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STRONG k-DEFORMATION RETRACT AND ITS APPLICATIONS

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ABSTRACT. In this paper, we study a strong k-deformation retract derived from a relative k-homotopy and investigate its properties in relation to both a k-homotopic thinning and the k-fundamental group. Moreover, we show that the k-fundamental group of a wedge product of closed k-curves not k-contractible is a free group by the use of some properties of both a strong k-deformation retract and a digital covering. Finally, we write an algorithm for calculating the k-fundamental group of a closed k-curve by the use of a k-homotopic thinning.

1. Introduction

A (binary) digital image $(X, k) \subset \mathbf{Z}^n$ in computer science is exactly a discrete topological space $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of $\mathbf{Z}^n$. During the past five years, in relation to digital (k_0, k_1)-covering theory, digital k-curve and closed k-surface theory, and digital k-graph theory, there are many papers including [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 28, 29]. In order to study these areas, we have developed many basic notions and terminologies such as the k-adjacency relations of $\mathbf{Z}^n$, (k_0, k_1)-continuity, (k_0, k_1)-homeomorphism, digital k-graph, (k_0, k_1)-isomorphism, (strongly) local (k_0, k_1)-isomorphism, (relative) (k_0, k_1)-homotopy, strong k-deformation, (graph) (k_0, k_1)-homotopy equivalence, k-homotopic thinning, universal (k_0, k_1)-covering, digital product image, digital fundamental group, generalized digital lifting theorem, discrete Deck's transformation group, Euler characteristic of a digital image, and so forth. The paper [19] (see also [11, 21]) developed the notions of graph (k_0, k)-isomorphism, graph (k_0, k)-homotopy, graph (k_0, k)-homotopy equivalence, and graph (k_0, k)-covering from the view point of digital k-graph theory. Furthermore, in relation to the study of a digital image (X, k) in $\mathbf{Z}^n$ (or a discrete topological space $X \subset \mathbf{Z}^n$ with k-adjacency),

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the paper [19, 25] (see also [11]) invoked the utility of a digital graph theoretical approach as well as a discrete topological approach considering the adjacency relations of $\mathbf{Z}^n$ in [8] (see also [10, 11, 12, 13, 15, 16]). Indeed, a digital image (X, k) in $\mathbf{Z}^n$ can be recognized to be a digital graph G_k on $\mathbf{Z}^n$ [19, 25] (see also [11, 13, 21]). Furthermore, the papers [11, 19] proposed the method how to realize a digital image into a simplicial complex. Besides, the paper [19] (see also [11]) showed that a digitally (k_0, k_1)-continuous map could lead to both a graph (k_0, k_1)-homomorphism and a simplicial map. Consequently, it turns out that the recognition of a digital image in terms of a digital k-graph or a simplicial complex gives some benefits in studying the Euler characteristic of a digital image [11]. The main advantages lie so convenient and efficient for dealing with a discrete topological space with k-adjacency (X, k) by the use of various tools derived from classical graph theory. There are some errors and insufficient presentations in relation to some examples of a closed k-surface in the papers [12, 16, 20, 21], which are corrected in this paper (see Remark 4.3). The study of discrete objects in $\mathbf{Z}^n$ has proceeded in order to find their discrete topological characterizations such as 3D Jordan theorem, a strong homotopy, some local properties of a strong 18- or 26- surface, a thinning algorithm within a digital Jordan surface, the digital k-topological number, and the digital k-linking number [1, 28, 29].

Up to now, the study of discrete objects in $\mathbf{Z}^n$ has proceeded with the following approaches.

- The digital (or discrete) topological approach was introduced in [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 31] with the adjacency relations of $\mathbf{Z}^n$.
- The connected order topological space was introduced in [24], which recovers the structure of a topology.
- The cell complex approach was developed in [27] by which an object is recognized to be a structure consisting of different dimensional cells. Also, this approach can recover the structure of a topology.
- The Alexandroff topological approach was established with Khalimsky continuity and a special kind of homeomorphism [5, 7].

In this paper, we use both the discrete topological approach and the digital k-graph theoretical one to study a discrete object $X \subset \mathbf{Z}^n$ with one of the k-adjacency of $\mathbf{Z}^n$. For a set $X \subset \mathbf{Z}^n$, consider the discrete topological subspace (X, D_X) induced from the discrete topology on $(\mathbf{Z}^n, D)$. Furthermore, consider a k_0-adjacency of $(X, D_X) \subset (\mathbf{Z}^{n_0}, D)$ and a k_1-adjacency of $(Y, D_Y) \subset (\mathbf{Z}^{n_1}, D)$. Hereafter, by (X, k_0) and (Y, k_1) we denote the discrete topological spaces $(X, D_X) \subset (\mathbf{Z}^{n_0}, D)$ with k_0-adjacency and $(Y, D_Y) \subset (\mathbf{Z}^{n_1}, D)$ with k_1-adjacency, respectively. Then, for a standard continuous map $f : (X, k_0) \to (Y, k_1)$, we easily see that f need not preserve the k_0-connectivity of (X, k_0) into the k_1-connectivity of (Y, k_1). Thus, we established the (k_0, k_1)-continuity in Proposition 2.1 and Remark 2.2. Besides, the notions of (k_0, k_1)-isomorphism, and relative (k_0, k_1)-homotopy, k-fundamental group, and (k_0, k_1)-covering were established for the study of some topological properties of some discrete objects in $\mathbf{Z}^n$ [6, 7, 8, 9, 10, 11, 12, 13].

