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L-FUZZY UNIFORM SPACES

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ABSTRACT. The aim of this paper is to study L-fuzzy uniformizable spaces. A new kind of topological fuzzy remote neighborhood system is defined and used for investigating the relationship between L-fuzzy co-topology and L-fuzzy (quasi-)uniformity. It is showed that this fuzzy remote neighborhood system is different from that in [23] when $\mathcal{U}$ is an L-fuzzy quasi-uniformity and they will be coincident when $\mathcal{U}$ is an L-fuzzy uniformity. It is also showed that each L-fuzzy co-topological space is L-fuzzy quasi-uniformizable.

1. Introduction

It is well-known that uniformity is a very important concept close to topology and a convenient tool for investigating topology. Fuzzy versions of (quasi-) uniformity theory were established by B. Hutton [8], R. Lowen [13], U. Höhle [5] and Shi [18–19], etc. Fuzzy (quasi-)uniformity in Hutton's sense has been accepted by many authors and has attracted wide attention in the literature. Up till now there are many spectacular and creative works about the theory of Hutton uniformities (See [2, 4, 8, 10, 14, 17, 21, 23–25]) and Zhang [25] gave a comparison of various uniformities in fuzzy topology.

Extension of Hutton's quasi-uniformities—I-fuzzy uniformity—was considered in [2]. Later, in [21] fuzzy uniformities for lattices more general than I, namely, so called (L, K)-fuzzy uniformities were considered. Finally, in [4], the paper specially devoted to the analysis of different approaches to the theory of fuzzy uniformities in the context of fuzzy sets, an essentially more general concept of an L-valued uniformity was studied by using a filter approach. Further, in [17], there is a significant extension of Hutton for quasi-uniformities without using filters explicitly and without any distributivity and with general tensor products generating the intersection axiom. In [23], fuzzy remote neighborhood system was used for studying L-fuzzy quasi-uniformity.

The aim of this paper is to define another topological fuzzy remote neighborhood system from a given L-fuzzy quasi-uniformity and use it to study the

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relationship between L-fuzzy co-topology and L-fuzzy quasi-uniformity. We show that this fuzzy remote neighborhood system is different from that in [23] when $\mathcal{U}$ is an L-fuzzy quasi-uniformity. However, when $\mathcal{U}$ is an L-fuzzy uniformity, they will be coincident. We prove that each L-fuzzy co-topological space is L-fuzzy quasi-uniformizable. In this article, all lattices are assumed to be complete.

2. Preliminaries

Let a, b be elements in L. An element $a \in L$ is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $c(L)$. We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$) or $b \gg a$ ($b \rhd a$), if for every directed (arbitrary) subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. Clearly if $a \in L$ is a coprime, then $a \ll b$ if and only if $a \lhd b$. A complete lattice L is said to be continuous (completely distributive) if every element in L is the supremum of all the elements way below (wedge below) it.

Proposition 2.1 ([3]). *Let L be a complete lattice. The following conditions are equivalent:*

(1) *L is completely distributive;*

(2) *L is distributive continuous lattice with enough coprimes;*

(3) *The operator $\bigvee : Low(L) \to L$ sending every lower set to its supremum has a left adjoint β, and in this case $\beta(a) = \{b \mid b \lhd a\}$.*

From (3) in the above proposition it is easy to see that the wedge below relation has the interpolation property in a completely distributive lattice, this is to say, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$.

Throughout this paper, L and M are two completely distributive lattices and there is an order reversing involution $'$ on L. L^X is the set of all L-fuzzy sets on X. $A' \in L^X$ defined by $A'(x) = (A(x))'$. The set of all coprimes of L^X is denoted by $c(L^X)$. Let $e|A$ denote the set $\{B \in L^X | e \not\leq B \geq A\}$ for $e \in c(L^X)$ and $A \in L^X$. Let $F : X \to Y$ be an ordinary mapping, define L-fuzzy mapping $F_L^{\to} : L^X \to L^Y$ and its L-fuzzy reverse mapping $F_L^{\leftarrow} : L^Y \to L^X$ by $F_L^{\to}(A)(y) = \bigvee\{A(x) \mid x \in X, f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and $F_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ (following the notation in [15]), respectively.

Definition 2.2 ([6, 7, 11, 16, 20]). An L-fuzzy co-topology is a mapping $\eta : L^X \to M$ such that

(FCT1) $\eta(1_X) = \eta(0_X) = 1$;

(FCT2) $\eta(A \vee B) \geq \eta(A) \wedge \eta(B)$ for all $A, B \in L^X$;

(FCT3) $\eta(\bigwedge_{j\in J} A_j) \geq \bigwedge_{j\in J} \eta(A_j)$ for every family $\{A_j | j \in J\} \subseteq L^X$.

The pair (L^X, η) is called an L-fuzzy co-topological space. A mapping $F : (L^X, \eta) \to (L^Y, \eta_1)$ is said to be fuzzy continuous with respect to η and η_1 if $\eta(F_L^{\leftarrow}(B)) \geq \eta_1(B)$ for all $B \in L^Y$. Let L-**FCTOP** denote the category of L-fuzzy co-topological spaces and fuzzy continuous mappings.

If η is an L-fuzzy co-topology on X, then τ is an L-fuzzy topology on X, where $\tau : L^X \to M$ is defined by $\tau(A) = \eta(A')$. The converse is also true. For convenience, in this paper, we use L-fuzzy co-topology. For undefined notions about category, please refer to [1].

