AN EQUIVALENCE FORM OF THE BRUNN-MINKOWSKI INEQUALITY FOR VOLUME DIFFERENCES

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Abstract. In this paper, we establish an equivalence form of the Brunn-Minkowski inequality for volume differences. As an application, we obtain a general and strengthened form of the dual Kneser-Süss inequality.

1. Introduction

If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then there is convex body $K + L$ such that

$$S(K + L, \cdot) = S(K, \cdot) + S(L, \cdot),$$

where $S(K, \cdot)$ denotes the surface area measure of $K$. This is a Minkowski's existence theorem; see [3] or [9]. The operation $+$ is called Blaschke addition.

Theorem A (The Kneser-Süss inequality [9]). If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then

$$V(K + L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are homothetic.

The volume differences function of convex bodies $K$ and $L$ was defined by Leng [5]:

$$Dv(K, D) = V(K) - V(D), \quad D \subset K.$$

In [5], Leng established the following Brunn-Minkowski inequality for volume differences.
Theorem B. If $K, L,$ and $D$ are convex bodies in $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ is a homothetic copy of $D$, then
\[ Dv(K + L, D + D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(L, D')^{1/n} \]
with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

If $p \geq 1$ and $K$ and $L$ contain the origin in their interiors, a convex body $K +_p L$ can be defined by
\[ h(K +_p L, u)^p = h(K, u)^p + h(L, u)^p \]
for $u \in S^{n-1}$. The operation $+_p$ is called the $p$-Minkowski addition. Firey [2] proved the following inequality.

Theorem C1. If $K$ and $L$ are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, $p \geq 1$, and $0 \leq i \leq n$, then
\[ W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)} \]
Furthermore, when $p > 1$, the equality holds if and only if $K$ and $L$ are dilates of each other.

Firey's ideas were transformed into a remarkable extension of the Brunn-Minkowski theory, called the Brunn-Minkowski-Firey theory, by Lutwak [6], [7]. Lutwak found the appropriate $p$-analog $S_p(K, \cdot)$, $p \geq 1$, of the surface area measure of a convex body $K$ in $\mathbb{R}^n$ containing the origin in its interior. In [6], Lutwak generalized Firey's inequality (3). He also generalized Minkowski's existence theorem, deduced the existence of a convex body $K +_p L$ for which
\[ S_p(K +_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot) \]
when $K$ and $L$ are origin-symmetric convex bodies, and proved the following result.

Theorem C2 (Lutwak's $p$-surface area measure inequality). If $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $n \neq p \geq 1$, then
\[ V(K +_p L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n}. \]
Furthermore, when $p > 1$, the equality holds if and only if $K$ and $L$ are dilates of each other.

In [8], Lutwak established the following dual Brunn-Minkowksi inequality.

Theorem D. If $K, L$ are star bodies in $\mathbb{R}^n$, then
\[ V(K + L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \]
with equality if and only if $K$ and $L$ are dilates of each other.

The aim of this paper is to extend Kneser-Süss inequality (Theorem A) to the context of volume differences, which is in turn proved to be equivalent to Leng's result (Theorem B). We then extend Lutwak's $p$-surface area measure inequality (Theorem C2) to the context of volume differences. Finally, a general
dual Brunn-Minkowski inequality which strengthens Lutwak’s result (Theorem D) is also given.

2. Definitions and preliminaries

The setting of this paper is $n$-dimensional Euclidean space $\mathbb{R}^n \ (n > 2)$. Let $\mathcal{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors and the letter $B$ for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. We denote by $V(K)$ the $n$-dimensional volume of a convex body $K$. Let $h_K : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, i.e., $h_K(u) = \text{Max}\{u \cdot x : x \in K\}, u \in S^{n-1}$, where $u \cdot x$ denotes the usual inner product of $u$ and $x$ in $\mathbb{R}^n$.

Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = \text{Max}\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body. Let $\varphi^n$ denote the set of star bodies in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_{\infty}$, where $| \cdot |_{\infty}$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ on $S^{n-1}$.