By the use of the relative (k_0, k_1)-homotopy in [21] (see also [9, 11, 12, 13]), we can establish the notions of strong k-deformation retract and k-homotopic thinning. In this paper, we study the k-fundamental group by the use of the $(2, k)$-homotopy in [3] and (k_0, k_1)-covering theory in [10, 12] (see also [13, 14, 18, 19, 21]), the digital k-homotopy equivalence in [16], and the (digital graph) (k_0, k_1)-homotopy equivalence in [19]. Besides, the unique digital lifting theorem in [10, 18], the digital homotopy lifting theorem in [10, 12, 13, 21], an automorphism group of a digital (k_0, k_1)-covering map in [13], and the universal (k_0, k_1)-covering theorem in [13] can be used to calculate the k-fundamental group of some discrete space in $\mathbf{Z}^n$.

Motivated by the covering theory in algebraic topology, (digital) (k_0, k_1)-covering theory was developed in [10, 18] (see also [12, 13, 19, 21]) and has been often used for the calculation of the k-fundamental group of some discrete object with k-adjacency of $\mathbf{Z}^n$ and the classification of discrete spaces with k-adjacency.

The paper is organized as follows. Section 2 provides some basic notions including the origins of a (k_0, k_1)-isomorphism and a geometric realization. Section 3 sheds light on the digital k-graph theoretical methods in digital topology. Section 4 studies a strong k-deformation retract, a k-homotopic thinning, a digital k-homotopy equivalence, and a (digital graph) (k_0, k_1)-homotopy equivalence. Section 5 describes the notion of strongly local (k_0, k_1)-isomorphism and investigates its applications. Section 6 investigates some properties of a (digital graph) (k_0, k_1)-covering. Section 7 calculates the digital k-fundamental group of a closed k-curves by the use of some properties of a digital covering and a strong k-deformation retract. Section 8 investigates some digital topological properties of a wedge product in digital topology and calculates the k-fundamental group of a wedge product of closed k-curves with some hypothesis. In Section 9, we write an algorithm for calculating the k-fundamental group of any closed k-curve. Section 10 concludes the paper with a summary.

2. Preliminaries

Let $\mathbf{Z}$ and $\mathbf{N}$ represent the sets of integers and natural numbers, respectively. A *digital picture* is commonly represented as a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, $(X, D_X) \dot{\subset} (\mathbf{Z}^n, D)$ is a discrete topological space depicted, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$ [26, 31]. From now on, an n-dimensional discrete topological space $X \subset \mathbf{Z}^n$ is considered with one of the k-adjacency relations of $\mathbf{Z}^n$ in a digital picture $(\mathbf{Z}^n, k, \bar{k}, X)$, $n \geq 1$, where $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$, and each k is one of the k-adjacency relations of $\mathbf{Z}^n$ in (2.1). Obviously, the space $X \subset \mathbf{Z}$ is considered in a digital picture $(\mathbf{Z}, 2, 2, X)$. We say that the pair (X, k) is a *space with k-adjacency* (briefly, *space* if not confused).

As a generalization of the commonly used 4- and 8-adjacency of $\mathbf{Z}^2$, and 6-, 18-, and 26-adjacency of $\mathbf{Z}^3$, the adjacency relations of $\mathbf{Z}^n$, $n \geq 1$, were established in [10] (see also [6, 7, 8, 9, 11, 12, 13]) as follows.

For a positive integer m with $1 \leq m \leq n$, two distinct points $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n) \in \mathbf{Z}^n$ are adjacent according to m if

- there are at most m distinct indices i such that $|p_i - q_i| = 1$; and
- for all indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In the following, this criterion consisting of the two conditions is called $(CON\star)$ [8, 10, 18] (see also [6, 7, 11, 12, 14]). More precisely, by $N_k(p)$ we denote the set of the points $q \in \mathbf{Z}^n$ which are adjacent to a given point p according to $(CON\star)$ and the number $k := k(m,n)$ is the cardinal number of $N_k(p)$ called the *k-neighbors of p* [26]. Consequently, given a natural number m in $(CON\star)$ with $1 \leq m \leq n$ determines each of the following k-adjacency relations of $\mathbf{Z}^n$ in terms of $(CON\star)$ [8] (see also [6, 7, 9, 10, 11, 12, 13]).

$$k \in \left\{ \begin{matrix} 2n, & n \geq 1, \\ 3^n - 1, & n \geq 2, \\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1, 2 \leq r \leq n-1, n \geq 3. & \end{matrix} \right\} \tag{2.1}$$

For example, $(n,k) \in \{(4,8),\ (4,32),\ (4,64),\ (4,80); (5,10), (5,50), (5,130), (5,210), (5,242)\}$ [6, 7, 8, 9, 10, 11].