Definition 2.3 ([22]). A topological fuzzy remote neighborhood system is a set $\mathcal{R} = \{R_e | e \in c(L^X)\}$ of mappings $R_e : L^X \to M$ such that
(FRN1) $R_e(1_X) = 0, R_e(0_X) = 1$;
(FRN2) $R_e(A) > 0 \Rightarrow e \not\leq A$;
(FRN3) $R_e(A \vee B) = R_e(A) \wedge R_e(B)$;
(FRN4) $R_e(A) = \bigvee_{B \in e|A} \bigwedge_{a \not\leq B} R_a(B)$.

Lemma 2.4 ([22]). *Let $\eta : L^X \to M$ be an L-fuzzy co-topology. Then we have*
(1) *$\mathcal{R}^\eta = \{R_e^\eta | e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system, where R_e^η is defined by*

$$R_e^\eta(A) = \begin{cases} \bigvee_{B \in e|A} \eta(B), & e \not\leq A, \\ 0, & e \leq A \end{cases}$$

for $e \in c(L^X)$ and $A \in L^X$.
(2) *If η and ζ are two L-fuzzy co-topologies which determine the same topological fuzzy remote neighborhood system, then $\eta = \zeta$.*

Lemma 2.5 ([22]). *Let $\mathcal{R} = \{R_e | e \in c(L^X)\}$ be a topological fuzzy remote neighborhood system and $\eta : L^X \to M$ be defined by $\eta(A) = \bigwedge_{e \not\leq A} R_e(A)$ for all $A \in L^X$. Then η is an L-fuzzy co-topology. Furthermore, if $\mathcal{R}$ and $\int$ are two topological fuzzy remote neighborhood systems which determine the same L-fuzzy co-topology, then $\mathcal{R} = \int$.*

Lemma 2.6 ([23]). *Let $\mathcal{R} = \{R_e | e \in c(L^X)\}$ be a set satisfying* (FRN1)–(FRN3). *Then the following two statements are equivalent*
(FRN4) $R_e(A) = \bigvee_{B \in e|A} \bigwedge_{a \not\leq B} R_a(B)$;
(FRN4*) $R_e(A) = \bigvee_{B \in e|A} (R_e(B) \wedge \bigwedge_{a \not\leq B} R_a(A))$.

3. L-fuzzy uniform spaces

In this section, a new kind of topological fuzzy remote neighborhood system is defined by a given L-fuzzy quasi-uniformity. We will use this kind of remote neighborhood system to study the relationship between L-fuzzy quasi-uniformity and L-fuzzy co-topology.

Let $H(L^X)$ denote the family of all mappings $f : L^X \to L^X$ such that:
(1) $A \leq f(A)$ for all $A \in L^X$;
(2) $f(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} f(A_j)$ for $\{A_j\}_{j \in J} \subseteq L^X$.
f_1 denotes the biggest element of $H(L^X)$, i.e., $f_1(A) = 0_X$ when $A = 0_X$ and $f_1(A) = 1_X$ otherwise. For $f, g \in H(L^X)$, we have that $f \wedge g \in H(L^X)$ and

$f \circ g \in H(L^X)$, where

$$f \wedge g(A) = \bigwedge_{B \vee C = A} f(B) \vee g(C) \quad \text{and} \quad f \circ g(A) = f(g(A)).$$

For each $f \in H(L^X)$, let $f^{\lhd}(B) = \bigwedge\{C \in L^X | f(C') \leq B'\}$, then we have the following proposition.

Proposition 3.1 ([8, 12]). (1) $f^{\lhd} \in H(L^X)$;
(2) $f \leq g$ *implies* $f^{\lhd} \leq g^{\lhd}$;
(3) $(f^{\lhd})^{\lhd} = f$;
(4) $(f \circ g)^{\lhd} = g^{\lhd} \circ f^{\lhd}$;
(5) $(f \wedge g)^{\lhd} = f^{\lhd} \wedge g^{\lhd}$;
(6) $(\bigvee_{t \in T} f_t)^{\lhd} = \bigvee_{t \in T} f_t^{\lhd}$.

Suppose $F : X \to Y$ is a mapping, $f \in H(L^Y)$, define $F^{\Leftarrow}(f) : L^X \to L^X$ by $F^{\Leftarrow}(f)(A) = F_L^{\leftarrow} \circ f \circ F^{\rightarrow}(A)$ for all $A \in L^X$, then we have

Proposition 3.2 ([8, 12, 14, 24]). (1) $F^{\Leftarrow}(f) \in H(L^X)$;
(2) $f \leq g$ *implies* $F^{\Leftarrow}(f) \leq F^{\Leftarrow}(g)$;
(3) $F^{\Leftarrow}(f^{\lhd}) = (F^{\Leftarrow}(f))^{\lhd}$;
(4) $F^{\Leftarrow}(f \circ g) \leq F^{\Leftarrow}(f) \circ F^{\Leftarrow}(g)$.

