r}.$$

Theorem 6. Let $G(x, y) = F(x, y) - \frac{1+x+y}{(1-x)^2} = F(x, y) - U_0(x) - U_1(x)y$, where $U_0(x) = \frac{1}{1-x}$, $U_1(x) = \frac{1}{(1-x)^2}$, then $G(x, y)$ satisfies the following functional equation:

$$G(x, -y) = -G\left(\frac{1}{x}, y\right) \frac{1}{x^3}.$$

Proof. Computation from $F(x, y) = (1+x + \frac{xy}{1-x}) \frac{e^y}{1-x^2e^y}$ gives the desired result. □

Note that $G(x, y) = \sum_{t=2}^\infty U_t(x) \frac{y^t}{t!}$. From the Theorem 3 and Theorem 5, we see that each $U_t(x)$ is a rational function in x . Now let $U_t(x) = \frac{p_\lambda(x)}{q_\mu(x)}$, where $p_\lambda(x)$ and $q_\mu(x)$ are reduced monic polynomials with degree λ and μ , respectively. Note that $F(0, y) = e^y$, $U_t(0) = 1$ and $\lambda < \mu$. For $t \geq 2$, fixed, by Theorem 6,

$$\frac{p_\lambda(x)}{q_\mu(x)} = U_t(x) = (-1)^{t+1} U_t\left(\frac{1}{x}\right) \frac{1}{x^3} = (-1)^{t+1} \frac{p_\lambda\left(\frac{1}{x}\right)}{q_\mu\left(\frac{1}{x}\right) x^3}.$$

Now, consider a rational polytope

$$P = \{(x_1, x_2, \dots, x_t) \in \mathbf{R}^t \mid x_i \geq 0 \text{ and } x_i + x_j \leq 1 \text{ for all } i, j = 1, 2, \dots, t \text{ with } i \neq j\}$$

(Refer to pages 235–238 of Stanley [7] for a full description of details about the rational polytope.) Let $V(P)$ be the set of all vertices in the polytope P contained in $(\mathbf{Q} \cap [0, 1])^t$, and let $\text{den}(\alpha)$ for $\alpha \in V(P)$ be the least common denominator of the rational entries of α with reduced expression. It turned out (see the Lemma 3.5 of Bona, Ju and Yoshida [4]) that the set

$$\{\text{den}(\alpha) \mid \alpha \in V(P)\} = \{1, 2\}.$$

This implies that $q_\mu(x) = (1-x)^{t+1}(1+x)^{\mu-t-1}$. (Refer also to 4.6.25 Theorem on page 237 in Stanley [7].) Hence they have to satisfy the next relations;

$$p_\lambda(x) = x^\lambda p_\lambda\left(\frac{1}{x}\right) \text{ and } q_\mu(x) = (-1)^{t+1} q_\mu\left(\frac{1}{x}\right) x^{\lambda+3}.$$

This implies that $\mu = \lambda + 3$, and

$$U_t(x) = \frac{p_\lambda(x)}{(1-x)^{t+1}(1+x)^{\lambda+2-t}}.$$

From Theorem 3 it must be that $\lambda = 2t - 4$ and

$$U_t(x) = \frac{p_{2t-4}(x)}{q_{2t-1}(x)} = \frac{p_{2t-4}(x)}{(1-x)^3(1-x^2)^{t-2}},$$

where $p_{2t-4}(x)$ is a monic symmetric polynomial of degree $2t-4$ with $p_{2t-4}(-1) \neq 0$. From the formula (12) and Theorem 3

$$\begin{aligned}
 p_{2t-4}(x) &= \frac{1}{(1+x)^2} \sum_{r=1}^t S(t,r)(r-1)!(r+tx)x^{2(r-1)}(1-x^2)^{t-r} \\
 &= \frac{A_t(x^2) + txA_{t-1}(x^2)}{(1+x)^2},
 \end{aligned}$$

as will be provided in the next section (the corresponding numerator of $U_t(x)$ for $t = 1, 2, \dots$).