For $a, b \in \mathbf{Z}$ with $a \precsim b$, the set $[a,b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \leq n \leq b\}$ is called a *digital interval* [3]. We say that a *k-path* from x to y in X is a sequence $(x = x_0, x_1, x_2, \dots, x_{m-1}, x_m = y)$ in X such that each point x_i is k-adjacent to x_{i+1} for $m \geq 1$ and $i \in [1, m-1]_{\mathbf{Z}}$ [26]. The number m is called the *length* of this path [23]. If $x_0 = x_m$, then the k-path is said to be *closed k-curve* [26]. For a space (X,k), two points $x, y \in X$ are k-connected [31] if there is a k-path from x to y in X, and if any two points in X are k-connected, then X is called *k-connected.* For an adjacency relation k, a *simple k-path* with m elements in $\mathbf{Z}^n$ is assumed to be a sequence $(x_i)_{i \in [0,m-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [26]. Furthermore, a *simple closed k-curve* with l elements in $\mathbf{Z}^n$ is a k-sequence $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$ derived from a simple k-curve $(x_i)_{i \in [0,l]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k-adjacent if and only if $j = i+1 \pmod{l}$ or $i = j+1 \pmod{l}$ [3] (see also [6, 7, 8, 9, 10, 11, 12]). We denote by $SC_k^{n,l}$ a simple closed k-curve with l elements in $\mathbf{Z}^n$ [7, 18] (see also [9, 10, 18]).

The following *digital k-neighborhood* has been used to define digital (k_0, k_1)-continuity [8, 18] (see also [9, 10, 11, 12, 13, 14, 15, 16, 17]).

Definition 1 ([10, 18], see also [6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17]). For a space (X,k) in $\mathbf{Z}^n$ and $\varepsilon \in \mathbf{N}$, we say that the k-neighborhood of $x_0 \in X$ with radius ε is the set

$$N_k(x_0, \varepsilon) = \{x \in X | \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a *shortest simple k-path* from x_0 to x in X.

Motivated by the digital continuity in [3, 31], the following characterizes digital continuity in a fashion used later in the paper and plays an important role in studying (k_0, k_1)-covering theory [10, 12, 13] and digital k-curve and closed k-surface theory [1, 3, 12, 13, 28, 29, 31].

Proposition 2.1 ([8, 18], see also [9, 10, 11, 12]). *Let (X, k_0) and (Y, k_1) be spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x_0 \in X, \varepsilon \in \mathbf{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $\delta \in \mathbf{N}$ such that the corresponding $N_{k_0}(x_0, \delta) \subset X$ satisfies $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$.*

The (k_0, k_1)-continuity in Proposition 2.1 implies the digital (k_0, k_1)-continuity in [3]. Furthermore, (k_0, k_1)-continuity has uniform continuity property [13]: Let $f : X \to Y$ be a (k_0, k_1)-continuous map. Then, for every point $x_0 \in X$, $\varepsilon \in \mathbf{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $N_{k_0}(x_0, 1) \subset X$ satisfying $f(N_{k_0}(x_0, 1)) \subset N_{k_1}(f(x_0), \varepsilon)$. Consequently, by the uniform continuity of the digital (k_0, k_1)-continuity in [13], we obtain the following.

Remark 2.2. [13] In Proposition 2.1, we may take $\delta = 1 = \varepsilon$.

3. Digital k-graph theoretical methods in digital topology

In [19] (see also [11, 13, 21]), digital graph versions of the (k_0, k_1)-continuity, the (k_0, k_1)-homeomorphism, the (k_0, k_1)-covering, and the (k_0, k_1)-homotopy in digital topology were developed. Let us introduce some necessary terminology for digital k-graph theory. A digital graph G on $\mathbf{Z}^n$ is considered in a quadruple $(\mathbf{Z}^n, k, \bar{k}, G)$, where $n \in \mathbf{N}$, G is a digital graph depicted on $\mathbf{Z}^n$, k represents an adjacency relation for G [19]. Indeed, a space (X, k) can be recognized to be a *digital graph with k-adjacency* called *a digital k-graph* [19]. In other words, we say that a *digital k-graph* is a graph on $\mathbf{Z}^n$ with k-adjacency and write it as $G_k = (V_k, E_k)$ consisting of both V_k and E_k which are the sets of vertices and *k-edges uv*, respectively, where the k-edge uv is considered in such a way: $u \in N_k(v) = \{u | u \text{ is } k\text{-adjacent to } v\}$ and $N_k(v)$ is the k-neighbors of v in $\mathbf{Z}^n$ [26]. Recently, in [19] (see also [11, 13]), for a space (X, k) in $\mathbf{Z}^n$, motivated by the geometric realization of $X \subset \mathbf{Z}^3$ in [4], the notion of *geometric realization* of (X, k) was developed in $\mathbf{Z}^n, n \geq 3$. The geometric realization of (X, k) in $\mathbf{Z}^n$ is the simplicial complex $S(G_k)$ realized by the digital k-graph G_k derived from (X, k) (see (3.1)). Moreover, the paper [19] (see also [11]) showed that a (k_0, k_1)-continuous map $f : (X, k_0) \to (Y, k_1)$ induces the (k_0, k_1)-homomorphism $G(f) : G_{k_0} \to G_{k_1}$ characterizing the simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$ [19]. Consequently, the following (3.1) shows the process of the geometric realization in [11, 13, 19].