Definition 3.3 ([2, 4, 21]). An L-fuzzy quasi-uniformity is a mapping $\mathcal{U} : H(L^X) \to M$ such that
(FQU1) $\mathcal{U}(f_1) = 1$;
(FQU2) $\mathcal{U}(f \wedge g) = \mathcal{U}(f) \wedge \mathcal{U}(g)$ for all $f, g \in H(L^X)$;
(FQU3) $\mathcal{U}(f) = \bigvee_{g \circ g \leq f} \mathcal{U}(g)$ for all $f \in H(L^X)$.
The pair $(L^X, \mathcal{U})$ is called an L-fuzzy quasi-uniform space. An L-fuzzy quasi-uniformity is called an L-fuzzy uniformity if it also satisfies the following condition:
(FQU4) $\mathcal{U}(f) = \mathcal{U}(f^{\lhd})$ for all $f \in H(L^X)$;
A mapping $F : (L^X, \mathcal{U}) \to (L^Y, \mathcal{U}_1)$ is called fuzzy (quasi-)uniformly continuous if $\mathcal{U}(F^{\Leftarrow}(g)) \geq \mathcal{U}_1(g)$ for all $g \in H(L^Y)$. The category of L-fuzzy quasi-uniform spaces and continuous mappings is denoted by L-**HuQUnif**.

Lemma 3.4 ([23]). *Let* $(L^X, \mathcal{U})$ *be an* L*-fuzzy quasi-uniform space and* $R_e^{(\mathcal{U})} : L^X \to M$ *be defined by* $R_e^{(\mathcal{U})}(A) = \bigvee_{e \not\leq f(A)} \mathcal{U}(f)$ *for all* $A \in L^X$. *Then* $\mathcal{R}^{(\mathcal{U})} = \{R_e^{(\mathcal{U})} | e \in c(L^X)\}$ *is a topological fuzzy remote neighborhood system.*

In L-topology, given an L-quasi-uniformity $\mathcal{U}$, we know that the mapping $i : L^X \to L^X$ defined by

$$i(A) = \bigvee\{C \in L^X | \ \exists f \in \mathcal{U}, f(C) \leq A\}$$

is an interior operator and $c : L^X \to L^X$ defined by

$$c(A) = \bigwedge\{f(A) | \ f \in \mathcal{U}\}$$

is a closure operator. Hence, i and c generate two L-topologies δ_i and δ_c, respectively. For L-quasi-uniform space, δ_i is not necessary coincident with δ_c. But $\delta_i = \delta_c$ is valid when $\mathcal{U}$ is an L-uniformity. In Lemma 3.4, one kind of topological fuzzy remote neighborhood system is generated by L-fuzzy quasi-uniformity. In fact, the idea of this topological remote neighborhood system is due to the definition of the closure operator above. In the following discussion, we will define another topological remote neighborhood system on account of the interior operator. We show that the two topological remote neighborhood systems may not be coincident when $\mathcal{U}$ is an L-fuzzy quasi-uniformity and they will be coincident when $\mathcal{U}$ is an L-fuzzy uniformity.

Theorem 3.5. *Let $(L^X,\mathcal{U})$ be an L-fuzzy quasi-uniform space and $R_e^{\mathcal{U}} : L^X \to M$ be defined by*

$$\forall A \in L^X,\ \ R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq C}\ \bigvee_{f(C') \leq A'} \mathcal{U}(f).$$

Then $\mathcal{R}^{\mathcal{U}} = \{R_e^{\mathcal{U}} | e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.

Proof. We need to check (FRN1)–(FRN4). (FRN1), (FRN2) and (FRN3) are straightforward, what remains is to prove (FRN4). From Lemma 2.6, we know that it is equivalent to check (FRN4*). Since $R_e^{\mathcal{U}}(A) \geq \bigvee_{B \in e|A}(R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A))$ is obvious, it suffices to show that $R_e^{\mathcal{U}}(A) \leq \bigvee_{B \in e|A}(R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A))$. Let $\alpha \lhd R_e^{\mathcal{U}}(A)$, i.e.,

$$\alpha \lhd R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq C}\ \bigvee_{f(C') \leq A'} \mathcal{U}(f) = \bigvee_{e \not\leq C}\ \bigvee_{f(C') \leq A'}\ \bigvee_{g \circ g \leq f} \mathcal{U}(g).$$

Then there exist $C \in L^X$, $f \in H(L^X)$ and $g \in H(L^X)$ such that

$$e \not\leq C \geq (g(C'))' \geq (g \circ g(C'))' \geq f(C')' \geq A \ \text{ and } \ \alpha \leq \mathcal{U}(g).$$

Let $B = (g(C'))'$. Then $B \in e|A$. Furthermore, we have

$$R_e^{\mathcal{U}}(B) = \bigvee_{e \not\leq D}\ \bigvee_{h(D') \leq B'} \mathcal{U}(h) \geq \bigvee_{h(C') \leq B'} \mathcal{U}(h) \geq \mathcal{U}(g) \geq \alpha$$

and

$$\bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A) = \bigwedge_{a \not\leq B}\ \bigvee_{a \not\leq D}\ \bigvee_{h(D') \leq A'} \mathcal{U}(h) \geq \bigwedge_{a \not\leq B}\ \bigvee_{h(B') \leq A'} \mathcal{U}(h) \geq \bigwedge_{a \not\leq B} \mathcal{U}(g) \geq \alpha.$$

Then $\alpha \leq R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A)$. Therefore,

$$\alpha \leq \bigvee_{B \in e|A} (R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A)).$$

From the arbitrariness of α, we have $R_e^{\mathcal{U}}(A) \leq \bigvee_{B \in e|A}(R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A))$. Thus the conclusion holds. □

Theorem 3.6. *Let $(L^X, \mathcal{U})$ be an L-fuzzy quasi-uniform space. Then $R_e^{\mathcal{U}}$ can also be written as follows:*

(1) $R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq C} \bigvee_{f \circ f(C') \leq A'} \mathcal{U}(f)$;

(2) $R_e^{\mathcal{U}}(A) = \bigvee_{C \in L^X} \bigvee_{e \not\leq f(C')' \geq (f \circ f(C'))' \geq A} \mathcal{U}(f)$;

(3) $R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq f^{\triangleleft}(A)} \mathcal{U}(f)$.

