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A LAW OF THE ITERATED LOGARITHM FOR l^p-VALUED GAUSSIAN RANDOM FIELDS

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Abstract. A general law of the iterated logarithm (LIL) is established for l^p-valued Gaussian random fields under explicit conditions.

1. Introduction and results

Csörgő and Shao [6] investigated the moduli of continuity, the almost sure path behavior of large increments and the law of the iterated logarithm for l^p-valued, $1 \le p < \infty$, Gaussian processes with stationary increments, on the basis of the well-known Borell inequality [2], and a Fernique type inequality for large and small increments of l^p-valued stochastic processes, due to Csáki, Csörgő and Shao [5] (cf. Lemma 2.1 of the just mentioned paper). Here we are motivated by the trend setting results of Csörgő and Shao [6].

For our purpose, we introduce one of main theorems in Csörgő and Shao [6]: Let $\{Y(t),\, t \ge 0\} = \{X_k(t),\, t \ge 0\}_{k=1}^{\infty}$ be a sequence of independent and centered Gaussian processes with stationary increments $\sigma_k^2(h) := E\{X_k(t+h) - X_k(t)\}^2$, where $\sigma_k(h)$ is assumed to be a nondecreasing continuous function for each $k \ge 1$. Assume that $Y(t)$ takes values in the l^p-space almost surely, where $1 \le p < \infty$. Put

$$
\begin{aligned}
\sigma(p,h) &= \Big(\sum_{k=1}^{\infty} \sigma_k^p(h)\Big)^{1/p}, \quad 1 \le p < \infty, \\
\sigma^*(h) &= \sup_{k \ge 1} \sigma_k(h), \\
\tilde{\sigma}(p,h) &= \begin{cases} \sigma\Big(\frac{2p}{2-p}, h\Big), & \text{if } 1 \le p < 2, \\ \sigma^*(h), & \text{if } 2 \le p < \infty, \end{cases} \\
\delta_p &= \Big(E|N(0,1)|^p\Big)^{1/p}, \quad 1 \le p < \infty,
\end{aligned} \tag{1.1}
$$

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where $N(0,1)$ denotes the standard normal random variable. We recall that a function $Q(x)$ of $x > 0$ is said to be *quasiincreasing* if there is a positive constant c such that $Q(x) \le c\,Q(y)$ for all $0 < x < y < \infty$. In this context and under these assumptions, Csörgő and Shao [6] proved the following *law of the iterated logarithm* for $\{Y(t),\ t \ge 0\}$.

Theorem A. *Assume that $\tilde{\sigma}(p,h)/h^\alpha$ and $\sigma(p,h)/h^\alpha$ are quasiincreasing on $(0,\infty)$ for some $\alpha > 0$ and also that, as $T \to \infty$,*

(a) $\sigma(p,T) = o\Big(\tilde{\sigma}(p,T)\big(\log\log T\big)^{1/2}\Big)$, $p \ge 1$ *and, for each* $\varepsilon > 0$,

(b) $$\lim_{a\to\infty} \max_{j\ge \log a} \max_{k\ge 1} \Big\{\sigma_k(a)\sigma_k(ja)\Big\}^{-1} \times E\Big\{\Big(X_k(a) - X_k(0)\Big)\Big(X_k(ja) - X_k(a)\Big)\Big\} \le 0.$$

Then we have

$$\limsup_{T\to\infty} \sup_{0\le s\le T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p,T)(2\log\log T)^{1/2}} = 1 \quad \text{a.s.},$$
$$\limsup_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p,T)(2\log\log T)^{1/2}} = 1 \quad \text{a.s.} \tag{1.2}$$

In this paper, we establish a general law of the iterated logarithm for l^p-valued Gaussian random fields under explicit conditions in place of the non-positivity condition (b) of Theorem A.

We first introduce some notations for use in this paper. Let $\mathbf{t} = (t_1, \dots, t_N)$ and $\mathbf{s} = (s_1, \dots, s_N)$ be vectors in $[0,\infty)^N$. For convenience, we define

$$\begin{aligned}
&\mathbf{0} = (0,\dots,0) \ \text{ and } \ \mathbf{1} = (1,\dots,1) \ \text{ in } \ [0,\infty)^N,\\
&(\mathbf{t},\,\mathbf{s}) = (t_1,\dots,t_N,s_1,\dots,s_N) \in [0,\infty)^{2N},\\
&\mathbf{t} \le \mathbf{s} \ \text{ if } \ t_i \le s_i \ \text{ for all integers } \ 1 \le i \le N,\\
&\mathbf{t} \pm \mathbf{s} = (t_1 \pm s_1,\dots,t_N \pm s_N), \quad \mathbf{ts} = (t_1 s_1,\dots,t_N s_N),\\
&a\mathbf{t} = (at_1,\dots,at_N) \text{ for a real number } a.
\end{aligned}$$