$$(X, k) \to G_k \to S(G_k) := |X|. \tag{3.1}$$

$|X| := S(G_k)$ in (3.1) is called the *geometric realization* of (X, k).

Remark 3.1 ([19], see also [11, 13, 21]). A space (X, k) can be recognized to be a digital k-graph $G_k = (V_k, E_k)$ or a simplicial complex $|X| := S(G_k)$ geometrically realized by (X, k).

By Remark 3.1, we can represent a (k_1, k_0)-homoemeomorphism in [3] (see also [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]) from a digital k-graph theoretical point of view, as follows.

Definition 2 ([19], see also [3, 11, 13, 21]). For two spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism* and use the notation $X \approx_{k_0} Y$.

In the following, by Remark 3.1, we obtain the following.

Remark 3.2 ([19], see also [11, 13, 21]). Since the digital graph theoretical approach is so convenient to study a space, hereafter, we use a (k_0, k_1)-isomorphism instead of a (k_0, k_1)-homeomorphism.

By some properties of a (k_0, k_1)-isomorphism, we obtain the following.

Theorem 3.3 ([13]). *If a (k_0, k_1)-isomorphism $h : X \to Y$ has $h(N_{k_0}(x, \delta)) = N_{k_1}(h(x), \varepsilon)$ for some $N_{k_0}(x, \delta) \subset X$, then $\delta = \varepsilon$.*

By Remarks 3.1 and 3.2, we may consider that the graph (k_0, k_1)-homotopy in [19] is equivalent to (k_0, k_1)-homotopy in [3, 10, 11, 12, 13]. For (X, k), consider a subset $(A, k) \subset (X, k)$. In relation to the strong k-deformation retract in [12, 21], motivated by the pointed (k_0, k_1)-homotopy in [3], the following notion of (k_0, k_1)-*homotopy relative to* A was established in [9] (see also [12, 21]). For a space (X, k) and a set $A \subset X$, we call $((X, A), k)$ a space pair with k-adjacency. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed space [3]. For a subspace $(A, k) \subset (X, k)$, we obtain a *discrete* (*or digital*) *homotopy relative to* A. Furthermore, the current notion of digital homotopy relative to A can be used to establish a strong k-deformation retract related to a k-homotopic thinning.

Definition 3 ([12, 13, 21]). Let (X, k_0) and (Y, k_1) be spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively, and $A \subset X$. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose that there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, k_1)$-continuous for all $t \in [0, m]_{\mathbf{Z}}$;
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is (k_0, k_1)-continuous for all $x \in X$.

Then, F is called a (k_0, k_1)-homotopy between f and g, and f and g are (k_0, k_1)-homotopic in Y and further, we use the notation $f \simeq_{(k_0,k_1)} g$ and $f \simeq_{k_0} g$ if $k_0 = k_1$.

• Furthermore, for all $t \in [0,m]_{\mathbf{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0,m]_{\mathbf{Z}}$.

Then, we call F a (k_0, k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0, k_1)-homotopic relative to A in Y. Besides, we use the notation $f \simeq_{(k_0,k_1)rel.A} g$.

Remark 3.4. In order to make the notion of (k_0, k_1)*-homotopy relative to some space* (A, k) clear, the fourth bullet item in Definition 2 of [9] and Definition 1 of [12] was changed into the current fourth bullet item in Definition 3.

In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy at $\{x_0\}$ [3]. If the identity map 1_X is (k,k)-homotopic relative to $\{x_0\}$ in X to a constant map with image consisting of some $x_0 \in X$, then we say that (X, x_0) is *pointed k-contractible* [3]. Owing to the $(3^n - 1)$-contractibility of $SC^{n,4}_{3^n-1}$ [17], we observe that the current k-contractibility is different from the contractibility in Euclidean topology [3, 17, 18].

