J. Korean Math. Soc. **44** (2007), No. 6, pp. 1417–1428

WEIGHTED L^p ESTIMATES FOR $\bar\partial$ ON A CONVEX DOMAIN WITH PIECEWISE SMOOTH BOUNDARY IN $\mathbb{C}^2$

Hong Rae Cho and Jong-Do Park

Abstract. We obtain weighted L^p estimates ($1 \le p < \infty$) for $\bar\partial$ on convex domains with piecewise smooth boundaries in $\mathbb{C}^2$ by using explicit formulas of solutions introduced by Berndtsson and Andersson.

1. Introduction and statement of the result

In this paper we investigate weighted L^p estimates ($1 \le p < \infty$) for solutions of the $\bar\partial$-equation on convex domains with piecewise smooth boundaries in $\mathbb{C}^2$. For each $j = 1, \ldots, N$ let $D_j = \{z \in \mathbb{C}^2 : \rho_j(z) < 0\}$ and assume that $\rho_j \in C^2(U_j)$, where U_j is a neighborhood of bD_j. Further assume that $d\rho_j \neq 0$ on U_j, $j = 1, \ldots, N$. By a convex domain with piecewise C^2-boundary in $\mathbb{C}^2$, we mean a bounded convex domain $D = D_1 \cap \cdots \cap D_N$ in $\mathbb{C}^2$ satisfying the transversal condition: $\partial\rho_{i_1} \wedge \cdots \wedge \partial\rho_{i_\ell} \neq 0$ on $\cap_{k=1}^{\ell} U_{i_k}$ for $1 \le i_1 < \cdots < i_\ell \le N$. For $z \in D$ we define $\rho(z)$ by

$$\frac{1}{\rho(z)} = \sum_{j=1}^{N} \frac{1}{\rho_j(z)}.$$

Then $\rho \in C^2(D)$ and

$$\frac{1}{N} \inf_{1 \le j \le N} (-\rho_j) \le -\rho \le \inf_{1 \le j \le N} (-\rho_j).$$

Hence $D = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $-\rho(z) \sim \delta(z), z \in D$, where $\delta(z)$ is the distance function from z to the boundary of D.

For $1 \le p < \infty$ and $\alpha > 0$, we define the weighted L^p space

$$L^{p,\alpha}(D) = \{f; \|f\|_{L^{p,\alpha}(D)} < \infty\},$$

where $\|\cdot\|_{L^{p,\alpha}(D)}$ is the weighted L^p-norm with respect to the weighted measure $|\rho|^{\alpha-1} dV$. In the piecewise smooth case $D = \cap_{j=1}^{N} D_j$, usually we consider the

Received April 5, 2006.

2000 *Mathematics Subject Classification*. 32W05, 32A26.

Key words and phrases. weighted L^p estimates for $\bar\partial$, convex domain, piecewise smooth boundary, totally convex.

This work was supported by the Korea Research Foundation Grant funded by Korea Government(MOEHRD, Basic Research Promotion Fund)(KRF-2005-070-C00007).

weighted measure $\prod_{j=1}^{N}|\rho_j|^{\alpha-1}dV$. However in this paper we use the weighted measure $|\rho|^{\alpha-1}dV$. Since $|\rho|$ is equivalent to the distance function, it is perhaps worthwhile to consider our weighted measure. Estimates for the case of weight $\prod_{j=1}^{N}|\rho_j|^{\alpha-1}dV$ are simple like as them of the smooth case. However, estimates for the case of weight $|\rho|^{\alpha-1}dV$ are difficult (see Lemma 4.1 and Lemma 4.2). It will be the first time to consider this weighted measure for the weighted L^p-norms on piecewise smooth domains.

Definition 1.1. We say that a convex domain $D = \{z \in \mathbb{C}^n; \rho(z) < 0\}$ is totally convex at the boundary point ζ_0 in the complex directions if

$$\overline{D} \cap (H_{\zeta_0}(\partial D) + \{\zeta_0\}) = \{\zeta_0\},$$

where $H_{\zeta_0}(\partial D)$ is the complex tangential space of ∂D at ζ_0.

In [12], Range introduced the total convexity to study the Carathéodory metric and holomorphic mappings. A bounded convex domain in $\mathbb{C}^n$ with real analytic boundary is totally convex at each boundary point in the complex directions. The following is the main theorem in this paper.

Theorem 1.2. *Let D be a convex domain with piecewise C^2-boundary in $\mathbb{C}^2$. Let f be a $\overline{\partial}$-closed $(0,1)$-form in $L^{p,\alpha}_{(0,1)}(D)$, $1 \le p < \infty$, $\alpha > 0$. For $1 \le j \le N$, we let $\sigma_j = \{z \in \mathbb{C}^2; \rho_j(z) = 0, \rho_k(z) \le 0 \quad \text{for} \quad k \ne j\}$. If D_j are totally convex at each point in σ_j in the complex direction, then there exists a linear operator $\mathsf{S} : L^{p,\alpha}_{(0,1)}(D) \to L^{p,\alpha}(D)$ satisfying $\overline{\partial}(\mathsf{S}f) = f$ and $\|\mathsf{S}f\|_{p,\alpha} \le C_{p,\alpha}\|f\|_{p,\alpha}$.*

Polking [9] and Range [10] obtained L^p $(1 < p < \infty)$ and Hölder estimates for $\overline{\partial}$ on convex domains with smooth boundaries in $\mathbb{C}^2$, respectively. However, for L^p estimates of the cases $p = 1$ and ∞, Range in his survey paper [11] proposed the problem (Problem 4): Are there L^1 and L^∞ estimates for $\overline{\partial}$ on convex domains in $\mathbb{C}^2$? As a special case of Theorem 1.2 our result solves L^1 estimates for $\overline{\partial}$ on convex domains in $\mathbb{C}^2$ (see [1], also).