Throughout the paper, we assume the following conditions. Let $\mathbb{T} = \{\mathbf{t} : \mathbf{t} = (t_1,\dots,t_N) \in [0,\infty)^N\}$ be a real N-dimensional parameter space with the Euclidean norm $\|\cdot\|$ such that $\|\mathbf{t}\| = \big(\sum_{i=1}^N t_i^2\big)^{1/2}$. Assume that $\{X_k(\mathbf{t}),\ \mathbf{t} \in [0,\infty)^N\}_{k=1}^\infty$ is a sequence of independent separable and centered Gaussian random fields with $X_k(\mathbf{0}) = 0$ and $\sigma_k(\|\mathbf{h}\|) := \sqrt{E\{X_k(\mathbf{t}+\mathbf{h}) - X_k(\mathbf{t})\}^2}$ for $\mathbf{h} > \mathbf{0}$, where $\sigma_k(h)$ are nondecreasing continuous functions of $h > 0$.

Let $\{\mathbf{X}(\mathbf{t}) := (X_1(\mathbf{t}), X_2(\mathbf{t}), \dots),\ \mathbf{t} \in [0,\infty)^N\}$ be an infinite dimensional Gaussian random field taking values in l^p-space with l^p-norm $\|\cdot\|_{l^p}$. Assume that $\mathbf{h} = (h_1,\dots,h_N)$ is in $\big(0, 1/\sqrt{N}\big)^N$ to get $0 < \|\mathbf{h}\| < 1$. For each $i = 1,\dots,N$, let $f_i(\mathbf{h})$ be positive continuous functions of $\mathbf{h} > \mathbf{0}$, and put

$$\mathbf{f_h} = (f_1(\mathbf{h}),\dots,f_N(\mathbf{h})), \quad \gamma(\mathbf{h}) = \Big\{2\log|\log\|\mathbf{f_h}\||\Big\}^{1/2},$$

where $\log x = \ln(\max\{x, 1\})$. Here and hereafter we continue using the notations for $\sigma(p,h), \tilde{\sigma}(p,h)$ and δ_p as in (1.1), in our present context and under our present assumptions as well.

Since $E\|\mathbf{X}(\mathbf{t}+\mathbf{h}) - \mathbf{X}(\mathbf{t})\|_{l^p}^p = \delta_p^p(\sigma(p, \|\mathbf{h}\|))^p$ for $1 \le p < \infty$, it follows that $\mathbf{X}(\mathbf{t}+\mathbf{h})-\mathbf{X}(\mathbf{t}) \in l^p$ almost surely for fixed $\mathbf{t}$ and $\mathbf{h}$ if and only if $\sigma(p, \|\mathbf{h}\|) < \infty$.

A positive function $R(h)$ of $h > 0$ is said to be *regularly varying* with exponent $\alpha > 0$ at $b \ge 0$ if $\lim_{h\to b}\{R(xh)/R(h)\} = x^\alpha$ for $x > 0$. Recall that a function $L(h)$ of $h > 0$ is said to be *slowly varying* at $b \ge 0$ if $\lim_{h\to b}\{L(xh)/L(h)\} = 1$ for $x > 0$. Thus the regularly varying function $R(h)$ can be written as $R(h) = h^\alpha L(h)$.

The main results are as follows.

Theorem 1.1. *Let $X_k(\mathbf{t})$ and $\sigma_k(\|\mathbf{h}\|)$ be as above, and let $\tilde{\sigma}(p,h)$ defined in* (1.1) *be a regularly varying function of $h > 0$ with exponent $\tilde{\alpha}$ $(0 < \tilde{\alpha} < 1)$. For each $i = 1, \ldots, N$, let $f_i(\mathbf{h})$ be a positive increasing (or decreasing) and continuous function of $\mathbf{h} > \mathbf{0}$. Then we have*

$$\limsup_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{X}(\mathbf{f_h})\|_{l^p}}{\tilde{\sigma}(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})} \ge 1 \qquad a.s. \tag{1.3}$$

In Theorems 1.1 and 1.2, we do not need the nonpositive condition (b) of Theorem A. Nevertheless we attempt new approach for their proofs.