Motivated by the digital k-fundamental group in [23], the (digital) k-fundamental group was induced from the pointed $(2,k)$-homotopy in [3]. To be specific, for a pointed space (X, x_0), a k-loop based at x_0 is a $(2,k)$-continuous function $f : [0,m]_{\mathbf{Z}} \to X$ with $f(0) = x_0 = f(m)$. The number m depends on the loop; different loops are allowed to have different digital interval domains. More precisely, we compare the homotopy properties of loops whose domains *may have different cardinality. Namely, if $m_f \leq m_{f'}$, we can obtain a trivial extension* of a loop $f : [0,m_f]_{\mathbf{Z}} \to X$ to a loop $f' : [0,m_{f'}]_{\mathbf{Z}} \to X$ in by

$$f'(t) = \begin{cases} f(t) & \text{if } \ 0 \leq t \leq m_f; \\ f(m_f) & \text{if } \ m_f \leq t \leq m_{f'}. \end{cases}$$

We have $g \in [f]$ if and only if there is a homotopy, holding the endpoints fixed, between *trivial extensions* F, G of f, g, respectively [3]. Let $F^k(X, x_0) = \{f | f$ is a k-loop based at $x_0\}$. For members $f : [0,m_1]_{\mathbf{Z}} \to X$, $g : [0,m_2]_{\mathbf{Z}} \to X$ in $F^k(X, x_0)$, in [23], we obtain a map $f * g : [0, m_1 + m_2]_{\mathbf{Z}} \to X$ given by

$$f * g(t) = \begin{cases} f(t), & 0 \leq t \leq m_1; \\ g(t - m_1), & m_1 \leq t \leq m_1 + m_2. \end{cases}$$

The k-homotopy class of a pointed loop f is denoted by $[f]$. The Khamlimsky operation $*$ preserves homotopy classes in the sense that if $f_1, f_2, g_1, g_2 \in F^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, i.e., $[f_1 * g_1] = [f_2 * g_2]$ [3, 23]. Then,

$$\pi^k(X, x_0) = \{[f] | f \in F^k(X, x_0)\}$$

is a group with the operation $[f] \cdot [g] = [f * g]$ [3, 23] which is called the *k-fundamental group* of (X, x_0) [3].

If x_0 and x_1 belong to the same k-connected component of X, then $\pi^k(X, x_0)$ and $\pi^k(X, x_1)$ are isomorphic to each other [3]. Furthermore, it is shown that a (k_0, k_1)-isomorphism $h : (X, x_0) \to (Y, y_0)$ induces a digital fundamental group isomorphism $h_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ defined by $h_*([f]) = [h \circ f]$ for $[f] \in \pi^{k_0}(X, x_0)$ [3]. It is clear that a (k_0, k_1)-isomorphism preserves the pointed k_0-contractibility to the pointed k_1-contractibility. Also, it was proved that if X is pointed k-contractible, then $\pi^k(X, x_0)$ is trivial [3]. Besides, a k-connected space (X, k) is called *simply k-connected* if $\pi^k(X)$ is trivial [18].

4. Strong k-deformation retract and k-homotopic thinning

As a generalization of a closed spline in $\mathbf{Z}^3$, let $C_k^{n,l}$ be a closed k-curve with l points in $\mathbf{Z}^n$. In this paper, we study $C_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ as a sequence such that c_i and c_j are *k-adjacent* if $j = i \pm 1(mod\, l)$ and with a further condition that for each $i \in [0, l-1]_{\mathbf{Z}}$ each of the index sets

$$(\bigstar) \qquad I_k(i) = \{t | c_t \in N_k(c_i, 1) \subset C_k^{n,l}\} \text{ is consecutive modulo } l,$$

where $N_k(c_i, 1)$ is the k-neighborhood of c_i with radius 1 in Definition 1. For instance, consider the space $X := (c_i)_{i\in[0,11]_{\mathbf{Z}}}$ in Figure 1(a). Let us examine the two points c_2 and c_7, then we see that

$$I_8(2) = \{1, 2, 3, 7\}, I_8(7) = \{1, 2, 3, 6, 7, 8\}$$

are not consecutive modulo 12. In this paper, such a kind of the space X in Figure 1(a) will not be considered as a closed 8-curve in relation to Theorems 4.9 and 7.1, Remark 4.10, and so forth.

Example 4.1. The space in Figure 1(b) is a typical closed 8-curve instead of a simple closed 8-curve.