Proof. (1) and (2) are trivial. (3) can be obtained by the definition of $f^{\triangleleft}$. □

Remark 3.7. When $\mathcal{U}$ is an L-fuzzy quasi-uniformity, $\mathcal{R}^{\mathcal{U}}$ is not necessary coincident with $\mathcal{R}^{(\mathcal{U})}$. The following example can show this (also see it in [12]).

Example 3.8. Let $X = [0,1]$, $L = M = \{0,1\}$, $\mathcal{D} = \{(x,y) \in X \times X \mid 0 \leq x \leq y \leq 1\}$ and define $f_{\mathcal{D}} : 2^X \to 2^X$ by $f_{\mathcal{D}}(U) = \{y \in X \mid \exists x \in U, \text{s.t.}, (x,y) \in \mathcal{D}\}$. Now define $\mathcal{U} : H(2^X) \to \{0,1\}$ as follows:

$$\mathcal{U}(f) = \begin{cases} 1, & f \geq f_{\mathcal{D}}, \\ 0, & \text{others.} \end{cases}$$

Then it is easy to verify that $\mathcal{U}$ is an L-fuzzy quasi-uniformity (in fact, it is a crisp quasi-uniformity). Since $f_{\mathcal{D}}([0,\frac{1}{2}]) = [0,1]$, we have

$$R_e^{(\mathcal{U})}([0,\frac{1}{2}]) = \bigvee_{e \notin f([0,\frac{1}{2}])} \mathcal{U}(f) = 0$$

for all $e \in X$. But

$$R_e^{\mathcal{U}}([0,\frac{1}{2}]) = \bigvee_{e \notin f^{\triangleleft}([0,\frac{1}{2}]) = \bigcap\{C \mid f(C') \subseteq (\frac{1}{2},1]\}} \mathcal{U}(f) = 1$$

for all $e \notin [0,\frac{1}{2}]$. Hence $R_e^{(\mathcal{U})} \neq R_e^{\mathcal{U}}$. Therefore, the two topologies they generated are not coincident.

However, for L-fuzzy uniform spaces, the two topological remote neighborhood systems are the same, just as the following theorem shows.

Theorem 3.9. *Let $\mathcal{U}$ be an L-fuzzy uniformity, then $\mathcal{R}^{\mathcal{U}} = \mathcal{R}^{(\mathcal{U})}$.*

Proof. Since $\mathcal{U}$ is an L-fuzzy uniformity, we have $\mathcal{U}(f) = \mathcal{U}(f^{\triangleleft})$ for all $f \in H(L^X)$. Hence,

$$\bigvee_{e \not\leq f^{\triangleleft}(A)} \mathcal{U}(f) = \bigvee_{e \not\leq f^{\triangleleft}(A)} \mathcal{U}(f^{\triangleleft}) \leq \bigvee_{e \not\leq g(A)} \mathcal{U}(g).$$

This is to say $R_e^{\mathcal{U}}(A) \leq R_e^{(\mathcal{U})}(A)$.

Conversely, by $(f^{\triangleleft})^{\triangleleft} = f$, we have

$$\bigvee_{e \not\leq g(A)} \mathcal{U}(g) = \bigvee_{e \not\leq (g^{\triangleleft})^{\triangleleft}(A)} \mathcal{U}(g^{\triangleleft}) \leq \bigvee_{e \not\leq f^{\triangleleft}(A)} \mathcal{U}(f).$$

This is to say $R_e^{\mathcal{U}}(A) \geq R_e^{(\mathcal{U})}(A)$. Therefore, $R_e^{\mathcal{U}}(A) = R_e^{(\mathcal{U})}(A)$. □

Let $(L^X, \mathcal{U})$ be an L-fuzzy quasi-uniform space and $\eta_{\mathcal{U}} : L^X \to M$ be defined by

$$\eta_{\mathcal{U}}(A) = \bigwedge_{e \not\leq A} R_e^{\mathcal{U}}(A) = \bigwedge_{e \not\leq A} \bigvee_{e \not\leq f^{\lhd}(A)} \mathcal{U}(f)$$

for all $A \in L^X$. From Lemma 2.5, we know that $\eta_{\mathcal{U}}$ is an L-fuzzy co-topology on X and call it the generated L-fuzzy co-topology by $\mathcal{U}$.

Theorem 3.10. *Let (L^X, η) be an L-fuzzy co-topological space. Then there is one L-fuzzy quasi-uniformity $\mathcal{U}_\eta$ on X such that the generated L-fuzzy co-topology by $\mathcal{U}_\eta$ is just η, i.e., $\eta = \eta_{\mathcal{U}_\eta}$. This is to say that each L-fuzzy co-topological space is L-fuzzy quasi-uniformizable.*

Proof. Let $U \in L^X$ and $f_U : L^X \to L^X$ be defined as follows:

$$f_U(A) = \begin{cases} 1_X, & A \not\leq U, \\ U, & 0_X \neq A \leq U, \\ 0_X, & A = 0_X. \end{cases}$$

Then $f_U \in H(L^X)$ and $f_U \circ f_U = f_U$. Define $\mathcal{U}_\eta : H(L^X) \to M$ by

$$\mathcal{U}_\eta(f) = \bigvee\{\wedge_{i=1}^{i=n}\eta(U_i)|\ f \geq \wedge_{i=1}^{i=n} f_{U_i'}, n \in N\}.$$

It is easy to verify that $\mathcal{U}_\eta$ is an L-fuzzy quasi-uniformity on X.