Theorem 1.2. *Let $X_k(\mathbf{t})$ and $\sigma_k(\|\mathbf{h}\|)$ be as in* Theorem 1.1, *and let $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ defined in* (1.1) *be regularly varying functions of $h > 0$ with positive exponents α and $\tilde{\alpha}$, respectively. For each $i = 1, \ldots, N$, let $f_i(\mathbf{h})$ be a positive increasing (or decreasing) and continuous function of $\mathbf{h} > \mathbf{0}$. Then we have*

$$\limsup_{\mathbf{h}\to\mathbf{0}} \sup_{\|\mathbf{t}\|\le\|\mathbf{f_h}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{f_h}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \|\mathbf{f_h}\|) + \tilde{\sigma}(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})} \le 1 \quad \text{a.s.} \tag{1.4}$$

Combining Theorems 1.1 and 1.2 yields the following law of the iterated logarithm for l^p-valued Gaussian random fields.

Corollary 1.1 (law of the iterated logarithm). *Let $X_k(\mathbf{t})$ and $\sigma_k(\|\mathbf{h}\|)$ be as in* Theorem 1.1. *Suppose that $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ are regularly varying functions of $h > 0$ with exponents α and $\tilde{\alpha}$ $(0 < \alpha, \tilde{\alpha} < 1)$, respectively. For each $i = 1, 2, \ldots, N$, let $f_i(\mathbf{h})$ be a positive increasing (or decreasing) and continuous function of $\mathbf{h} > \mathbf{0}$. Further, if the condition*

(i) $\quad \sigma(p, \|\mathbf{f_h}\|) = o\Big(\tilde{\sigma}(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})\Big), \quad \mathbf{h} \to \mathbf{0}$

is satisfied, then we have

$$\limsup_{\mathbf{h}\to\mathbf{0}} \sup_{\|\mathbf{t}\|\le\|\mathbf{f_h}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{f_h}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\tilde{\sigma}(p, \|\mathbf{f_h}\|)\ \gamma(\mathbf{h})} = 1 \qquad a.s., \tag{1.5}$$

$$\limsup_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{X}(\mathbf{f_h})\|_{l^p}}{\tilde{\sigma}(p, \|\mathbf{f_h}\|)\ \gamma(\mathbf{h})} = 1 \qquad a.s. \tag{1.6}$$

The structures of main theorems above and the new techniques for their proofs can be applied to develop a limit theory on l^p-valued random fields with respect to the following stochastic processes: Ornstein-Uhlenbeck processes (Csáki et al. [4]), Gaussian processes (Choi and Hwang [3]), Lin and Lu [9]), Lévy Brownian motion (Lin and Choi [7], Zhang [10]) and many other related ones in various papers.

As can be seen in the following examples, the law of the iterated logarithm for l^p-valued Gaussian random fields is independent of the size of time parameter T or h (converging to infinity or zero, respectively).

Example 1.1. Consider the case $N = 1$ in Corollary 1.1. Let $\{Y(t), t \geq 0\} = \{X_k(t), t \geq 0\}_{k=1}^{\infty}$ be a sequence of independent separable and centered Gaussian processes with $X_k(0) = 0$ and stationary increments $\sigma_k^2(h) := E\{X_k(t+h) - X_k(t)\}^2$, where $\sigma_k(h)$ is assumed to be a nondecreasing continuous function for each $k \geq 1$. Assume that $Y(t)$ takes values in the l^p-space almost surely, where $1 \leq p < \infty$. Set $N = 1$ in Corollary 1.1 and put

$$\mathbf{h} = h_1 = h = 1/T \quad \text{for } 1 < T < \infty,$$
$$\mathbf{f_h} = f_1(h) = f_1(1/T) = T.$$

Then, it follows from Corollary 1.1 with $\mathbf{t} = t = 0$ that conditions of Theorem A (without the condition (b)) for

$$0 < \alpha < 1, \quad \|\mathbf{f_h}\| = T, \quad \mathbf{h} \to \mathbf{0} \Longleftrightarrow T \to \infty,$$
$$\tilde{\sigma}(p, \|\mathbf{f_h}\|) = \tilde{\sigma}(p, T) \quad \text{and} \quad \gamma(\mathbf{h}) = \sqrt{2 \log\log T}$$

are satisfied. Thus we obtain (1.2) of Theorem A.

Example 1.2. Let $\{X_k(\mathbf{t}),\ \mathbf{t} \in [0,\infty)^N\}_{k=1}^{\infty}$ be a sequence as in Example 1.1. Set $N = 1$ in (1.6) of Corollary 1.1 and put

$$\mathbf{f_h} = f_1(\mathbf{h}) = \mathbf{h} = h \quad \text{for } 0 < h < 1.$$

Then, it follows from Corollary 1.1 with $\gamma(\mathbf{h}) = \sqrt{2 \log\log(1/h)}$ that conditions of Remark 4.1 in Csörgő and Shao [6] (without the condition (4.37)) are satisfied. Thus we obtain (4.38) of Remark 4.1 in the just mentioned paper.