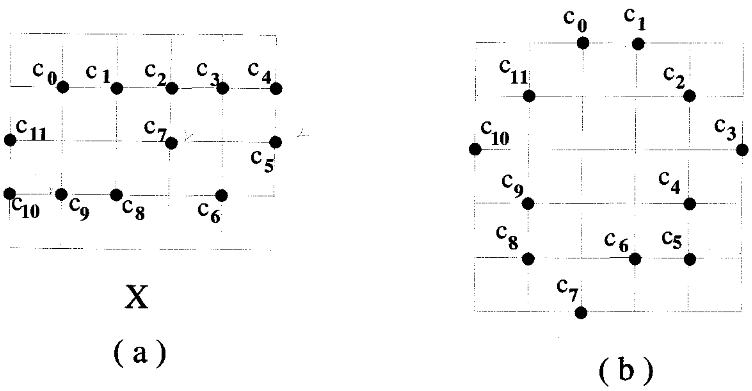

FIGURE 1. Configuration of $C_8^{2,12}$

Remark 4.2. In general, considering Cartesian products $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$ and $C_{k_1}^{n_1,l_1} \times C_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$ with some k-adjacency of $\mathbf{Z}^{n_1+n_2}$, we always assume that $SC_{k_j}^{n_j,l_j} := (c_i)_{i\in[0,l_j-1]_{\mathbf{Z}}}$ and $C_{k_j}^{n_j,l_j} := (c_i)_{i\in[0,l_j-1]_{\mathbf{Z}}}$ satisfy the following: For each $i \in [0, l_j - 1]_{\mathbf{Z}}$ and $j \in \{1, 2\}$, each of the index sets

$$I_k(i) = \{t|c_t \in N_k(c_i, 1)\} \text{ is consecutive modulo } l_j, \tag{4.1}$$

where a k-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ and $C_{k_1}^{n_1,l_1} \times C_{k_2}^{n_2,l_2}$ can be considered and is determined by some number m with $m \in [\max\{m_1, m_2\}, n_1 + n_2]_{\mathbf{Z}}$ via $(CON\star)$ and each number $m_j \in \mathbf{N}$ is taken from the k_j-adjacency of $SC_{k_j}^{n_j,l_j}$ and $C_{k_j}^{n_j,l_j}$ via $(CON\star)$. In addition, if the number $m \geq n_j$, then take $I_k(i) = I_{3^{n_j}-1}(i)$ in (4.1). In particular, we remind that the k-adjacency of this product set is different from the compatible k-adjacency of a digital product image in [18].

In [2], the notion of closed k-surface in $\mathbf{Z}^3$ was established. Let us now introduce some necessary terminology. A point $x \in X$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [2]. The k-corner x is called *simple* if y, z are not k-corners and if x is the only point k-adjacent to both y, z [2]. X is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve [29]. For a k-connected digital image (X, k) in $\mathbf{Z}^3$, we recall the following: $|X|^x := N_{26}^*(x) \cap X$, $N_{26}^*(x) = \{x'|x$ and x' are 26-adjacent$\}$ [2]. Thus we can restate $|X|^x := N_{26}(x, 1) - \{x\}$ in $\mathbf{Z}^3$ by Definition 1.

Definition 4 ([2]). Let (X, k) be a k-connected digital image in $\mathbf{Z}^n, n = 3$, and $\bar{X} = \mathbf{Z}^n - X$. Then X is called a *closed k-surface* if it satisfies the following.

(1) In case that $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$, where $k \neq 3^n - 2^n - 1$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one k-component k-adjacent to x;

(b) $|\bar{X}|^x$ has exactly two $\bar{k}$-components $\bar{k}$-adjacent to x; we denote by $C^{x\,x}$ and $D^{x\,x}$ these two components; and

(c) for any point $y \in N_k(x) \cap X$, $N_{\bar{k}}(y) \cap C^{x\,x} \neq \phi$ and $N_{\bar{k}}(y) \cap D^{x\,x} \neq \phi$, where $N_k(x)$ means the k-neighbors of x.

Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.

(2) In case that $(k, \bar{k}) = (3^n - 2^n - 1, 2n)$, then for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve. Furthermore, if the image $|X|^x$ is a simple closed k-curve, then the closed k-surface X is called simple.

Remark 4.3. (correcting of an example of a closed k-surface)

(1) As a generalization the digital k-surface in Definition 4, in [21], Definition 4 was stated for $n \in \mathbf{N}$ with $n \geq 3$. Furthermore, in [21] (see Theorem 5.2), it was asserted that a Cartesian product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ is a simple closed

k-surface in $\mathbf{Z}^{n_1+n_2}$ if $m_1 = m_2$, where the k-adjacency of $\mathbf{Z}^{n_1+n_2}$ is determined by the number m_2 and $(CON\star)$, and m_i is also determined by the k_i-adjacency of $\mathbf{Z}^{n_i}$ via $(CON\star)$, $i \in \{1,2\}$. But, there is an error of the assertion because it cannot satisfy the condition (1)(b). For instance, examine that $SC_8^{2,6} \times SC_8^{2,6}$ cannot be a closed 32-surface. Thus, in Theorem 5.2 in [21], Theorem 5.4 in [20], and Example 2 in [12], Theorems 6.1 and 6.2 of [21], the terminology '*a closed k-surface*' should be corrected into a '*digital image*' (X, k).

(2) (correcting the $(3^{n_1+n_2}-1)$-adjacency in [12]) The paper [12] calculated $\pi^{3^{n_1+n_2}-1}(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$ with a $(3^{n_1+n_2}-1)$-homotopic thinning and some property of a digital covering map (see Theorems 2, 3, and 4 in [12]). Then, we remind that for $SC_{3^{n_j}-1}^{n_j,l_j} := (c_i)_{i\in[0,l_j-1]_{\mathbf{Z}}}, j \in \{1,2\}$, and each $i \in [0, l_j - 1]_{\mathbf{Z}}$, we assume that each $I_{3^{n_j}-1}(i) = \{t \mid c_t \in N_{3^{n_j}-1}(c_i, 1)\}$ is consecutive modulo l_j.