Now we prove that $\eta = \eta_{\mathcal{U}_\eta}$. Noting that $f_{A'}^{\lhd}(A) = A$, from the definition of $\eta_{\mathcal{U}_\eta}$, we have

$$\eta_{\mathcal{U}_\eta}(A) = \bigwedge_{e \not\leq A} \bigvee_{e \not\leq f^{\lhd}(A)} \bigvee\{\wedge_{i=1}^{i=n}\eta(U_i)|f \geq \wedge_{i=1}^{i=n} f_{U_i'}, n \in N\} \geq \bigwedge_{e \not\leq A} \eta(A) = \eta(A).$$

This is to say $\eta_{\mathcal{U}_\eta} \geq \eta$.

On the other hand, we have

$$\begin{aligned}
\eta_{\mathcal{U}_\eta}(A) &= \bigwedge_{e \not\leq A} \bigvee_{e \not\leq f^{\lhd}(A)} \bigvee\{\wedge_{i=1}^{i=n}\eta(U_i)|\ f \geq \wedge_{i=1}^{i=n} f_{U_i'}, n \in N\} \\
&\leq \bigwedge_{e \not\leq A} \bigvee_{e \not\leq f^{\lhd}(A)} \bigvee\{\wedge_{i=1}^{i=n}\eta(U_i)|\ f^{\lhd} \geq \wedge_{i=1}^{i=n} f_{U_i'}^{\lhd}, n \in N\} \\
&\leq \bigwedge_{e \not\leq A} \bigvee_{e \not\leq f^{\lhd}(A)} \bigvee\{\wedge_{i=1}^{i=n}\eta(U_i)|\ f^{\lhd}(A) \geq \wedge_{i=1}^{i=n} f_{U_i}(A), n \in N\} \\
&\leq \bigwedge_{e \not\leq A} \bigvee\{\eta(\wedge_{j=1}^{j=m} U_j)|\ e \not\leq \wedge_{j=1}^{j=m} U_j \geq A, m \in N\} \\
&\leq \bigwedge_{e \not\leq A} \bigvee\{\eta(B)|\ e \not\leq B \geq A\} \\
&= \eta(A).
\end{aligned}$$

This concludes the proof. □

Example 3.11. We consider the L-fuzzy unit interval $I(L)$ in L-topological spaces. Let X be the L-fuzzy unit interval $I(L)$ and $\tau(I(L))$ be the usual L-topology on $X = I(L)$. For more detail about L-fuzzy unit interval, please refer to [11, 12 , 14]. Define $\eta : L^X \to [0,1]$ by

$$\eta(A) = \begin{cases} 1 & A' \in \tau(I(L)), \\ 0 & \text{others.} \end{cases}$$

In fact, η is just the characteristic function of usual co-topology on $I(L)$. From [12, 14], we know that $I(L)$ is (quasi-)unformizable. Hence it is also L-fuzzy (quasi-)uniformizable.

Theorem 3.12. *If $F : (L^X, \mathcal{U}) \to (L^Y, \mathcal{U}_1)$ is fuzzy quasi-uniformly continuous, then $F : (L^X, \eta_{\mathcal{U}}) \to (L^Y, \eta_{\mathcal{U}_1})$ is fuzzy continuous.*

Proof. Let $B \in L^Y$ and $\alpha \lhd \eta_{\mathcal{U}_1}(B)$. Since $F : (L^X, \mathcal{U}) \to (L^Y, \mathcal{U}_1)$ is fuzzy quasi-uniformly continuous, we have $\mathcal{U}_1(f) \le \mathcal{U}(F^{\Leftarrow}(f))$ for all $f \in H(L^Y)$. Hence,

$$\alpha \lhd \eta_{\mathcal{U}_1}(B) = \bigwedge_{e \not\le B} \bigvee_{e \not\le f^{\lhd}(B)} \mathcal{U}_1(f) \le \bigwedge_{e \not\le B} \bigvee_{e \not\le f^{\lhd}(B)} \mathcal{U}(F^{\Leftarrow}(f)).$$

Noting that $F_L^{\rightarrow}(a) \not\le B$ when $a \not\le F_L^{\leftarrow}(B)$, we can find some $f_{(a)} \in H(L^Y)$ such that $F_L^{\rightarrow}(a) \not\le f_{(a)}^{\lhd}(B)$ and $\alpha \le \mathcal{U}(F^{\Leftarrow}(f_{(a)}))$. Now let $h_{(a)} = F^{\Leftarrow}(f_{(a)})$. Then $h_{(a)} \in H(L^X)$ and $a \not\le h_{(a)}^{\lhd}(F_L^{\leftarrow}(B))$. Hence,

$$\alpha \le \bigwedge_{a \not\le F_L^{\leftarrow}(B)} \mathcal{U}(h_{(a)}) \le \bigwedge_{a \not\le F_L^{\leftarrow}(B)} \bigvee_{a \not\le h^{\lhd}(F_L^{\leftarrow}(B))} \mathcal{U}(h) = \eta_{\mathcal{U}}(F_L^{\leftarrow}(B)).$$