2. Proofs

The following Lemmas 2.1-2.2 are essential for the proof of Theorem 1.1, and Lemma 2.1 is a well-known version of the second Borel-Cantelli lemma.

Lemma 2.1. *Let $\{A_k, k \geq 1\}$ be a sequence of events. If*

(a) $\displaystyle\sum_{k=1}^{\infty} P(A_k) = \infty,$

(b) $\displaystyle\liminf_{n\to\infty} \sum_{1 \leq j < k \leq n} \frac{P(A_j \cap A_k) - P(A_j)P(A_k)}{\left(\sum_{j=1}^{n} P(A_j)\right)^2} \leq 0,$

then $P(A_n,\ \text{i.o.}) = 1.$

Lemma 2.2 (Berman [1]). *Let $\{X_j, j = 1, 2, \ldots, n\}$ be centered and stationary normal random variables with $E(X_iX_j) = r_{ij}$ and $r_{ii} = 1$. Let $I_c^{+1} = [c, \infty)$ and $I_c^{-1} = (-\infty, c)$. Denote by F_j the event $\{X_j \in I_{c_j}^{\varepsilon_j}\}$ for $c_j \in (-\infty, \infty)$, $j = 1, 2, \ldots, n$, where ε_j is either $+1$ or -1. Let $K \subset \{1, 2, \ldots, n\}$. If $\{K_l, l = 1, 2, \ldots, s\}$ is a partition of K, then*

$$\left|P\Big\{\bigcap_{j\in K} F_j\Big\} - \prod_{l=1}^{s} P\Big\{\bigcap_{j\in K_l} F_j\Big\}\right| \le \sum_{1\le l<m\le s}\ \sum_{i\in K_l}\ \sum_{j\in K_m} |r_{ij}|\phi(c_i, c_j; r_{ij}^*),$$

where $\phi(x, y; r)$ is the standard bivariate normal density with correlation r, and r_{ij}^* is a number between 0 and r_{ij}.

Since Theorem 1.1 does not possess such nonpositive condition as (b) of Theorem A or (2.9) of Theorem 2.1 in Csörgő and Shao [6], we must calculate the correlation function as in (2.2) below in order to apply Lemma 2.1. Also Lemma 2.2 is used to estimate an upper bound for the left hand of the condition (b) in Lemma 2.1 as can be seen in the proof of Theorem 1.1 below.

Proof of Theorem 1.1. First, consider the case $1 \le p < 2$. From (3.2) in Csörgő and Shao [6], we have

$$\frac{\|\mathbf{X}(\mathbf{f_h})\|_{l^p}}{\tilde\sigma(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})} \ge \frac{\sum_{i=1}^{\infty}\Big(\sigma_i(\|\mathbf{f_h}\|)\Big)^{2(p-1)/(2-p)} X_i(\mathbf{f_h})}{\tilde\sigma(p, \|\mathbf{f_h}\|)^{p/(2-p)}\gamma(\mathbf{h})} =: \frac{Y(\mathbf{h})}{\gamma(\mathbf{h})}.$$

Note that $Y(\mathbf{h})$ is a standard normal random variable. Let

$$\{\mathbf{h}_k = (h_{1k}, \ldots, h_{Nk})\}_{k=1}^{\infty}$$

be a decreasing sequence in $(0, 1/\sqrt{N})^N$ such that $\mathbf{0} < \mathbf{h}_{k+1} < \mathbf{h} \le \mathbf{h}_k$ and $\lim_{k\to\infty} \mathbf{h}_k = \mathbf{0}$. Define

$$\mathbf{f}_k = \mathbf{f}_{\mathbf{h}_k} = (f_1(\mathbf{h}_k), \ldots, f_N(\mathbf{h}_k)) \in (0, \infty)^N, \quad k \ge 1.$$

The proof of Theorem 1.1 is completed by showing that

$$\limsup_{k\to\infty} \frac{Y(\mathbf{h}_k)}{\sqrt{2\log|\log\|\mathbf{f}_k\||}} \ge 1 \qquad \text{a.s.}$$