2.1. Mixed volume and dual mixed volume

If $K_i \in \mathcal{K}^n \ (i = 1, 2, \ldots, r)$ and $\lambda_i \ (i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by

$$V(\sum_{i_1, \ldots, i_n}^r \lambda_i K_i) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1} \cdots K_{i_n}),$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V(K_{i_1} \cdots K_{i_n})$, which is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (6). If $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \cdots K_n)$ is usually written as $V_i(K, L)$.

From (6), we easily get: If $K, L, M \in \mathcal{K}^n$ and $\alpha, \mu \geq 0$, then

$$V_i(M, \alpha K + \mu L) = \alpha V_i(M, K) + \mu V_i(M, L).$$

Further, from (6) it follows immediately that

$$\lim_{\varepsilon \to 0} V(K + \varepsilon L) - V(K) \varepsilon = n V_i(K, L).$$

If $K_1, \ldots, K_n \in \varphi^n$, then the dual mixed volume of $K_1, \ldots, K_n$ is written as $\tilde{V}(K_1, \ldots, K_n)$. If $K_1 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = L$, then $\tilde{V}(K_1, \ldots, K_n)$ is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume
\( \tilde{V}(K, B) \) is written as \( \tilde{W}_i(K) \) and is called the \textit{i-th dual Quermassintegral} of \( K \).

### 2.2. The Blaschke addition and the radial Blaschke addition

If \( K, L \) and \( \alpha, \mu \geq 0 \), then the Theorem of Fenchel-Jessen-Alexandrov tells that there exists a convex body, unique up to translation, which we denote by \( \alpha \cdot K + \mu \cdot L \), such that

\[
S(\alpha \cdot K + \mu \cdot L, \cdot) = \alpha S(K, \cdot) + \mu S(L, \cdot).
\]

This addition is called \textit{Blaschke addition}.

The following result will be used later: If \( K, L, M \in K^n \) and \( \alpha, \mu \geq 0 \), then

\[
V_1(\alpha K + \mu L, M) = \alpha V_1(K, M) + \mu V_1(L, M).
\]

As an aside, we note that corresponding to (8) one has for \( K, L \in K^n \),

\[
\lim_{\varepsilon \to 0} \frac{V(L + \varepsilon K) - V(L)}{\varepsilon} = \frac{n}{n-1} V_1(K, L).
\]

See Goikkman [4].

If \( K, L \in \varphi^n \) and \( \alpha, \mu \geq 0 \), then the radial Blaschke linear combination, \( \alpha \cdot K + \mu \cdot L \), is the star body whose radial function is given by

\[
\rho(\alpha \cdot K + \mu \cdot L, \cdot)^{n-1} = \alpha \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}.
\]

We shall call the addition \textit{radial Blaschke addition}.

### 3. Lemmas

The following well-known results will be required to prove our main theorems.

**Lemma 1** (Bellman's inequality). \( \text{Let } a = \{a_1, \ldots, a_n\} \text{ and } b = \{b_1, \ldots, b_n\} \text{ be two sequences of positive real numbers and } p > 1 \text{ such that } a_i^p - \sum_{i=2}^{n} a_i^p > 0 \text{ and } b_i^p - \sum_{i=2}^{n} b_i^p > 0 \text{, then}

\[
\left( \frac{a_i^p - \sum_{i=2}^{n} a_i^p}{a_i^p} \right)^{1/p} + \left( \frac{b_i^p - \sum_{i=2}^{n} b_i^p}{b_i^p} \right)^{1/p} \leq \left( \frac{(a_1 + b_1)^p - \sum_{i=2}^{n} (a_i + b_i)^p}{(a_1 + b_1)^p} \right)^{1/p}
\]

with equality if and only if \( a = vb \) where \( v \) is a constant.

**Lemma 2** (Minkowski's inequality for integrals). \( \text{If } f_j \geq 0 (j = 1, \ldots, m) \), \( p > 1 \), then

\[
\left( \int_{S^{n-1}} \left( \sum_{j=1}^{m} f_j(u) \right)^p dS(u) \right)^{1/p} \leq \sum_{j=1}^{m} \left( \int_{S^{n-1}} f_j^p(u) dS(u) \right)^{1/p},
\]

with equality if and only if \( f_j \) are effectively proportional.