Therefore, $\eta_{\mathcal{U}_1}(B) \le \eta_{\mathcal{U}}(F_L^{\leftarrow}(B))$ from the arbitrariness of α. So $F : (L^X, \eta_{\mathcal{U}}) \to (L^Y, \eta_{\mathcal{U}_1})$ is fuzzy continuous. □

Theorem 3.13. *If $F : (L^X, \eta) \to (L^Y, \eta_1)$ is fuzzy continuous, then $F : (L^X, \mathcal{U}_\eta) \to (L^Y, \mathcal{U}_{\eta_1})$ is fuzzy quasi-uniformly continuous.*

Proof. Let $F : (L^X, \eta) \to (L^Y, \eta_1)$ be fuzzy continuous. From the definition of $\mathcal{U}_{\eta_1}$, we know that

$$\mathcal{U}_{\eta_1}(f) = \bigvee \{\wedge_{i=1}^{i=n} \eta_1(U_i) | f \ge \wedge_{i=1}^{i=n} f_{U_i'}, n \in N\}.$$

Moreover, if $f \ge \wedge_{i=1}^{i=n} f_{U_i'}$, then we have

$$F^{\Leftarrow}(f) \ge F^{\Leftarrow}(\wedge_{i=1}^{i=n} f_{U_i'}) = \wedge_{i=1}^{i=n}(F^{\Leftarrow}(f_{U_i'})) = \wedge_{i=1}^{i=n} f_{F_L^{\leftarrow}(U_i)'}$$

Since $F : (L^X, \eta) \to (L^Y, \eta_1)$ is fuzzy continuous, we have $\wedge_{i=1}^{i=n} \eta_1(U_i) \le \wedge_{i=1}^{i=n} \eta(F_L^{\leftarrow}(U_i))$. Hence, $\mathcal{U}_{\eta_1}(f) \le \mathcal{U}_\eta(F^{\Leftarrow}(f))$. Therefore, $F : (L^X, \mathcal{U}_\eta) \to (L^Y, \mathcal{U}_{\eta_1})$ is fuzzy quasi-uniformly continuous. □

Theorem 3.14. *Let* **G** : L-**FCTOP** $\to$ L-**HuQUnif** *be defined by*

$$\mathbf{G}((L^X, \tau)) = (L^X, \mathcal{U}_\tau).$$

Then **G** *is an embedding functor from* L-**FCTOP** *to* L-**HuQUnif**.

Remark 3.15. In [9], Kim studied the relationship between L-fuzzy quasi-uniformities and L-fuzzy topologies and showed that each L-fuzzy topological space is L-fuzzy quasi-uniformizable if L is an order dense chain. From Theorem 3.10, we know that each L-fuzzy co-topological space is L-fuzzy quasi-uniformizable and the condition that L is an order dense chain is not needed. Kim's approach is different from ours. The following example shows this.

Example 3.16. Let X be a single-point set and let $L = \{0, 1, a, b\}$ be the diamond-type lattice, that is, $a \vee b = 1$, $a \wedge b = 0$, $a' = b$ and $b' = a$. Hence, we know that $c(L) = \{a, b\}$ and we do not distinguish L and L^X. Define $\tau : L \to M$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \lambda \in \{0,1\}, \\ b, & \lambda = a, \\ a, & \lambda = b. \end{cases}$$

Then τ is an L-fuzzy topology and its corresponding L-fuzzy co-topology is

$$\eta(\lambda) = \tau(\lambda') = \begin{cases} 1, & \lambda \in \{0,1\}, \\ a, & \lambda = a, \\ b, & \lambda = b. \end{cases}$$

It is easy to verify that $H(L^X)$ is the set $\{f_1, f^2, f^3, f^4\}$, where f_1, f^2, f^3, f^4 are defined as follows:

$$f_1(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda = 1, \\ 1, & \lambda = a, \\ 1, & \lambda = b. \end{cases} \qquad f^2(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda = 1, \\ a, & \lambda = a, \\ b, & \lambda = b. \end{cases}$$

and

$$f^3(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda = 1, \\ a, & \lambda = a, \\ 1, & \lambda = b. \end{cases} \qquad f^4(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda = 1, \\ 1, & \lambda = a, \\ b, & \lambda = b. \end{cases}$$

It is easy to check that $f_1^{\lhd} = f_1$, $(f^2)^{\lhd} = f^2$, $(f^3)^{\lhd} = f^4$ and $(f^4)^{\lhd} = f^3$. Furthermore,

$$f_a(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda \in \{1, b\}, \\ a, & \lambda = a. \end{cases} = f^3, \qquad f_b(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda \in \{1, a\}, \\ b, & \lambda = b. \end{cases} = f^4$$

Hence from Theorem 4.2 in [5] and Theorem 3.10 in this paper, we have

$$\mathcal{U}_\eta(f) = \begin{cases} 0, & f = f^2, \\ b, & f = f^3, \\ a, & f = f^4, \\ 1, & f = f_1. \end{cases} \qquad \mathcal{U}_\tau(f) = \begin{cases} 0, & f = f^2, \\ b, & f = f^3, \\ a, & f = f^4, \\ 1, & f = f_1. \end{cases}$$

By Theorem 3.5 in [9], we have

$$I_{\mathcal{U}_\tau}(\lambda,a)=\begin{cases}1, & \lambda=1,\\ 0, & \text{others.}\end{cases}\qquad I_{\mathcal{U}_\tau}(\lambda,b)=\begin{cases}1, & \lambda=1,\\ 0, & \text{others.}\end{cases}$$

Therefore, from Theorem 3.6 in [9], we can get that

$$\tau_{\mathcal{U}_\tau}(\lambda)=\begin{cases}1, & \lambda\in\{0,1\},\\ 0, & \text{others.}\end{cases}$$

Then $\tau_{\mathcal{U}_\tau}\neq\tau$.