For each $i = 1, 2, \ldots, N$, let $f_i(\mathbf{h})$ be increasing functions of $\mathbf{h} > \mathbf{0}$, and put $\|\mathbf{f}_k\| = \theta^{-k}$ for $\theta > 1$ and $k \ge 1$. Setting $A_k = \{Y(\mathbf{h}_k) > x_k\}$, where $x_k = \sqrt{2(1-\varepsilon)\log\log\theta^k}$, $0 < \varepsilon < 1$, then

$$P(A_k) \ge \frac{1}{\sqrt{2\pi}} \frac{\exp\big(-(1-\varepsilon)\log\log\theta^k\big)}{2(1-\varepsilon)\log\log\theta^k} \ge c\,k^{-(1-\varepsilon/2)}$$

for sufficiently large k, where $c > 0$ is a constant. Thus we have $\sum_{k=1}^{\infty} P(A_k) = \infty$. Next, it suffices to show that condition (b) of Lemma 2.1 is satisfied. For

$k = 1, 2, \ldots$, let

$$Y_k = Y(\mathbf{h}_k) = \frac{\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\Big)^{2(p-1)/(2-p)} X_i(\mathbf{f}_k)}{\tilde{\sigma}(p, \|\mathbf{f}_k\|)^{p/(2-p)}}.$$

Then, for $k > l$,

$$\begin{aligned} E(Y_k Y_l) &= \tilde{\sigma}(p, \|\mathbf{f}_k\|)^{-p/(2-p)} \tilde{\sigma}(p, \|\mathbf{f}_l\|)^{-p/(2-p)} \\ &\quad \times E\Big\{\sum_{i=1}^{\infty} \big(\sigma_i(\|\mathbf{f}_k\|)\big)^{2(p-1)/(2-p)} X_i(\mathbf{f}_k) \\ &\quad \times \sum_{i=1}^{\infty} \big(\sigma_i(\|\mathbf{f}_l\|)\big)^{2(p-1)/(2-p)} X_i(\mathbf{f}_l)\Big\} \\ &= \tilde{\sigma}(p, \|\mathbf{f}_k\|)^{-p/(2-p)} \tilde{\sigma}(p, \|\mathbf{f}_l\|)^{-p/(2-p)} \\ &\quad \times \sum_{i=1}^{\infty} \big(\sigma_i(\|\mathbf{f}_k\|)\big)^{2(p-1)/(2-p)} \big(\sigma_i(\|\mathbf{f}_l\|)\big)^{2(p-1)/(2-p)} \\ &\quad \times E\Big\{X_i(\mathbf{f}_k) X_i(\mathbf{f}_l)\Big\} \end{aligned}$$

and

$$\begin{aligned} \big|E\{X_i(\mathbf{f}_k)X_i(\mathbf{f}_l)\}\big| &= \frac{1}{2}\big|\sigma_i^2(\|\mathbf{f}_k\|) + \sigma_i^2(\|\mathbf{f}_l\|) - \sigma_i^2(\|\mathbf{f}_k - \mathbf{f}_l\|)\big| \\ &\le \frac{1}{2}\left(\sigma_i^2(\|\mathbf{f}_k\|) + \left(1 - \frac{\|\mathbf{f}_l - \mathbf{f}_k\|^2}{\|\mathbf{f}_l\|^2}\right)\sigma_i^2(\|\mathbf{f}_l\|)\right) \\ &< \frac{1}{2}\left(\sigma_i^2(\|\mathbf{f}_k\|) + \frac{2\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|}\sigma_i^2(\|\mathbf{f}_l\|)\right). \end{aligned} \tag{2.1}$$

Consequently, using the Hölder inequality and the regular variation of $\tilde{\sigma}$, we arrive at

$$\begin{aligned} |r_{kl}| := |E(Y_k Y_l)| &< \tilde{\sigma}(p, \|\mathbf{f}_k\|)^{-p/(2-p)} \tilde{\sigma}(p, \|\mathbf{f}_l\|)^{-p/(2-p)} \\ &\quad \times \sum_{i=1}^{\infty} \big(\sigma_i(\|\mathbf{f}_k\|)\big)^{2(p-1)/(2-p)} \big(\sigma_i(\|\mathbf{f}_l\|)\big)^{2(p-1)/(2-p)} \\ &\quad \times \left\{\frac{1}{2}\sigma_i^2(\|\mathbf{f}_k\|) + \frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|}\sigma_i^2(\|\mathbf{f}_l\|)\right\} \\ &\le \Big(\tilde{\sigma}(p, \|\mathbf{f}_k\|)\tilde{\sigma}(p, \|\mathbf{f}_l\|)\Big)^{-p/(2-p)} \\ &\quad \times \left\{\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\sigma_i(\|\mathbf{f}_l\|)\Big)^{2(p-1)/(2-p)} \big(\sigma_i(\|\mathbf{f}_k\|)\big)^2 \right. \\ &\quad \left. + \frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|} \sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\sigma_i(\|\mathbf{f}_l\|)\Big)^{2(p-1)/(2-p)} \big(\sigma_i(\|\mathbf{f}_l\|)\big)^2 \right\} \end{aligned}$$