In smooth convex domains we need not the total convexity for estimates for $\overline{\partial}$. However, in piecewise smooth convex domains we need the total convexity for our estimates in the proofs of Lemma 4.1 and Lemma 4.2.

In this paper, for L^p estimates, we use the weighted Cauchy-Fantappiè kernel constructed by Berndtsson and Andersson [2], which was also used by Menini [6] to prove L^p estimates in domains with piecewise smooth strictly pseudoconvex boundaries. For other cases of the $\overline{\partial}$-problem on domains with piecewise smooth boundaries we can refer ([4], [5], [7], [8], [13], [14]).

2. An example of total convexity

For $s > 0$, let

$$D_s = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + e^A e^{-1/|z_2|^s} < 1\},$$

where $A = 1 + 2/s$.

Lemma 2.1. *Then D_s is a smoothly bounded convex domain in $\mathbb{C}^2$.*

Before giving the proof we consider the following characterization of the convexity.

Lemma 2.2 ([3]). *A real-valued C^2-function λ on an open set $U \subset \mathbb{C}$ is convex on U if and only if*

$$\Delta_{\mathbb{C}}\lambda(z) := \frac{\partial^2\lambda}{\partial z\partial\bar z}(z) - \left|\frac{\partial^2\lambda(z)}{\partial z^2}\right| \geq 0 \quad \text{for} \quad z \in U. \tag{2.1}$$

Proof of Lemma 2.1. Let $\sigma_s(t) = e^A e^{-\frac{1}{t^{s/2}}}, 0 \leq t \leq (\frac{s}{s+2})^{2/s}$. Then we have

$$\sigma_s'(t) > 0, \quad 0 < t \leq \left(\frac{s}{s+2}\right)^{2/s}, \tag{2.2}$$

$$2\sigma_s''(t)t + \sigma_s'(t) > 0, \quad 0 < t \leq \left(\frac{s}{s+2}\right)^{2/s}. \tag{2.3}$$

Let $\lambda(z) = \sigma_s(|z|^2)$ for $z \in \mathbb{C}$ with $|z| \leq (\frac{s}{s+2})^{1/s}$. Then we have

$$\begin{aligned}\Delta_{\mathbb{C}}\lambda(z) &= \frac{\partial^2\lambda}{\partial z\partial\bar z}(z) - \left|\frac{\partial^2\lambda(z)}{\partial z^2}\right| \\ &= \sigma_s''(|z|^2)|z|^2 + \sigma_s'(|z|^2) - |\sigma_s''(|z|^2)\bar z^2| \\ &= \begin{cases}\sigma_s'(|z|^2) & \text{if} \quad \sigma_s''(|z|^2) \geq 0 \\ 2\sigma_s''(|z|^2)|z|^2 + \sigma_s'(|z|^2) & \text{if} \quad \sigma_s''(|z|^2) \leq 0.\end{cases}\end{aligned}$$

By (2.2) and (2.3), we get $\Delta_{\mathbb{C}}\lambda(z) \geq 0$. By Lemma 2.2, D_s is convex. □

Lemma 2.3. *D_s is totally convex at each boundary point in the complex directions.*

Proof. Let $\rho_s(z) = |z_1|^2 + \sigma_s(|z_2|^2) - 1$. Then ρ_s is a boundary defining function for D_s. Let

$$\phi_s(\zeta, z) = \langle \partial\rho_s(\zeta), \zeta - z\rangle.$$

Note that $H_{\zeta_0}(bD) + \{\zeta_0\} = \{z \in \mathbb{C}^2; \phi_s(\zeta_0, z) = 0\}$ and

$$\begin{aligned}2 \operatorname{Re} \phi_s(\zeta, z) = {} & \rho_s(\zeta) - \rho_s(z) + |\zeta_1 - z_1|^2 + \sigma_s(|z_2|^2) - \sigma_s(|\zeta_2|^2) \\ & + 2 \operatorname{Re} \left[\frac{\partial\sigma_s}{\partial\zeta_2}(|\zeta_2|^2)(\zeta_2 - z_2)\right].\end{aligned}$$

For $\lambda(z) = \sigma_s(|z|^2)$, it is enough to prove that

$$\lambda(z) - \lambda(\zeta) + 2 \operatorname{Re} \left[\frac{\partial\lambda}{\partial\zeta}(\zeta)(\zeta - z)\right] > 0 \quad \text{for} \quad \zeta \neq z.$$

By the Taylor expansion, there exists $0 < \theta < 1$ such that

$$\lambda(z) - \lambda(\zeta) + 2 \operatorname{Re} \left[\frac{\partial\lambda}{\partial\zeta}(\zeta)(\zeta - z)\right] = H_Z\lambda(z - \zeta),$$

where $H_Z\lambda$ is the real Hessian of λ at Z and $Z = \zeta + \theta(z-\zeta)$. Then we have

$$\begin{aligned} H_Z\lambda(z-\zeta) &= \mathrm{Re}\left[\frac{\partial^2\lambda}{\partial\zeta^2}(Z)(z-\zeta)^2\right] + \frac{\partial^2\lambda}{\partial\zeta\partial\overline{\zeta}}(Z)|z-\zeta|^2 \\ &= \mathrm{Re}\left[\sigma_s''(|Z|^2)\overline{Z}^2(z-\zeta)^2\right] + (\sigma_s''(|Z|^2)|Z|^2 + \sigma_s'(|Z|^2))|z-\zeta|^2 \\ &\geq \begin{cases} \sigma_s'(|Z|^2)|z-\zeta|^2 & \text{if } \sigma_s''(|Z|^2) \geq 0 \\ (\sigma_s'(|Z|^2) + 2|Z|^2\sigma_s''(|Z|^2))|z-\zeta|^2 & \text{if } \sigma_s''(|Z|^2) \leq 0. \end{cases} \end{aligned}$$

By (2.2) and (2.3), we obtain the desired result for the case $Z \neq 0$.