According to our approach in this paper, we have

$$R_a^{\mathcal{U}_\eta}(\lambda)=\begin{cases}0, & \lambda\in\{1,a\},\\ 1, & \lambda=0,\\ b, & \lambda=b.\end{cases}\qquad R_b^{\mathcal{U}_\eta}(\lambda)=\begin{cases}0, & \lambda\in\{1,b\},\\ 1, & \lambda=0,\\ a, & \lambda=a.\end{cases}$$

Hence

$$\eta_{\mathcal{U}_\eta}(\lambda)=\begin{cases}1, & \lambda\in\{0,1\},\\ a, & \lambda=a,\\ b, & \lambda=b.\end{cases}$$

Therefore, $\eta_{\mathcal{U}_\eta}=\eta$.

Question 3.17. In L-topology, we know that an L-topological space (L^X,δ) is L-uniformizable if and only if it is Hutton completely regular. We do not know how to define completely regular separation axiom in L-fuzzy topological spaces so that it can be compatible with L-uniformizable space.

4. L-fuzzy pointwise uniformities

Shi [18, 19] studied pointwise L-(quasi-)uniformities. One remarkable advantage of Shi's (quasi-)uniformity is that it can directly reflect the characteristics of pointwise L-topology, i.e., the relations between a point and its quasi-coincident neighborhood or remote neighborhood. In [23], we studied the extension of Shi's quasi-uniformity in a Kubiak-Šostak sense. Similar to those in section 3, the purpose of this section is to define another topological remote neighborhood system by a pointwise L-fuzzy quasi-uniformity. First, we recall some notions and results in [18, 19, 23].

Let $D(L^X)$ denote the set of all mappings $d: c(L^X)\to L^X$ such that $e\not\leq d(e)$ for all $e\in c(L^X)$. d_0 is the smallest element of $D(L^X)$, i.e., $d_0(e)=0$ for all $e\in c(L^X)$. For $d,g\in D(L^X)$, we define

(1) $d\leq g$ if and only if $d(e)\leq g(e)$ for all $e\in c(L^X)$,

(2) $(d\vee g)(e)=d(e)\vee g(e)$ for all $e\in c(L^X)$

(3) $(d\diamond g)(e)=\bigwedge\{d(a)\mid a\in c(L^X),\ a\not\leq g(e)\}$ for all $e\in c(L^X)$.

Then $d\vee g\in D(L^X)$, $d\diamond g\in D(L^X)$, $d\diamond g\leq d$, $d\diamond g\leq g$ and the operations $\vee$ and $\diamond$ satisfy associative law.

Definition 4.1 ([18, 19]). An order-preserving mapping $d\in D(L^X)$ is said to be symmetric, if for all $\lambda,\mu\in c(L^X)$, there exists $a\in c(L^X)$ such that $a\not\leq\lambda'$ and $\mu\not\leq d(a)$ implies that there exists $b\in c(L^X)$ such that $b\not\leq\mu'$ and $\lambda\not\leq d(b)$.

Let $D_s(L^X)$ denote the set of all symmetric mappings in $D(L^X)$.

Definition 4.2 ([23]). A pointwise L-fuzzy quasi-uniformity is a mapping $\mathcal{U} : D(L^X) \to M$ such that

(FQU1) $\mathcal{U}(d_0) = 1$;

(FQU2) $\mathcal{U}(d \vee g) = \mathcal{U}(d) \wedge \mathcal{U}(g)$ for all $d, g \in D(L^X)$;

(FQU3) $\mathcal{U}(d) = \bigvee_{g \diamond g \geq d} \mathcal{U}(g)$ for all $d \in D(L^X)$.

If $\mathcal{U}$ is a pointwise L-fuzzy quasi-uniformity on X, the pair $(L^X, \mathcal{U})$ is called a pointwise L-fuzzy quasi-uniform space.

Definition 4.3 ([22]). A mapping $\mathcal{B} : D(L^X) \to M$ is called a base of one pointwise L-fuzzy quasi-uniformity if it satisfies:

(FB1) $\mathcal{B}(d_0) = 1$;

(FB2) $\mathcal{B}(d \vee g) \geq \mathcal{B}(d) \wedge \mathcal{B}(g)$ for all $d, g \in D(L^X)$;

(FB3) $\mathcal{B}(d) \leq \bigvee_{g \diamond g \geq d} \mathcal{B}(g)$ for all $d \in D(L^X)$.