This inequality is reversed if \( 0 < p < 1 \) or \( p < 0 \).
Lemma 3. If $K$, $L$, and $D$ are convex bodies in $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ is a homothetic copy of $D$, then

$$Dv(K+L, D+D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n}$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

Proof. We will prove the lemma using the method of Leng [5].

Applying the Kneser-Süss inequality (1), we obtain

$$V(K+L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}$$

with equality if and only if $K$ and $L$ are homothetic, and

$$V(D+D')^{(n-1)/n} = V(D)^{(n-1)/n} + V(D')^{(n-1)/n}.$$  

From (15) and (16), we obtain

$$Dv(K+L, D+D') \geq [V(K)^{(n-1)/n} + V(L)^{(n-1)/n}]^{n/(n-1)}$$

$$- [V(D)^{(n-1)/n} + V(D')^{(n-1)/n}]^{n/(n-1)}.$$  

From (17) and applying inequality (12), we have

$$Dv(K+L, D+D')^{(n-1)/n} \geq (V(K) - V(D))^{(n-1)/n} + (V(L) - V(D'))^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

Remark 1. In the special case where $D$ and $D'$ are single points, inequality (14) becomes the classical Kneser-Süss Inequality.

4. Main results

We next observe that Lemma 3 is actually equivalent to Leng’s result (Theorem B).

Theorem 1. If $K$, $L$, and $D$ are convex bodies in $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ is a homothetic copy of $D$, then

$$Dv(K+L, D+D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n}$$

$$\Leftrightarrow Dv(K+L, D+D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(L, D')^{1/n},$$

where the conditions of equality are also equivalent.

Proof. ($\Rightarrow$) Suppose that

$$Dv(K+L, D+D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.
From (10), we obtain
\[
\frac{n}{n-1} (V_1(K, L) - V_1(D, D'))
\]
\[
= \lim_{\varepsilon \to 0} \frac{Dv(L + \varepsilon K, D' + \varepsilon D) + Dv(D', L)}{\varepsilon}
\]
\[
\geq \lim_{\varepsilon \to 0} \frac{(Dv(L, D'))^{(n-1)/n} + \varepsilon Dv(K, D)^{(n-1)/n} + Dv(D', L)}{\varepsilon},
\]
with equality if and only if \(K\) and \(L\) are homothetic and \((V(K), V(D)) = \mu(V(L), V(D'))\), where \(\mu\) is a constant.

On the other hand, from (19) and in view of L'Hôpital's rule, we have
\[
V_1(K, L) - V_1(D, D')
\]
\[
\geq \lim_{\varepsilon \to 0} (Dv(L, D')^{(n-1)/n} + \varepsilon Dv(K, D)^{(n-1)/n})^{1/(n-1)} \geq 0
\]
\[
= Dv(L, D')^{1/n} Dv(K, D)^{(n-1)/n}.
\]

Suppose that \(M, N \in \mathcal{K}^n\) and \(N \subset M\), from (7) and (20), it follows that
\[
V_1(M, K + L) - V_1(N, D + D')
\]
\[
= (V_1(M, K) - V_1(N, D)) + (V_1(M, L) - V_1(N, D'))
\]
\[
\geq (Dv(K, D)^{1/n} + Dv(L, D')^{1/n}) Dv(M, N)^{(n-1)/n}.
\]
If we take \(M = K + L\) and \(N = D + D'\) in (21), in view of \(V(K, \ldots, K) = V(K)\), we have
\[
Dv(K + L, D + D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(K, D)^{1/n},
\]
with equality if and only if \(K\) and \(L\) are homothetic and \((V(K), V(D)) = \mu(V(L), V(D'))\), where \(\mu\) is a constant.

\((\Leftarrow)\) Suppose that
\[
Dv(K + L, D + D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(L, D')^{1/n},
\]
with equality if and only if \(K\) and \(L\) are homothetic and \((V(K), V(D)) = \mu(V(L), V(D'))\), where \(\mu\) is a constant.