In case $Z = 0$, we have $\zeta = -\theta(z-\zeta)$. Thus we have

$$\frac{\partial\lambda}{\partial\zeta}(\zeta)(\zeta - z) = e^A e^{-1/|\zeta|^s}\frac{s\overline{\zeta}(\zeta-z)}{2|\zeta|^{s+2}} = e^A e^{-1/|\zeta|^s}\frac{s}{2\theta|\zeta|^s}.$$

Since $|\zeta|^s \leq s/(s+2)$ and $\zeta \neq 0$, it follows that

$$\begin{aligned} \lambda(z) - \lambda(\zeta) + 2\,\mathrm{Re}\left[\frac{\partial\lambda}{\partial\zeta}(\zeta)(\zeta-z)\right] &\geq -\lambda(\zeta) + 2\mathrm{Re}\left[\frac{\partial\lambda}{\partial\zeta}(\zeta)(\zeta-z)\right] \\ &= e^A e^{-1/|\zeta|^s}\left(\frac{s}{\theta|\zeta|^s} - 1\right) \\ &\geq e^A e^{-1/|\zeta|^s}\left(\frac{s}{\theta} + \frac{2}{\theta} - 1\right) > 0, \end{aligned}$$

if $Z = 0$ and $\zeta \neq z$. □

3. Construction of the solution for the $\bar{\partial}$-equation

Before proving Theorem 1.2, we need some notations. Let

$$s(\zeta, z) = \sum_{j=1}^{2}(\zeta_j - z_j)dz_j$$

and $Q^k(\zeta, z) = -\dfrac{\partial\rho_k(\zeta)}{\rho_k(\zeta)}$ for $k = 1, \ldots, N$. For any $r > 1$, we define the real-valued function $G(t) = (1+t)^{-r}$ and write $G_k^{(\alpha_k)} = G^{(\alpha_k)}(\langle Q^k, \zeta - z\rangle)$, where $G^{(\alpha_k)}(t)$ is the α_k-th derivative of $G(t)$. For any $r > 1$ we define

$$K^r(\zeta, z) = \sum_{j=1}^{2}\sum_{\alpha_1+\cdots+\alpha_N = 2-j} c_\alpha\left(\prod_{k=1}^{N} G_k^{(\alpha_k)}\right)\frac{s\wedge(\bar{\partial}s)^{j-1}}{\langle s, \zeta - z\rangle^j}\bigwedge_{k=1}^{N}(\bar{\partial}_\zeta Q^k)^{\alpha_k},$$

which was introduced by Berndtsson and Andersson [2]. Then for a continuous $(0,1)$-form f in $\overline{D}$ with $\overline{\partial}f = 0$, this kernel $K^r(\zeta, z)$ gives a solution operator

$$\mathsf{S}f(z) = \int_{\zeta\in D} f(\zeta)\wedge K^r(\zeta, z),\ z \in D$$

such that

$$f = \overline{\partial}(\mathsf{S}f).$$

For a smooth form f, this formula holds for any $r > 0$. Define $\phi_k(\zeta, z) = \langle \partial\rho_k(\zeta), \zeta - z\rangle - \rho_k(\zeta)$. The following estimate is a well-known consequence of the convexity of D_k:

$$2\mathrm{Re}\, \phi_k(\zeta, z) \geq -\rho_k(\zeta) - \rho_k(z) \quad \text{for all} \quad \zeta, z \in \overline{D}_k. \tag{3.1}$$

Note that

$$\bar\partial_\zeta Q^k(\zeta, z) = -\frac{\bar\partial\partial\rho_k(\zeta)}{\rho_k(\zeta)} + \frac{\bar\partial\rho_k(\zeta)\wedge\partial\rho_k(\zeta)}{\rho_k(\zeta)^2} \tag{3.2}$$

and

$$G_k^{(\alpha_k)} = (-1)^{\alpha_k} r(r+1)\cdots(r+\alpha_k-1)\left(\frac{-\rho_k(\zeta)}{\phi_k(\zeta, z)}\right)^{r+\alpha_k}. \tag{3.3}$$

By (3.2) and (3.3), we see that

$$\mathsf{S}f(z) = \mathsf{S}_1 f(z) + \sum_{k=1}^{N}\Big(\mathsf{S}_{2,k} f(z) + \mathsf{S}_{3,k} f(z)\Big),$$

where

$$\mathsf{S}_1 f(z) = \int_{\zeta\in D} f(\zeta)\wedge\prod_{j=1}^{N}|\rho_j(\zeta)|^r K_1^r(\zeta, z), \tag{3.4}$$

$$\mathsf{S}_{2,k} f(z) = \int_{\zeta\in D} f(\zeta)\wedge\prod_{j\neq k}|\rho_j(\zeta)|^r \bar\partial\rho_k(\zeta)\wedge|\rho_k(\zeta)|^{r-1} K_{2,k}^r(\zeta, z), \tag{3.5}$$

$$\mathsf{S}_{3,k} f(z) = \int_{\zeta\in D} f(\zeta)\wedge\prod_{j=1}^{N}|\rho_j(\zeta)|^r K_{3,k}^r(\zeta, z) \quad \text{for} \quad k = 1, \ldots, N. \tag{3.6}$$

Here we have the following estimates of the kernels in (3.4), (3.5), and (3.6):

$$|K_1^r(\zeta, z)| \lesssim \frac{1}{|\zeta - z|^3\prod_{j=1}^{N}|\phi_j(\zeta, z)|^r} \tag{3.7}$$

$$|K_{\nu,k}^r(\zeta, z)| \lesssim \frac{1}{|\zeta - z|\prod_{j\neq k}|\phi_j(\zeta, z)|^r|\phi_k(\zeta, z)|^{r+1}}, \tag{3.8}$$

for $\nu = 2, 3$, $k = 1, \ldots, N$. In order to estimate (3.4), (3.5), and (3.6), we need two crucial integral estimates which play important parts in the proof of Theorem 1.2. We will prove them in the next section.