Definition 4.4. Let $\mathcal{U}$ be a pointwise L-fuzzy quasi-uniformity on X. $\mathcal{U}$ is called a pointwise L-fuzzy uniformity if there exists a mapping $\mathcal{B} : D(L^X) \to M$ with $\mathcal{B}(d) = 0$ for all $d \in D(L^X) - D_s(L^X)$ such that $\mathcal{B}$ is a base of $\mathcal{U}$, i.e., $\mathcal{B}$ satisfies (FB1)–(FB3) and $\mathcal{U}(d) = \bigvee_{g \geq d} \mathcal{B}(g)$.

Lemma 4.5 ([23]). *Let $(L^X, \mathcal{U})$ be a pointwise L-fuzzy quasi-uniform space and $R_e^{(\mathcal{U})} : L^X \to M$ be defined by*

$$\forall A \in L^X, \ R_e^{(\mathcal{U})}(A) = \bigvee_{A \leq d(e)} \mathcal{U}(d).$$

Then $\mathcal{R}^{(\mathcal{U})} = \{R_e^{(\mathcal{U})} | e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.

Theorem 4.6. *Let $(L^X, \mathcal{U})$ be a pointwise L-fuzzy quasi-uniform space and $R_e^{\mathcal{U}} : L^X \to M$ be defined by*

$$\forall A \in L^X, \ R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq \bigvee_{a \not\leq A'} d(a)'} \mathcal{U}(d).$$

Then $\mathcal{R}^{\mathcal{U}} = \{R_e^{\mathcal{U}} | e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.

Proof. We only check (FRN4*). Since

$$R_e^{\mathcal{U}}(A) \geq \bigvee_{B \in e|A} (R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A))$$

is obvious, it suffices to show that $R_e^{\mathcal{U}}(A) \leq \bigvee_{B \in e|A} (R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A))$. Let

$$\alpha \lhd R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq \bigvee_{a \not\leq A'} d(a)'} \mathcal{U}(d) = \bigvee_{e \not\leq \bigvee_{a \not\leq A'} d(a)'} \bigvee_{g \diamond g \geq d} \mathcal{U}(g).$$

Then there exist $d, g \in D(L^X)$ with $e \not\leq \bigvee_{a \not\leq A'} d(a)'$ such that $g \diamond g \geq d$ and $\alpha \leq \mathcal{U}(g)$. Let $B = \bigvee_{b \not\leq A'} g(b)'$. Then $B \in e|A$. Since

$$\bigvee_{c \not\leq B'} g(c)' = \bigvee_{c \not\leq \bigwedge_{b \not\leq A'} g(b)} g(c)' = \bigvee_{b \not\leq A'} \bigvee_{c \not\leq g(b)} g(c)' = \bigvee_{b \not\leq A'} (g \diamond g(b))' \leq \bigvee_{b \not\leq A'} (d(b))',$$

we have $e \not\leq \bigvee_{c \not\leq B'} g(c)'$. Hence,

$$R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq \bigvee_{a \not\leq B'} d(a)'} \mathcal{U}(d) \geq \mathcal{U}(g) \geq \alpha.$$

Furthermore, we have

$$\bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A) = \bigwedge_{a \not\leq B} \bigvee_{a \not\leq \bigvee_{b \not\leq A'} d(b)'} \mathcal{U}(d) \geq \bigwedge_{a \not\leq B} \mathcal{U}(g) \geq \alpha.$$

Then $\alpha \leq R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A)$. Therefore,

$$\alpha \leq \bigvee_{B \in e|A} (R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A)).$$

From the arbitrariness of α, we have

$$R_e^{\mathcal{U}}(A) \leq \bigvee_{B \in e|A} (R_e^{\mathcal{U}}(B) \wedge \bigwedge_{a \not\leq B} R_a^{\mathcal{U}}(A)).$$

Thus the conclusion holds. □

Theorem 4.7. *If $(L^X, \mathcal{U})$ is a pointwise L-fuzzy uniform space, then $R_e^{\mathcal{U}} \leq R_e^{(\mathcal{U})}$ for all $e \in c(L^X)$.*

Proof. Since $\mathcal{U}$ is a pointwise L-fuzzy uniformity, there exists a base $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}(d) = 0$ for all $d \in D(L^X) - D_s(L^X)$. Let

$$\alpha \lhd R_e^{\mathcal{U}}(A) = \bigvee_{e \not\leq \bigvee_{a \not\leq A'} d(a)'} \mathcal{U}(d) = \bigvee_{e \not\leq \bigvee_{a \not\leq A'} d(a)'} \bigvee_{d_* \geq d} \mathcal{B}(d_*).$$

Then there exist $d \in D(L^X)$ and $d_* \in D_s(L^X)$ such that $e \not\leq \bigvee_{a \not\leq A'} d(a)'$, $d_* \geq d$ and $\alpha \leq \mathcal{B}(d_*)$. Hence, $e \not\leq \bigvee_{a \not\leq A'} d_*(a)'$. Since d_* is symmetric, we have $d_*(e) \geq A$. Therefore, $\alpha \leq \mathcal{B}(d_*) \leq \bigvee_{A \leq d(e)} \mathcal{U}(d) = R_e^{(\mathcal{U})}(A)$. From the arbitrariness of α, we have $R_e^{\mathcal{U}}(A) \leq R_e^{(\mathcal{U})}(A)$, i.e., $R_e^{\mathcal{U}} \leq R_e^{(\mathcal{U})}$. □

Question 4.8. Is $R_e^{\mathcal{U}} = R_e^{(\mathcal{U})}$ valid when $(L^X, \mathcal{U})$ is a pointwise L-fuzzy uniform space?

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