The minimal simple closed k-curves in $\mathbf{Z}^n, n \geq 2$, are now investigated with relation to the k-fundamental group, a k-isomorphism, and k-contractibility [8, 9, 10, 11, 12, 13]. For example, in $\mathbf{Z}^2$ three types of minimal simple closed curves, MSC_4, MSC_8, and MSC_8' are shown as follows [6, 7, 8, 9, 10, 11, 12], where the *minimal space* means a space with the minimal cardinality in relation to both the k-contractibility and the space containing the given digital image: (1) Let MSC_4 be a set which is 4-isomorphic to the space,

$$\{(0,0),(1,0),(2,0),(2,1),(2,2),(1,2),(0,2),(0,1)\} := (a_i)_{i\in[0,7]_{\mathbf{Z}}}. \tag{4.2}$$

(2) Let MSC_8 be a set which is 8-isomorphic to the space,

$$\{(0,0),(1,-1),(2,-1),(3,0),(2,1),(1,1)\} := (b_i)_{i\in[0,5]_{\mathbf{Z}}}. \tag{4.3}$$

(3) Let MSC_8' be a set which is 8-isomorphic to the space,

$$\{(0,0),(1,1),(0,2),(-1,-1)\} := (c_i)_{i\in[0,3]_{\mathbf{Z}}}. \tag{4.4}$$

Indeed, while MSC_8 is not 8-contractible in [8, 9, 10, 11, 12], MSC_4 and MSC_8' are 8-contractible in [3, 18] (see also [6, 7, 8, 9, 10, 11, 13]). Thus, we see the following [3, 18]:

$$\pi^8(MSC_4) \quad \text{and} \quad \pi^8(MSC_8') \quad \text{are trivial.} \tag{4.5}$$

In order to classify spaces in $\mathbf{Z}^n$, the following notion of digital k-homotopy equivalence was developed in [16].

Definition 5 ([16], see also [19, 21]). For two spaces (X,k) and (Y,k) in $\mathbf{Z}^n$, if there are k-continuous maps $h : X \to Y$ and $l : Y \to X$ such that the compositions $l \circ h \simeq_{k\cdot h} 1_X$ and $h \circ l \simeq_{k\cdot h} 1_Y$, then the map $h : X \to Y$ is called a *k-homotopy equivalence*. Then, we use the notation, $X \simeq_{k\cdot h\cdot e} Y$.

By Remark 3.1, we can generalize the k-homotopy equivalence in [16], as follows.

Definition 6 ([19]). For two spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, if there are both a (k_0, k_1)-continuous map $h : X \to Y$ and a (k_1, k_0)-continuous map $l : Y \to X$ such that $l \circ h \simeq_{k_0 \cdot h} 1_X$ and $h \circ l \simeq_{k_1 \cdot h} 1_Y$, then the map $h : X \to Y$ is called a (k_0, k_1)-homotopy equivalence. Then, we use the notation, $X \simeq_{(k_0,k_1)\cdot h \cdot e} Y$. Besides, if $k_0 = k_1$, then we use the notation, $X \simeq_{k_0 \cdot h \cdot e} Y$.

By the use of the minimal simple closed 4- and 8-curves in (4.2), (4.3), and (4.4), we obtain the following.

Cartesian products in Example 4.4 are considered as digital images instead of closed k-surfaces in $\mathbf{Z}^4$.

Example 4.4 ([16, 20]). (1) A digital image $MSC_4 \times MSC_4 \subset \mathbf{Z}^4$ cannot be 8-homotopy equivalent to $MSC_8 \times MSC_4 \subset \mathbf{Z}^4$.

(2) $MSC'_8 \times MSC_4$ cannot be $(k, 32)$-homotopy equivalent to $MSC_8 \times MSC'_8$, $k \in \{8, 32\}$.

In relation to a *k-simple point* [1], we need to introduce several terminologies as follows. If (X, k) is finite in $(\mathbf{Z}^n, k, \bar{k}, X)$, the infinite $\bar{k}$-component of $\bar{X}$ is the $\bar{k}$-background. The other $\bar{k}$-connected components of $\bar{X}$ are called the $\bar{k}$-*cavities*. The presence of a *k-hole* in X is detected whenever there is a closed k-path in X that cannot be *k-deformed* in X to a simple path [28]. The notion of k-deformation in relation to the *k-hole* is taken from [28]. For example, in $\mathbf{Z}^3$ a solid k-torus has no $\bar{k}$-cavities and one k-hole, a hollow k-torus has one $\bar{k}$-cavity and two k-holes [29].