4. Integral estimates

Let $\tilde D_j = \{z \in D : -\rho_j(z) = \inf_{1\leq k\leq N}(-\rho_k(z))\}$. Then $|\rho_j(z)| \sim \delta(z)$ for $z \in \tilde D_j$ and $D = \cup_{j=1}^N \tilde D_j$. We recall that $\sigma_j = \{z \in \mathbb{C}^2 : \rho_j(z) = 0, \rho_k(z) \leq 0 \text{ for } k \neq j\}$.

Lemma 4.1. *Let $1 \leq \nu, \mu \leq N$ and let r be large enough and $\alpha > 0$.*

(i) *For* $0 \le \epsilon < \alpha + 1$ *there exists a constant* $C_{\alpha,\epsilon}$ *such that*

$$\int_{z\in\tilde{D}_\mu} \frac{|\rho_\mu(z)|^{\alpha-\epsilon}}{|\zeta - z|^3 |\phi_\nu(\zeta,z)|^r}\, dV(z) \le C_{\alpha,\epsilon} |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1}, \ \zeta \in \tilde{D}_\nu,$$

(ii) *For* $0 \le \epsilon < \alpha$ *there exists a constant* $C_{\alpha,\epsilon}$ *such that*

$$\int_{z\in\tilde{D}_\mu} \frac{|\rho_\mu(z)|^{\alpha-1-\epsilon}}{|\zeta - z| |\phi_\nu(\zeta,z)|^r}\, dV(z) \le C_{\alpha,\epsilon} |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1}, \ \zeta \in \tilde{D}_\nu.$$

Lemma 4.2. *Let* k *and* r *be same as in Lemma 4.1. For any* $\epsilon > 0$*, there exists a constant* C_ϵ *such that*

(i) $$\int_{\zeta\in\tilde{D}_\nu} \frac{|\rho_\nu(\zeta)|^{r-1-\epsilon}}{|\zeta - z|^3 |\phi_\nu(\zeta,z)|^r}\, dV(\zeta) \le C_\epsilon |\rho_\mu(z)|^{-\epsilon}, \ z \in \tilde{D}_\mu,$$

(ii) $$\int_{\zeta\in\tilde{D}_\nu} \frac{|\rho_\nu(\zeta)|^{r-2-\epsilon}}{|\zeta - z| |\phi_\nu(\zeta,z)|^r}\, dV(\zeta) \le C_\epsilon |\rho_\mu(z)|^{-\epsilon}, \ z \in \tilde{D}_\mu.$$

Proof of Lemma 4.1. For the proof of (i) of Lemma 4.1, it is enough to prove that for $(\zeta_0, z_0) \in \overline{\tilde{D}_\nu} \times \overline{\tilde{D}_\mu}$ there exist neighborhoods U_{ζ_0} of ζ_0 and U_{z_0} of z_0 such that

$$\begin{aligned} I_1(\zeta) :=& \int_{z\in\tilde{D}_\mu\cap U_{z_0}} \frac{|\rho_\mu(z)|^{\alpha-\epsilon}}{|\zeta - z|^3 |\phi_\nu(\zeta,z)|^r}\, dV(z) \\ \le& C_{\alpha,\epsilon} |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1} \quad \text{for} \quad \zeta \in \tilde{D}_\nu \cap U_{\zeta_0}. \end{aligned}$$

If $\phi_\nu(\zeta_0, z_0) \ne 0$, then we can assume that $|\phi_\nu(\zeta, z)| > 1$ in small neighborhoods U_{ζ_0} of ζ_0 and U_{z_0} of z_0. For fixed $\zeta \in U_{\zeta_0}$, we choose a local system $t(z) = (t_1, t_2, t_3, t_4)$ in U_{z_0} such that $t_1(z) = -\rho_\mu(z)$ and $t_2(\zeta) = t_3(\zeta) = t_4(\zeta) = 0$. If we write $t' = (t_3, t_4)$ then we have

$$I_1(\zeta) \lesssim \int_{|t'|<1} \int_{-1}^1 \int_0^1 \frac{|t_1|^{\alpha-\epsilon} dt_1 dt_2 dt'}{(|t_1| + |t_2| + |t'|)^3} \lesssim \int_0^1 |t_1|^{\alpha-\epsilon} |\log|t_1|| dt_1 \lesssim 1.$$

Now we consider the case $\phi_\nu(\zeta_0, z_0) = 0$. Since D_ν is totally convex at each boundary point of σ_ν in the complex direction, it follows that $\zeta_0 = z_0 \in \sigma_\nu \cap \sigma_\mu$ and

$$d_z\rho_\nu \wedge d_z \text{Im}\phi_\nu \wedge d_z\rho_\mu \wedge d_z \text{Im}\phi_\mu = -\partial_z\rho_\nu \wedge \bar{\partial}_z\rho_\nu \wedge \partial_z\rho_\mu \wedge \bar{\partial}_z\rho_\mu \ne 0 \quad \text{at} \quad \zeta_0 = z_0.$$

Thus for fixed $\zeta \in U_{z_0}$ we can choose a local coordinate system

$$\begin{aligned} t_1 + it_2 &= -\rho_\nu(z) + i\text{Im}\ \phi_\nu(\zeta, z) \\ t_3 + it_4 &= -\rho_\mu(z) + i\text{Im}\ \phi_\mu(\zeta, z) \end{aligned}$$

on $z \in U_{z_0}$. Note that

$$\begin{aligned} \text{Re}\ \phi_\nu(\zeta, z) &\gtrsim -\rho_\nu(\zeta) - \rho_\nu(z) \\ &\ge -\rho_\nu(\zeta) - \rho_\mu(z) \quad \text{for} \quad \zeta \in \tilde{D}_\nu, z \in \tilde{D}_\mu. \end{aligned}$$