$$
\begin{aligned}
&\le \Big(\tilde\sigma(p,\|\mathbf{f}_k\|)\tilde\sigma(p,\|\mathbf{f}_l\|)\Big)^{-p/(2-p)} \\
&\qquad \times \Bigg\{ \bigg(\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\sigma_i(\|\mathbf{f}_l\|)\Big)^{2p/(2-p)} \bigg)^{(p-1)/p} \\
&\qquad\qquad \times \bigg(\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\Big)^{2p/(2-p)} \bigg)^{(2-p)/p} \\
&\qquad + \frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|} \bigg(\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_k\|)\sigma_i(\|\mathbf{f}_l\|)\Big)^{2p/(2-p)} \bigg)^{(p-1)/p} \\
&\qquad\qquad \times \bigg(\sum_{i=1}^{\infty} \Big(\sigma_i(\|\mathbf{f}_l\|)\Big)^{2p/(2-p)} \bigg)^{(2-p)/p} \Bigg\} \\
&\le \Big(\tilde\sigma(p,\|\mathbf{f}_k\|)\tilde\sigma(p,\|\mathbf{f}_l\|)\Big)^{-p/(2-p)} \\
&\qquad \times \Bigg\{ \Big(\tilde\sigma(p,\|\mathbf{f}_k\|)\tilde\sigma(p,\|\mathbf{f}_l\|)\Big)^{2(p-1)/(2-p)} \Big(\tilde\sigma(p,\|\mathbf{f}_k\|)\Big)^2 \\
&\qquad + \frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|} \Big(\tilde\sigma(p,\|\mathbf{f}_k\|)\tilde\sigma(p,\|\mathbf{f}_l\|)\Big)^{2(p-1)/(2-p)} \Big(\tilde\sigma(p,\|\mathbf{f}_l\|)\Big)^2 \Bigg\} \\
&= \frac{\tilde\sigma(p,\|\mathbf{f}_k\|)}{\tilde\sigma(p,\|\mathbf{f}_l\|)} + \frac{\tilde\sigma(p,\|\mathbf{f}_l\|)}{\tilde\sigma(p,\|\mathbf{f}_k\|)} \frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|} \\
&\le \left(\frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|}\right)^{\alpha'} + \left(\frac{\|\mathbf{f}_k\|}{\|\mathbf{f}_l\|}\right)^{1-\alpha''} = \theta^{-(k-l)\alpha'} + \theta^{-(k-l)(1-\alpha'')} \\
&\le 2\theta^{-(k-l)\alpha_0} =: \eta_{kl},
\end{aligned} \tag{2.2}
$$

where $0 < \alpha' < \tilde\alpha < \alpha'' < 1$ and $\alpha_0 = \min\{\alpha', 1-\alpha''\}$.

On the other hand, if $f_i(\mathbf{h})$, $i = 1, \ldots, N$, are decreasing functions of $\mathbf{h} > 0$, then we put $\|\mathbf{f}_k\| = \theta^k$ for $k \ge 1$ and $\theta > 1$, and set $x_k = \sqrt{2(1-\varepsilon)\log\log\theta^k}$, $0 < \varepsilon < 1$, as before. By proceeding along the same lines as above, we can also obtain the following:

$$
\begin{aligned}
|r_{kl}| &\le \left(\frac{\|\mathbf{f}_l\|}{\|\mathbf{f}_k\|}\right)^{\alpha'} + \left(\frac{\|\mathbf{f}_l\|}{\|\mathbf{f}_k\|}\right)^{1-\alpha''} = \theta^{-(k-l)\alpha'} + \theta^{-(k-l)(1-\alpha'')} \\
&\le 2\,\theta^{-(k-l)\alpha_0} =: \eta_{kl}
\end{aligned}
$$

for $k > l$. Let $\xi_j = \left[\frac{2}{\alpha_0}\log_\theta j\right]$ for any j $(l < j \le k)$ satisfying ${x_j}^2 \ge x_l x_k$. It follows from Lemma 2.2 that, for sufficiently large n,