From (8), we have
\[
\frac{n}{n-1} (V_1(K, L) - V_1(D, D'))
\]
\[
= \lim_{\varepsilon \to 0} \frac{Dv(K + \varepsilon L, D + \varepsilon D') + Dv(D, K)}{\varepsilon}
\]
\[
\geq \lim_{\varepsilon \to 0} \frac{(Dv(K, D)^{1/n} + \varepsilon Dv(L, D')^{1/n})^{n} + Dv(D, K)}{\varepsilon},
\]
with equality if and only if \(K\) and \(L\) are homothetic and \((V(K), V(D)) = \mu(V(L), V(D'))\), where \(\mu\) is a constant.
On the other hand, from (22) and in view of L'Hôpital's rule, we have
\[ V_1(K, L) - V_1(D, D') \]
\[
\geq \lim_{\varepsilon \to 0} (Dv(K, D))^{1/n} + \varepsilon Dv(L, D')^{1/n} n^{-1} Dv(L, D')^{1/n} 
\]
\[
= Dv(K, D)^{(n-1)/n} Dv(L, D')^{1/n}. 
\]

From (9) and (23), for any \( M, N \in \mathcal{K}^n \) and \( N \subset M \), we have
\[ V_1(K \oplus L, M) - V_1(D \oplus D', N) \]
\[
= (V_1(K, M) - V_1(D, N)) + (V_1(L, M) - V_1(D', N)) 
\]
\[
\geq (Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n}) Dv(M, N)^{1/n}. 
\]

If we take \( M = K \oplus L \) and \( N = D \oplus D' \) in (24), and in view of \( V(K, \ldots, K) = V(K) \), we obtain inequality (14). \( \square \)

**Remark 2.** In the special case where \( D \) and \( D' \) are single points, Theorem 1 gives the following important result.

**Corollary 1.** The Knörrer-Süss inequality is equivalent to the Brunn-Minkowski inequality, namely, for \( K, L \in \mathcal{K}^n \),
\[ V(K \oplus L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n} \]
\[
\Leftrightarrow V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, 
\]
with equality if and only if \( K \) and \( L \) are homothetic.

Similarly, from the Lutwak's \( p \)-surface area measure inequality (4) and the Bellman's inequality, we can get the following result which is a general form of (4).

**Theorem 2.** If \( K, L, \) and \( D \) are origin-symmetric convex bodies in \( \mathbb{R}^n, D \subset K, \) and \( D' \subset L \) is a homothetic copy of \( D \), then for \( n \neq p \geq 1 \),
\[ Dv(K \oplus_p L, D \oplus_p D')^{(n-p)/n} \geq Dv(K, D)^{(n-p)/n} + Dv(L, D')^{(n-p)/n}. \]

Furthermore, when \( p > 1 \), the equality holds if and only if \( K \) and \( L \) are dilates of each other and \( (V(K), V(D)) = \mu (V(L), V(D')) \), where \( \mu \) is a constant.

**Remark 3.** Note that the Knörrer-Süss inequality (14) for volume differences corresponds to the case \( p = 1 \) in (25). On the other hand, if \( D \) and \( D' \) are single points, (25) reduces to the classical Knörrer-Süss inequality.

Finally, the following is a general and strengthened form of Lutwak's dual Brunn-Minkowski inequality.

**Theorem 3.** If \( K, L \in \varphi^n, \alpha \in [0, 1], \) then for \( i < 1 \),
\[ W_i(K \oplus L)^{(n-1)/(n-i)} \]
\[
\leq \tilde{W}_i(\alpha K \oplus(1 - \alpha)L)^{(n-1)/(n-i)} + \tilde{W}_i((1 - \alpha)K \oplus \alpha L)^{(n-1)/(n-i)} 
\]
\[
\leq \tilde{W}_i(K)^{(n-1)/(n-i)} + \tilde{W}_i(L)^{(n-1)/(n-i)}, 
\]
with equality if and only if $K$ and $L$ are dilates of each other.