3. Computational results using maple

Computations of $U_t(x)$ for $t = 0, 1, 2, \dots, 8$ are as follows:

$$\begin{aligned}
 U_0(x) &= \frac{1}{1-x}, & U_1(x) &= \frac{1}{(1-x)^2}, \\
 U_2(x) &= \frac{1}{(1-x)^3}, & U_3(x) &= \frac{1+x+x^2}{(1-x)^3(1-x^2)}, \\
 U_4(x) &= \frac{1+2x+6x^2+2x^3+x^4}{(1-x)^3(1-x^2)^2}, \\
 U_5(x) &= \frac{1+3x+19x^2+14x^3+19x^4+3x^5+x^6}{(1-x)^3(1-x^2)^3}, \\
 U_6(x) &= \frac{1+4x+48x^2+56x^3+142x^4+56x^5+48x^6+4x^7+x^8}{(1-x)^3(1-x^2)^4}, \\
 U_7(x) &= \frac{1+5x+109x^2+176x^3+730x^4+478x^5+730x^6}{(1-x)^3(1-x^2)^5} \\
 &\quad + \frac{176x^7+109x^8+5x^9+x^{10}}{(1-x)^3(1-x^2)^5}, \\
 U_8(x) &= \frac{1+6x+234x^2+486x^3+3087x^4+2868x^5+6796x^6}{(1-x)^3(1-x^2)^6} \\
 &\quad + \frac{2868x^7+3087x^8+486x^9+234x^{10}+6x^{11}+x^{12}}{(1-x)^3(1-x^2)^6}.
 \end{aligned}$$

Remark.

1. The sequence a_1, a_2, \dots, a_n of positive real numbers is called to be *unimodal* if there exists an index k such that $1 \leq k \leq n$, and $a_1 \leq a_2 \leq \dots \leq a_k \geq a_{k+1} \geq \dots \geq a_n$, and *log-concave* if $a_{k-1}a_{k+1} \leq a_k^2$ holds for all indices k . We can show that the log-concave sequence is unimodal. (See Bona [2] for the proof.) The sequence $\{A(t, k)\}_{k=0}^{t-1}$ of Eulerian numbers is log-concave so that it is unimodal for all t . The coefficients of numerator in $U_t(x)$ is unimodal for $t = 1, 2, \dots, 6$ but not always as shown in $U_7(x)$ and $U_8(x)$ above.

2. The sequence $\{WK_t(n)\}_{t,n}$ (for $t = 0, 1, 2, \dots, 5$) are provided below.

$$WK_0(n) = C(n, 0) = 1. \quad (\text{empty graph})$$

$$WK_1(n) = C(n + 1, 1) = n + 1. \quad (\bullet)$$

$$WK_2(n) = C(n + 2, 2) = \frac{1}{2}(n^2 + 3n + 2). \quad (\bullet \text{ --- } \bullet)$$

$$WK_3(n) = \frac{1}{16}(4n^3 + 18n^2 + 28n + 15 + (-1)^n). \quad \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \text{ --- } \bullet \end{array} \right)$$

(Remarks 4 of section 6 of Bona and Ju [3])

$$WK_4(n) = \frac{1}{16}(2n^4 + 12n^3 + 28n^2 + 30n + 13$$

$$+ (-1)^n(2n + 3)). \quad \left(\begin{array}{c} \bullet \text{ --- } \bullet \\ | \quad | \\ \bullet \text{ --- } \bullet \end{array} \right)$$

(Theorem 2.1(1) of section 2 of Ju [6])

$$WK_5(n) = \frac{1}{192}(12n^5 + 90n^4 + 280n^3 + 450n^2 + 374n + 129$$

$$+ (-1)^n(30n^2 + 90n + 63)).$$

(Theorem 3.1(1) of section 3 of Ju [6])

We provides a table for $\{WK_t(n)\}_{t,n}$ as follows:

	$n=0$	1	2	3	4	5	6	7	8	...
$t=0$	1	1	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	8	9	...
2	1	3	6	10	15	21	28	36	45	...
3	1	4	11	23	42	69	106	154	215	...
4	1	5	20	52	117	225	400	656	1025	...
5	1	6	37	117	328	733	1514	2996	4895	...
6	1	7	70	262	927	2385	5752	11896	23425	...
7	1	8	135	583	2642	7745	21942	50614	112355	...
8	1	9	264	1288	7593	25089	84062	215136	540225	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Table for $\{WK_t(n)\}_{t,n}$

Note that the sequence 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, 1, 5, 11, 10, 5, 1, ... has not been appeared in the web-site [On-Line Encyclopedia of Integer Sequences] until we have gotten our results.