$$
\begin{aligned}
&\sum_{1\le l<k\le n} \{P(A_l \cap A_k) - P(A_l)P(A_k)\} \\
&\le \sum_{l=1}^{n-1}\sum_{k=l+1}^{n} \frac{|r_{kl}|}{2\pi(1-{r^*_{kl}}^2)^{1/2}} \exp\left\{ -\frac{x_l^2 + x_k^2 - 2x_l x_k |r^*_{kl}|}{2(1-|r^*_{kl}|^2)} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{l=1}^{n-1}\sum_{k=l+1}^{l+\xi_j} \frac{|r_{kl}|}{2\pi(1-{r^*_{kl}}^2)^{1/2}} e^{-x_l^2/2} \exp\Big\{-\Big(\frac{1-|r^*_{kl}|}{1+|r^*_{kl}|}\Big)\frac{x_j^2}{2}\Big\} \\
&\quad + \sum_{l=1}^{n-1}\sum_{k=l+\xi_j+1}^{n} \frac{|r_{kl}|}{2\pi(1-{r^*_{kl}}^2)^{1/2}} e^{-x_k^2/2} e^{-x_l^2/2} \exp\Big\{\frac{1}{1-|r^*_{kl}|^2}|r^*_{kl}|x_j^2\Big\} \\
&= : S_1 + S_2.
\end{aligned}
$$

Consider the sum S_1. If r $(0 < r < 1)$ is the maximum of $|r_{kl}|$ for $1 \leq l < k \leq n$, then we have

$$
\begin{aligned}
S_1 &\leq \sum_{l=1}^{n-1}\sum_{k=l+1}^{l+\xi_j} \frac{r}{2\pi\sqrt{1-r^2}} e^{-x_l^2/2} \exp\Big\{-\Big(\frac{1-r}{1+r}\Big)\frac{x_j^2}{2}\Big\} \\
&\leq c\sum_{l=1}^{n-1} \xi_j\, e^{-x_l^2/2} \exp\Big\{-R\frac{2(1-\varepsilon)\log\log\theta^j}{2}\Big\} \\
&\leq c\sum_{l=1}^{n-1} e^{-x_l^2/2}\,(\log_\theta j)\, j^{-R(1-\varepsilon)} \leq c\sum_{l=1}^{n-1} e^{-x_l^2/2}\frac{1}{\sqrt{2\pi}\,x_l^2}\, j^{-R(1-\varepsilon)/2} \\
&\leq c\sum_{l=1}^{n} P(A_l)\, j^{-R(1-\varepsilon)/2},
\end{aligned} \tag{2.3}
$$

where $R := (1-r)/(1+r)$ and $c > 0$ is a constant. Next, consider the second sum S_2. For $k - l > \xi_j$, since

$$|r_{kl}|x_j^2 \leq \eta_{kl}x_j^2 \leq 4j^{-2}\log(j\,\log\theta),$$

we have

$$S_2 \leq c\,j^{-1}\sum_{l=1}^{n-1}\sum_{k=1}^{n}\frac{1}{2\pi {x_l}^2 {x_k}^2}\, e^{-({x_l}^2+{x_k}^2)/2} \leq c\,j^{-1}\Big(\sum_{l=1}^{n} P(A_l)\Big)^2. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$
\begin{aligned}
&\sum_{1\leq l<k\leq n} \{P(A_l\cap A_k) - P(A_l)P(A_k)\} \\
&\qquad\qquad \leq c\,j^{-R(1-\varepsilon)/2}\Bigg(\sum_{l=1}^{n} P(A_l) + \Big(\sum_{l=1}^{n} P(A_l)\Big)^2\Bigg),
\end{aligned}
$$

and condition (b) of Lemma 2.1 is satisfied for j large enough as $n \to \infty$. □

To prove Theorem 1.2, we need the following lemma which is a modification of Lemma 2.3 in Lin et al. [8].

Lemma 2.3. *Let $X_k(\mathbf{t})$ and $\sigma_k(||\mathbf{h}||)$ be as in* Theorem 1.2, *and let $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ defined in* (1.1) *be regularly varying functions with positive exponents α and $\tilde{\alpha}$, respectively. For each $i = 1,\ldots,N$, let $f_i(\mathbf{h})$ be a positive continuous* *function. Then, for any $0 < \varepsilon < 1$, there exists a positive constant c depending only on ε and N such that, for any $x \geq 1$,*

$$\begin{aligned} P\Big\{ \sup_{\|\mathbf{t}\| \leq \|\mathbf{f_h}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f_h}\|} &\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p} \\ &\geq (1+\varepsilon)\Big(\delta_p \sigma(p, \|\mathbf{f_h}\|) + x\tilde{\sigma}(p, \|\mathbf{f_h}\|)\Big)\Big\} \\ \leq c \exp\Big(-\frac{x^2}{2}\Big). \end{aligned}$$

Now Theorem 1.2 is proved by using Lemma 2.3.