These inequalities are reversed for $i > n$ or $1 < i < n$.

**Proof.** Noting that $\tilde{W}_i(K) = \int_{S^{n-1}} \rho(K)^{n-i} dS(u)$, and from (11), (13), we have for $i < 1$,

$$\tilde{W}_i(K^\alpha + L)^{(n-1)/(n-i)}$$

$$= \left( \frac{1}{n} \int_{S^{n-1}} \rho(K^\alpha + L, u)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$= \left( \frac{1}{n} \int_{S^{n-1}} (\rho(K, u)^{n-1} + \rho(L, u)^{n-1})^{(n-i)/(n-1)} dS(u) \right)^{(n-1)/(n-i)}$$

$$\leq \left( \frac{1}{n} \int_{S^{n-1}} (\rho(K, u)^{n-1} + (1 - \alpha) \rho(L, u)^{n-1})^{(n-i)/(n-1)} dS(u) \right)^{(n-1)/(n-i)}$$

$$+ \left( \frac{1}{n} \int_{S^{n-1}} ((1 - \alpha) \rho(K, u)^{n-1} + \alpha \rho(L, u)^{n-1})^{(n-i)/(n-1)} dS(u) \right)^{(n-1)/(n-i)}$$

$$= \left( \frac{1}{n} \int_{S^{n-1}} (\rho(\alpha \cdot K^\alpha + (1 - \alpha) \cdot L, u)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$+ \left( \frac{1}{n} \int_{S^{n-1}} (\rho((1 - \alpha) \cdot K^\alpha + \alpha \cdot L, u)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$= \tilde{W}_i(\alpha \cdot K^\alpha + (1 - \alpha) \cdot L)^{(n-1)/(n-i)} + \tilde{W}_i((1 - \alpha) \cdot K^\alpha + \alpha \cdot L)^{(n-1)/(n-i)}.$$  

On the other hand, for $i < 1$,

$$\tilde{W}_i(\alpha \cdot K^\alpha + (1 - \alpha) \cdot L)^{(n-1)/(n-i)}$$

$$= \left( \frac{1}{n} \int_{S^{n-1}} \rho(\alpha \cdot K^\alpha + (1 - \alpha) L)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$\leq \alpha \left( \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$+ (1 - \alpha) \left( \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{n-i} dS(u) \right)^{(n-1)/(n-i)}$$

$$= \alpha \tilde{W}_i(K)^{(n-1)/(n-i)} + (1 - \alpha) \tilde{W}_i(L)^{(n-1)/(n-i)}.$$  

Similarly, we get

$$\tilde{W}_i((1 - \alpha) \cdot K^\alpha + \alpha \cdot L)^{(n-1)/(n-i)} \leq (1 - \alpha) \tilde{W}_i(K)^{(n-1)/(n-i)} + \alpha \tilde{W}_i(L)^{(n-1)/(n-i)}.$$  

Hence,

$$\tilde{W}_i(\alpha \cdot K^\alpha + (1 - \alpha) \cdot L)^{(n-1)/(n-i)} + \tilde{W}_i((1 - \alpha) \cdot K^\alpha \cdot L)^{(n-1)/(n-i)}$$

$$\leq \tilde{W}_i(K)^{(n-1)/(n-i)} + \tilde{W}_i(L)^{(n-1)/(n-i)},$$

with equality if and only if $K$ and $L$ are dilates of each other.
The cases of $i > n$ and $1 < i < n$ are obtained analogously.

**Remark 4.** Taking $i = 0$, inequality (26) becomes the following strengthened form of the dual Kneser-Süss inequality.

**Corollary 2.** If $K, L \in \varphi^n, \alpha \in [0, 1]$, then

\[
V(K + \alpha L)^{(n-1)/n} \leq V((\alpha K + (1 - \alpha) L)^{(n-1)/n} + V((1 - \alpha) K + \alpha L)^{(n-1)/n} \\
\leq V(K)^{(n-1)/n} + V(L)^{(n-1)/n},
\]

with equality if and only if $K$ and $L$ are dilates of each other.

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