If we write $t' = (t_1, t_4)$ and introduce polar coordinates in t' with $\xi = |t'|$, then we have

$$I_1(\zeta) \lesssim \int_{\substack{|\xi|<1\\ \xi\in\mathbb{C}}} \int_0^1 \int_{-1}^1 \frac{|t_3|^{\alpha-\epsilon} dt_2 dt_3 d\xi}{(|t_2|+|t_3|+|\xi|)^3(|t_2|+|t_3|+|\rho_\nu(\zeta)|)^r}$$
$$\lesssim \int_0^1 \int_{-1}^1 \frac{|t_3|^{\alpha-\epsilon} dt_2 dt_3}{(|t_2|+|t_3|)(|t_2|+|t_3|+|\rho_\nu(\zeta)|)^r}.$$

If we make the change of variables $t_2 = |\rho_\nu(\zeta)|t_2'$ and $t_3 = |\rho_\nu(\zeta)|t_3'$, and omit the primes, then we have

$$I_1(\zeta) \lesssim |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1} \int_0^\infty \int_0^\infty \frac{t_3^{\alpha-\epsilon} dt_2 dt_3}{(t_2+t_3)(t_2+t_3+1)^r}.$$

If $\alpha-\epsilon \geq 0$, we choose r so that $r-\alpha+\epsilon > 1$ and we introduce polar coordinates in (t_2, t_3) with $\tau = |(t_2, t_3)|$. Then we have

$$I_1' := \int_0^\infty \int_0^\infty \frac{t_3^{\alpha-\epsilon} dt_2 dt_3}{(t_2+t_3)(t_2+t_3+1)^r} \lesssim \int_0^\infty \frac{d\tau}{(\tau+1)^{r-\alpha+\epsilon}} \lesssim 1.$$

If $-1 < \alpha - \epsilon < 0$, we have

$$I_1' \lesssim \int_0^\infty \int_0^\infty \frac{dt_2 dt_3}{t_2^{\epsilon-\alpha+\eta} t_3^{1-\eta}(t_2+t_3+1)^r}$$
$$\lesssim \int_0^\infty \frac{dt_2}{t_2^{\epsilon-\alpha+\eta}(t_2+1)^{r/2}} \int_0^\infty \frac{dt_3}{t_3^{1-\eta}(t_3+1)^{r/2}} \lesssim 1,$$

where we choose η and r so that $0 < \epsilon - \alpha + \eta < 1$ and $r > 2$.

(ii) We will prove that for $(\zeta_0, z_0) \in \sigma_\nu \cap \sigma_\mu$ with $\phi_\nu(\zeta_0, z_0) = 0$, there exist neighborhoods U_{ζ_0} of ζ_0 and U_{z_0} of z_0 such that

$$I_2(\zeta) := \int_{z\in\tilde D_\mu \cap U_{z_0}} \frac{|\rho_\mu(z)|^{\alpha-1-\epsilon}}{|\zeta - z||\phi_\nu(\zeta,z)|^r} dV(z)$$
$$\leq C_{\alpha,\epsilon}|\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1}, \quad \zeta \in \tilde D_\nu \cap U_{\zeta_0}.$$

Similarly as the case (i), we have

$$I_2(\zeta) \lesssim \int_{\substack{|\xi|<1\\ \xi\in\mathbb{C}}} \int_{-1}^1 \int_0^1 \frac{|t_3|^{\alpha-1-\epsilon} dt_2 dt_3 d\xi}{|\xi|(|t_2|+|t_3|+|\rho_\nu(\zeta)|)^r}$$
$$\lesssim \int_{-1}^1 \int_0^1 \frac{|t_3|^{\alpha-1-\epsilon} dt_2 dt_3}{(|t_2|+|t_3|+|\rho_\nu(\zeta)|)^r}$$
$$\lesssim |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1} \int_{-\infty}^\infty \int_0^\infty \frac{|t_3|^{\alpha-1-\epsilon} dt_2 dt_3}{(|t_2|+|t_3|+1)^r}$$
$$\lesssim |\rho_\nu(\zeta)|^{\alpha-\epsilon-r+1},$$

since $\alpha - \epsilon > 0$ and $r - \alpha + \epsilon > 1$. □

Proof of Lemma 4.2. (i) We will prove that for $(\zeta_0, z_0) \in \sigma_\nu \cap \sigma_\mu$ with $\phi_\nu(\zeta_0, z_0) = 0$, there exist neighborhoods U_{ζ_0} of ζ_0 and U_{z_0} of z_0 such that

$$J_1(z) := \int_{\zeta\in\tilde{D}_\nu\cap U_{\zeta_0}} \frac{|\rho_\nu(\zeta)|^{r-1-\epsilon}}{|\zeta - z|^3|\phi_\nu(\zeta,z)|^r}\, dV(\zeta) \leq C_\epsilon |\rho_\mu(z)|^{-\epsilon},\ z \in \tilde{D}_\mu \cap U_{z_0}.$$

Since D_ν is totally convex at each boundary point of σ_ν, it follows that $\zeta_0 = z_0$ and

$$d_\zeta\rho_\nu \wedge d_\zeta \mathrm{Im}\phi_\nu \wedge d_\zeta\rho_\mu \wedge d_\zeta\mathrm{Im}\phi_\mu = -\partial_\zeta\rho_\nu \wedge \bar{\partial}_\zeta\rho_\nu \wedge \partial_\zeta\rho_\mu \wedge \bar{\partial}_\zeta\rho_\mu \neq 0 \quad \text{at} \quad \zeta_0 = z_0.$$