Proof of Theorem 1.2. Let $\theta = (1+\varepsilon)^2$ for any given $\varepsilon > 0$. Define:

$$\begin{aligned} A_j &= \{\mathbf{h} : \theta^j \leq \|\mathbf{f_h}\| \leq \theta^{j+1}\}, \quad -\infty < j < \infty, \\ \|\mathbf{f}_{\mathbf{h}_j}\| &= \sup\{\|\mathbf{f_h}\| : \mathbf{h} \in A_j\}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\limsup_{\mathbf{h}\to \mathbf{0}} \sup_{\|\mathbf{t}\| \leq \|\mathbf{f_h}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f_h}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \|\mathbf{f_h}\|) + \tilde{\sigma}(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})} \\ &\leq \limsup_{|j|\to\infty} \sup_{\mathbf{h}\in A_j} \sup_{\|\mathbf{t}\| \leq \|\mathbf{f_h}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f_h}\|} \\ &\qquad\qquad \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \|\mathbf{f_h}\|) + \tilde{\sigma}(p, \|\mathbf{f_h}\|)\gamma(\mathbf{h})} \\ &\leq \limsup_{|j|\to\infty} \sup_{\|\mathbf{t}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \\ &\qquad\qquad \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \theta^j) + \tilde{\sigma}(p, \theta^j)\sqrt{2\log\log\theta^{|j|}}} \\ &\leq \theta^{\alpha_0} \limsup_{|j|\to\infty} \sup_{\|\mathbf{t}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \\ &\qquad\qquad \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \|\mathbf{f}_{\mathbf{h}_j}\|) + \tilde{\sigma}(p, \|\mathbf{f}_{\mathbf{h}_j}\|)\sqrt{2\log\log\theta^{|j|}}}, \end{aligned} \tag{2.5}$$

where $\alpha_0 = \max\{\alpha, \tilde{\alpha}\}$. Now, we will show that the right-hand side of (2.5) is less than or equal to θ^{α_0} almost surely. Applying Lemma 2.3, there exists $c_\varepsilon > 0$ such that

$$P\Big\{ \sup_{\|\mathbf{t}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\| \leq \|\mathbf{f}_{\mathbf{h}_j}\|} \|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p} > (1+\varepsilon)^2\Big(\delta_p\sigma(p, \|\mathbf{f}_{\mathbf{h}_j}\|)$$

$$+ \tilde{\sigma}(p, \|\mathbf{f}_{\mathbf{h}_j}\|)\sqrt{2\log\log\theta^{|j|}}\Big)\Big\}$$

$$\le P\Big\{ \sup_{\|\mathbf{t}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p} > (1+\varepsilon)\Big(\delta_p\sigma(p, \|\mathbf{f}_{\mathbf{h}_j}\|)$$

$$+ \sqrt{1+\varepsilon}\,\tilde{\sigma}(p, \|\mathbf{f}_{\mathbf{h}_j}\|)\sqrt{2\log\log\theta^{|j|}}\Big)\Big\}$$

$$\le c_\varepsilon \exp\Big(-(1+\varepsilon)\log\log\theta^{|j|}\Big) \le c\,|j \vee 1|^{-1-\varepsilon}$$

for $|j|$ large enough, where $j \vee 1 = \max\{j, 1\}$. Hence we have

$$\sum_{|j|=1}^{\infty} P\Big\{ \sup_{\|\mathbf{t}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}$$

$$> (1+\varepsilon)^2\Big(\delta_p\sigma(p, \|\mathbf{f}_{\mathbf{h}_j}\|) + \tilde{\sigma}(p, \|\mathbf{f}_{\mathbf{h}_j}\|)\sqrt{2\log\log\theta^{|j|}}\Big)\Big\} < \infty,$$

and the Borel-Cantelli lemma implies

$$\limsup_{|j|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{f}_{\mathbf{h}_j}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_{l^p}}{\delta_p\sigma(p, \|\mathbf{f}_{\mathbf{h}_j}\|) + \tilde{\sigma}(p, \|\mathbf{f}_{\mathbf{h}_j}\|)\sqrt{2\log\log\theta^{|j|}}} \le 1 \quad \text{a.s.} \tag{2.6}$$

Combining (2.6) with (2.5) yields (1.4) by the arbitrariness of θ. □

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