4. Further questions

There might be a correspondence between the set \mathcal{G} of all simple graphs and the set of the certain rational functions in $\mathbb{C}[[x]]$, where $\mathbb{C}[[x]]$ is a ring of formal power series. Suppose we call it the ρ -function. That is,

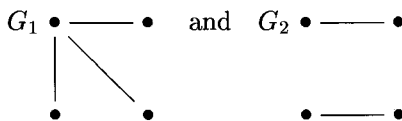
$$\rho : \mathcal{G} \rightarrow \mathbb{C}[[x]](G \mapsto \rho(G)),$$

where $\rho(G) = \sum_{n=0}^{\infty} WG(n)x^n$ is the corresponding generating function of the sequence $\{WG(n)\}_{n=0}^{\infty}$ constructed as in the section Introduction.

The question is what does the image $\rho(\mathcal{G})$ look like? It is not clear that what are those differences between $\rho(\mathcal{T})$ and $\rho(\mathcal{G} - \mathcal{T})$ if we let \mathcal{T} be the set of all trees. $\rho(\mathcal{T})$ is contained in the set of the rational functions of the form $\frac{p(x)}{(1-x)^{m+1}}$, where m is the number of vertices of the tree and $p(x)$ is a polynomial of degree less than m . In fact, if a graph G is the bipartite graph (equivalently, if the graph has no odd cycles, or the graph is bi-colorable) then the corresponding $\rho(G)$ is of the form $\frac{p(x)}{(1-x)^{m+1}}$. (See Bona, Ju and Yoshida [4] about this.) The question is that when the polynomial in the numerator of $\rho(G)$ is symmetric. We conjecture that it is symmetric if the graph is connected. The difference between trees and non-tree graphs is that whether they have a cycle(or cycles) of length at least 3. It seems to be true that for graphs G with cycle of odd length $WG(n)$ is a quasi-polynomial and the denominator of the $\rho(G)$ has a factor $1 - x^2$ because the entries of the vertices of the corresponding polytope has non-integral rational numbers(See Stanley [7], and Bona, Ju and Yoshida [4]).

The ρ -function is not injective.

For example, for graphs



$$\rho(G_1) = \rho(G_2) = \frac{1+4x+x^2}{(1-x)^5} \text{ since } \sum_{k=1}^{n+1} k^3 = (\sum_{k=1}^{n+1} k)^2.$$

It is obvious that for the graph G with the given number m of the vertices $WK_m(n) \leq WG(n) \leq (n + 1)^m = WD_m(n)$, where the graph D_m is the discrete graph with m vertices and no edges. We can raise the same questions for trees and binary trees. That is, find the tree T with m vertices that gives us maximum or minimum in $WT(n)$. This is related with the relative size of the leading coefficients of the (quasi-)polynomial $WG(n)$ as $n \rightarrow \infty$.

If P is a d -dimensional integral convex polytope, then the leading coefficient of the Ehrhart quasi-polynomial $i(P, n)$ ($WG(n)$ in our case) of the polytope P is equal to the volume of the polytope P . (See 4.6.30 Proposition on page 239 of Stanley [7].) Hence, our previous question is equivalent to find the graph (or the tree) with a maximal or minimal volume for the corresponding polytope.

References

- [1] M. Bona, *Walk Through Combinatorics*, World Scientific, 2002.
- [2] ———, *Combinatorics of Permutations*, Chapman & Hall /CRC, 2004.
- [3] M. Bona and H.-K. Ju, *Enumerating Solutions of a System of Linear Inequalities Related to Magic Squares*, *Annals of Combinatorics* **10** (2006), no. 2, 179–191.
- [4] M. Bona, H.-K. Ju, and R. Yoshida, *On the Enumeration of Certain Weighted Graphs*, *Discrete Appl. Math.* **155** (2007), 1481–1496.
- [5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics* 2nd ed., Addison Wesley, 1994.
- [6] H.-K. Ju, *Enumerating Solutions of BJ-problems for the Complete Graphs K_4 and K_5* , *Korean Annals of Math.* **22** (2005), no. 2, 173–188.
- [7] R. Stanley, *Enumerative Combinatorics*, vol.1, Cambridge University Press, 1999.

DEPARTMENT OF MATHEMATICS
CHONNAM NATIONAL UNIVERSITY
KWANGJU 500-757, KOREA
E-mail address: hkju@chonnam.ac.kr