Thus for fixed $z \in U_{z_0}$ we can choose a local coordinate system

$$u_1 + iu_2 = -\rho_\nu(\zeta) + i\mathrm{Im}\ \phi_\nu(\zeta, z)$$
$$u_3 + iu_4 = -\rho_\mu(\zeta) + i\mathrm{Im}\ \phi_\mu(\zeta, z)$$

on $\zeta \in U_{z_0}$. Then we have

$$\begin{aligned} J_1(z) &\lesssim \int_{\substack{|\xi|<1\\ \xi\in\mathbb{C}}} \int_{-1}^{1}\int_0^1 \frac{|u_1|^{r-1-\epsilon}du_1du_2d\xi}{(|\xi|+|u_1|+|u_2|)^3(|u_1|+|u_2|+|\rho_\mu(z)|)^r} \\ &\lesssim \int_{-1}^{1}\int_0^1 \frac{du_1du_2}{(|u_1|+|u_2|)(|u_1|+|u_2|+|\rho_\mu(z)|)^{1+\epsilon}} \\ &\lesssim |\rho_\mu(z)|^{-\epsilon}\int_{(u_1,u_2)\in\mathbb{R}^2} \frac{du_1du_2}{(|u_1|+|u_2|)(|u_1|+|u_2|+1)^{1+\epsilon}} \\ &\lesssim |\rho_\mu(z)|^{-\epsilon}\int_0^\infty \frac{d\tau}{(\tau+1)^{1+\epsilon}} \\ &\lesssim |\rho_\mu(z)|^{-\epsilon}. \end{aligned}$$

(ii) As the case (i), it holds that

$$\begin{aligned} J_2(z) &:= \int_{\zeta\in\tilde{D}_\nu\cap U_{\zeta_0}} \frac{|\rho_\nu(\zeta)|^{r-2-\epsilon}}{|\zeta - z||\phi_\nu(\zeta,z)|^r} dV(\zeta) \\ &\lesssim \int_{\substack{|\xi|<1\\ \xi\in\mathbb{C}}} \int_{-1}^{1}\int_0^1 \frac{|u_1|^{r-2-\epsilon}du_1du_2d\xi}{|\xi|(|u_1|+|u_2|+|\rho_\mu(z)|)^r} \\ &\lesssim \int_{-1}^{1}\int_0^1 \frac{|u_1|^{r-2-\epsilon}du_1du_2}{(|u_1|+|u_2|+|\rho_\mu(z)|)^r} \\ &\lesssim |\rho_\mu(z)|^{-\epsilon}\int_{-\infty}^{\infty}\int_0^\infty \frac{|u_1|^{r-2-\epsilon}du_1du_2}{(|u_1|+|u_2|+1)^r}. \end{aligned}$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty}\int_0^\infty \frac{|u_1|^{r-2-\epsilon}du_1du_2}{(|u_1|+|u_2|+1)^r} &\lesssim \int_0^\infty \frac{u_1^{r-2-\epsilon}du_1}{(u_1+1)^{r-1}} \\ &\lesssim \int_0^1 \frac{du_1}{(u_1+1)^{r-1}} + \int_1^\infty \frac{du_1}{(u_1+1)^{1+\epsilon}} \lesssim 1. \end{aligned}$$

□

5. L^p estimates for $\bar\partial$

Finally we note that Theorem 1.2 is a direct consequence of the next proposition.

Proposition 5.1. *For $1 \le p < \infty$ and $\alpha > 0$, there exists a constant $C_{p,\alpha} > 0$ such that*

$$\|\mathrm{S}_1 f\|_{p,\alpha} + \sum_{\nu=2}^{3}\sum_{k=1}^{N} \|\mathrm{S}_{\nu,k} f\|_{p,\alpha} \le C_{p,\alpha}\|f\|_{p,\alpha} \quad \text{for } \nu = 2,3 \text{ and } k = 1,\dots,N.$$

Proof. At first we consider the operator $\mathrm{S}_1 f$. We estimate the weighted L^p norm of $\mathrm{S}_1 f$ by using the inequality (3.7) and the formula (3.4). Fix k with $1 \le k \le N$. For the case $p = 1$, it follows from the inequality (3.1) and (i) of Lemma 4.1 that

$$\begin{aligned}
\|\mathrm{S}_1 f\|_{1,\alpha} &\lesssim \int_{\zeta\in D} |f(\zeta)| \prod_{j=1}^{N} |\rho_j(\zeta)|^r dV(\zeta) \int_{z\in D} \frac{|\rho(z)|^{\alpha-1}}{|\zeta - z|^3 \prod_{j=1}^{N} |\phi_j(\zeta,z)|^r} dV(z)\\
&\lesssim \sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)|^r dV(\zeta) \int_{z\in\tilde D_\mu} \frac{|\rho_\mu(z)|^{\alpha-1}}{|\zeta - z|^3 |\phi_\nu(\zeta,z)|^r} dV(z)\\
&\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)|^r |\rho_\nu(\zeta)|^{\alpha-r} dV(\zeta)\\
&\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)|^{\alpha} dV(\zeta).
\end{aligned}$$

If $p > 1$ and q is the conjugate exponent of p, Hölder inequality applied to $\mathrm{S}_1 f$ implies

$$\begin{aligned}
&|\mathrm{S}_1 f(z)|\\
&\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)| \prod_{j\ne\nu} |\rho_j(\zeta)|^r |\rho_\nu(\zeta)|^{r-1} |K_1^r(\zeta,z)||\rho_\nu(\zeta)|^{\epsilon}|\rho_\nu(\zeta)|^{-\epsilon} dV(\zeta)\\
&\lesssim \sum_{\nu=1}^{N} \left(\int_{\zeta\in\tilde D_\nu} (|f(\zeta)||\rho_\nu(\zeta)|)^p \prod_{j\ne\nu} |\rho_j(\zeta)|^r |\rho_\nu(\zeta)|^{r-1+\epsilon p} |K_1^r(\zeta,z)| dv(\zeta) \right)^{1/p}\\
&\qquad \times \left(\int_{\zeta\in\tilde D_\nu} \prod_{j\ne\nu} |\rho_j(\zeta)|^r |\rho_\nu(\zeta)|^{r-1-\epsilon q} |K_1^r(\zeta,z)| dV(\zeta) \right)^{1/q}.
\end{aligned}$$

By (3.7) and the inequality (ii) of Lemma 4.2, we have

$$\int_{\zeta\in\tilde D_\nu} \prod_{j\ne\nu} |\rho_j(\zeta)|^r |\rho_\nu(\zeta)|^{r-1-\epsilon q} |K_1^r(\zeta,z)| dV(\zeta)$$

$$\lesssim \int_{\zeta\in\tilde{D}_\nu} \frac{|\rho_\nu(\zeta)|^{r-1-\epsilon q}}{|\zeta - z|^3|\phi_\nu(\zeta,z)|^r} dV(\zeta) \lesssim |\rho_\mu(z)|^{-\epsilon q}.$$

If we choose small ϵ such that $\epsilon p < \alpha$, then

$$\begin{aligned}
&||\mathsf{S}_1 f||_{p,\alpha} \\
&\lesssim \sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde{D}_\nu} (|f(\zeta)||\rho_\nu(\zeta)|)^p \prod_{j\neq\nu} |\rho_j(\zeta)|^r |\rho_\nu(\zeta)|^{r-1+\epsilon p} dV(\zeta) \\
&\qquad \times \int_{z\in\tilde{D}_\mu} |K_1^r(\zeta,z)||\rho_\mu(z)|^{\alpha-1-\epsilon p} dV(z) \\
&\lesssim \sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde{D}_\nu} (|f(\zeta)||\rho_\nu(\zeta)|)^p |\rho_\nu(\zeta)|^{r-1+\epsilon p} dV(\zeta) \\
&\qquad \times \int_{z\in\tilde{D}_\mu} \frac{|\rho_\mu(z)|^{\alpha-1-\epsilon p}}{|\zeta - z|^3|\phi_\nu(\zeta,z)|^r} dV(z) \\
&\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde{D}_\nu} (|f(\zeta)||\rho_\nu(\zeta)|)^p |\rho_\nu(\zeta)|^{\alpha-1} dV(\zeta).
\end{aligned}$$

Now we consider the operator $\mathsf{S}_{2,k}f$. We estimate the weighted L^p norm of $\mathsf{S}_{2,k}f$ by using the inequality (3.8) and the formula (3.5). For the case $p = 1$ it follows from (3.5) that

$$\begin{aligned}
||\mathsf{S}_{2,k}f||_{1,\alpha} \lesssim &\sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde{D}_\nu} |f(\zeta)| dV(\zeta) \int_{z\in\tilde{D}_\mu} \frac{1}{|\zeta - z|} \\
&\times \prod_{j\neq k} \frac{|\rho_j(\zeta)|^r}{|\phi_j(\zeta,z)|^r} \frac{|\rho_k(\zeta)|^{r-1}}{|\phi_k(\zeta,z)|^{r+1}} |\rho_\mu(z)|^{\alpha-1} dV(z).
\end{aligned}$$

We will show that

$$A_k := \prod_{j\neq k} \frac{|\rho_j(\zeta)|^r}{|\phi_j(\zeta,z)|^r} \frac{|\rho_k(\zeta)|^{r-1}}{|\phi_k(\zeta,z)|^{r+1}} \lesssim \frac{|\rho_\nu(\zeta)|^{r-2}}{|\phi_\nu(\zeta,z)|^r}. \tag{5.1}$$

If $k = \nu$, then

$$A_\nu \lesssim \frac{|\rho_\nu(\zeta)|^{r-1}}{|\phi_\nu(\zeta,z)|^{r+1}} \lesssim \frac{|\rho_\nu(\zeta)|^{r-2}}{|\phi_\nu(\zeta,z)|^r}$$

for any $r > 2$. If $k \neq \nu$, then

$$A_k \lesssim \frac{|\rho_\nu(\zeta)|^r}{|\phi_\nu(\zeta,z)|^r} \frac{|\rho_k(\zeta)|^{r-1}}{|\phi_k(\zeta,z)|^{r+1}} \lesssim \frac{|\rho_\nu(\zeta)|^r}{|\phi_\nu(\zeta,z)|^r} \frac{1}{|\rho_k(\zeta)|^2} \lesssim \frac{|\rho_\nu(\zeta)|^{r-2}}{|\phi_\nu(\zeta,z)|^r}.$$

Thus we have

$$||\mathsf{S}_{2,k}f||_{1,\alpha} \lesssim \sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde{D}_\nu} |f(\zeta)||\rho_\nu(\zeta)|^{r-2} dV(\zeta) \int_{z\in\tilde{D}_\mu} \frac{|\rho_\mu(z)|^{\alpha-1}}{|\zeta - z||\phi_\nu(\zeta,z)|^r} dV(z)$$

$$\begin{aligned}&\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)|^{r-2}|\rho_\nu(\zeta)|^{\alpha-r+1}dV(\zeta)\\ &= \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)||\rho_\nu(\zeta)|^{\alpha-1}dV(\zeta).\end{aligned}$$

If $p > 1$ and q is the conjugate exponent of p, Hölder inequality applied to $\mathsf{S}_{2,k}f$ implies

$$\begin{aligned}&|\mathsf{S}_{2,k}f(z)|\\ &\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)| \prod_{j\neq k} |\rho_j(\zeta)|^r|\rho_k(\zeta)|^{r-1}|K^r_{2,k}(\zeta,z)||\rho_\nu(\zeta)|^{\epsilon}|\rho_\nu(\zeta)|^{-\epsilon}dV(\zeta)\\ &\lesssim \sum_{\nu=1}^{N}\left(\int_{\zeta\in\tilde D_\nu} |f(\zeta)|^p \prod_{j\neq k} |\rho_j(\zeta)|^r|\rho_k(\zeta)|^{r-1}|K^r_{2,k}(\zeta,z)||\rho_\nu(\zeta)|^{\epsilon p}dV(\zeta)\right)^{1/p}\\ &\quad\times\left(\int_{\zeta\in\tilde D_\nu} \prod_{j\neq k} |\rho_j(\zeta)|^r|\rho_k(\zeta)|^{r-1}|K^r_{2,k}(\zeta,z)||\rho_\nu(\zeta)|^{-\epsilon q}dV(\zeta)\right)^{1/q}.\end{aligned}$$

By using (3.8) and (5.1), we have

$$\begin{aligned}&\int_{\zeta\in\tilde D_\nu} \prod_{j\neq k} |\rho_j(\zeta)|^r|\rho_k(\zeta)|^{r-1}|K^r_{2,k}(\zeta,z)||\rho_\nu(\zeta)|^{-\epsilon q}dV(\zeta)\\ &\lesssim \int_{\zeta\in\tilde D_\nu} \frac{1}{|\zeta-z|}\prod_{j\neq k}\frac{|\rho_j(\zeta)|^r}{|\phi_j(\zeta,z)|^r}\frac{|\rho_k(\zeta)|^{r-1}}{|\phi_k(\zeta,z)|^{r+1}}|\rho_\nu(\zeta)|^{-\epsilon q}dV(\zeta)\\ &\lesssim \int_{\zeta\in\tilde D_\nu} \frac{|\rho_\nu(\zeta)|^{r-2-\epsilon q}}{|\zeta-z||\phi_\nu(\zeta,z)|^r}dV(\zeta) \lesssim |\rho_\mu(z)|^{-\epsilon q}.\end{aligned}$$

If we choose small ϵ such that $\epsilon p < \alpha$, then

$$\begin{aligned}&\|\mathsf{S}_{2,k}f\|^p_{p,\alpha}\\ &\lesssim \sum_{\nu,\mu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)|^p|\rho_\nu(\zeta)|^{r-2+\epsilon p}dV(\zeta)\int_{z\in\tilde D_\mu}\frac{|\rho_\mu(z)|^{\alpha-1-\epsilon p}}{|\zeta-z||\phi_\nu(\zeta,z)|^r}dV(z)\\ &\lesssim \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)|^p|\rho_\nu(\zeta)|^{r-2+\epsilon p}|\rho_\nu(\zeta)|^{\alpha-\epsilon p-r+1}dV(\zeta)\\ &= \sum_{\nu=1}^{N} \int_{\zeta\in\tilde D_\nu} |f(\zeta)|^p|\rho_\nu(\zeta)|^{\alpha-1}dV(\zeta).\end{aligned}$$

The estimate of $\mathsf{S}_{3,k}f$ easily can be reduced to that of $\mathsf{S}_{2,k}f$. □

References

[1] H. Ahn and H. R. Cho, *Zero sets of holomorphic functions in the Nevanlinna type class on convex domains in* $\mathbb{C}^2$, Japan. J. Math. **28** (2002), no. 2, 245–260.

[2] B. Berndtsson and M. Andersson, *Henkin-Ramirez formulas with weight factors*, Ann. Inst. Fourier **32** (1982), no. 3, 91–110.

[3] K. Diederich and J. E. Fornaess, *Support functions for convex domains of finite type*, Math. Z. **230** (1999), no. 1, 145–164.

[4] C. Laurent-Thiébaut and J. Leiterer, *Uniform estimates for the Cauchy-Riemann equation on q-convex wedges*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 2, 383–436.

[5] I. Lieb and R. M. Range, *Lösungsoperatoren für den Cauchy-Riemann-Komplex mit* C^k*−Abschätzungen*, Math. Ann. **253** (1980), no. 2, 145–164.

[6] C. Menini, *Estimations pour la résolution du* $\bar\partial$ *sur une intersection d'ouverts srtictment pseudoconvexes,* Math. Z. **225** (1997), no. 1, 87–93.

[7] J. Michel and A. Perotti, C^k*-regularity for the* $\bar\partial$*-equation on strictly pseudoconvex domains with piecewise smooth boundaries*, Math. Z. **203** (1990), no. 3, 415–427.

[8] J. Michel and M.-C. Shaw, *The* $\bar\partial$ *problem on domains with piecewise smooth boundaries with applications,* Trans. Amer. Math. Soc. **351** (1999), no. 11, 4365–4380.

[9] J. C. Polking, *The Cauchy-Riemann equation in convex domains,* Proc. Symp. Pure Math. **52** (1991), 309–322.

[10] R. M. Range, *On Hölder and BMO estimates for* $\bar\partial$ *on convex domains in* $\mathbb{C}^2$, J. Geom. Anal. **2** (1992), no. 6, 575–584.

[11] ———, *On* $\bar\partial$*-problems on (pseudo)-convex domains,* Topics in complex analysis (Warsaw, 1992), 311–320, Banach Center Publ., 31, Polish Acad. Sci., Warsaw, 1995.

[12] ———, *The Carathéodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pacific J. Math. **78** (1978), no. 1, 173–188.

[13] R. M. Range and Y.-T. Siu, *Uniform estimates for the* $\bar\partial$*-equation on domains with piecewise smooth strictly pseudoconvex boundaries,* Math. Ann. **206** (1973), 325–354.

[14] H. Ricard, *Estimations* C^k *pour l'opérateur de Cauchy-Riemann sur des domaines à coins q-convexes et q-concaves*, Math. Z. **244** (2003), no. 2, 349–398.

HONG RAE CHO
DEPARTMENT OF MATHEMATICS
PUSAN NATIONAL UNIVERSITY
PUSAN 609-735, KOREA
E-mail address: chohr@pusan.ac.kr

JONG-DO PARK
DEPARTMENT OF MATHEMATICAL SCIENCES
SEOUL NATIONAL UNIVERSITY
SEOUL 151-747, KOREA
E-mail address: jdpark@math.snu.ac.kr

CURRENT ADDRESS:
DEPARTMENT OF MATHEMATICS
POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY
POHANG 790-784, KOREA
E-mail address: jongdopark@gmail.com