Definition 7 ([1], see also [21]). A simple k-point in a space $(X, k) \subset \mathbf{Z}^n$ is a point $x \in X$ the deletion of which preserves the topology of the space X, *i.e.*, there are bijections between the k-components, the $\bar{k}$-cavities, the k-holes of X and those of $X - \{x\}$, respectively.

For a space (X, k), a deleting process of a simple k-point of (X, k) is called a *k-thinning* in [26]. In this paper, for C_k^{n,l_1} not k-contractible, a k-homotopic thinning algorithm for calculating the k-fundamental group of C_k^{n,l_1} is established in terms of the deletion of simple k-curve points in C_k^{n,l_1}. Let us now state a simple k-curve point as follows. We again remind that each $C_k^{n,l}$ in this paper is assumed to satisfy the above condition ★.

Definition 8. For $C_k^{n,l}$, we say that a point $x \in C_k^{n,l}$ is a simple k-curve point if it is a simple k-point in $C_k^{n,l}$.

The deletion of simple k-curve points in $C_k^{n,l}$ not k-contractible is called a *k-thinning* of $C_k^{n,l}$ [26]. We can easily see the following.

Theorem 4.5. *A point $c_i \in C_k^{n,l} := (c_i)_{i \in [0,m-1]_{\mathbf{Z}}}$ is a simple k-curve point if $\sharp(N_k(c_{i-1(mod\, l)}, 1) \cap N_k(c_{i+1(mod\, l)}, 1)) \geq 3$, where $\sharp$ means the cardinal number of the set.*

Proof. For $C_k^{n,l} := (c_i)_{i\in[0,m-1]_{\mathbf{Z}}}$, delete each point $c_i \in C_k^{n,l}$ such that

$$\sharp(N_k(c_{i-1(mod\, l)}, 1) \cap N_k(c_{i+1(mod\, l)}, 1)) \geq 3.$$

Then, the remaining set

$$C_k^{n,l} - \{c_i | \sharp(N_k(c_{i-1(mod\, l)}, 1) \cap N_k(c_{i+1(mod\, l)}, 1)) \geq 3\}$$

is obviously a simple closed k-curve in $\mathbf{Z}^n$. Thus this implies the proof of the assertion. □

The current theorem will be used to write an algorithm for calculating the k-fundamental group of a closed k-curve in $\mathbf{Z}^n$ (see Section 9). Consider the space in Figure 1(b). Then, we see that the point c_5 is a simple 18-curve point in $C_8^{2,12}$.

Suppose that (X, A) is a space pair with k-adjacency and $i : A \to X$ is the inclusion map, A is called a *k-retract* of X if and only if there is a k-continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ [3]. Then, the map r is called a k-retraction of X onto A. While the paper [15] referred to a k-deformation, the presentation of the notion was insufficient. The correct one is the following.

Definition 9 ([21], correcting of the (strong) k-deformation in [15]). For a space pair $((X, A), k)$, A is said to be a strong k-deformation retract of X if there is a k-retraction r of X onto A such that $F : i \circ r \simeq_{k \cdot rel. A} 1_X$.

Then, a point $x \in X - A$ is called strong k-deformation retractable.

In view of Definition 9, Theorem 3 in [15] should be represented as follows.

Theorem 4.6 ([12, 15, 21]). *If (A, x_0) is a strong k-deformation retract of (X, x_0), then $\pi^k(X, x_0)$ is isomorphic to $\pi^k(A, x_0)$.*

Definition 10 ([21]). For a space (X, k), we can delete strong k-deformation retractable points from X. This processing is called a k-homotopic thinning.

Example 4.7. See the space Y in Figure 2 in Section 8. Then, the space $\{a_0, a_4, a_5, a_6, a_7, a_8\} := Z$ in Figure 2 is an 8-homotopic thinning space from Y (see Remark 4.12).

Remark 4.8. A k-homotopic thinning is different from a k-thinning. Precisely, a k-homotopic thinning implies a k-thinning (see Figure 1(a)). But, the converse need not hold.

The current notion of strong k-deformation retract can be used to calculate the k-fundamental group of a space (X, k) in $\mathbf{Z}^n$ (see Theorems 7.1 and 8.4) and further, to write an algorithm for calculating the k-fundamental group in Section 8.

Theorem 4.9. *For C_k^{n,l_1} not k-contractible, $SC_k^{n,l} \subset C_k^{n,l_1}$ with $l \leq l_1$ is obtained by a k-homotopic thinning.*

Proof. Obviously, each simple k-curve point can be deleted from C_k^{n,l_1} by a k-homotopic thinning. Furthermore, by the processing we see that $C_k^{n,l_1} - \{c_i | c_i$ is a simple k-curve point$\}$ is a strong k-deformation retract of C_k^{n,l_1}. □

Example 4.10. (1) Consider the set X in Figure 1(a). Then, *the set* $\{c_2, c_4, c_{10}\}$ can be deleted in terms of an 8-thinning from the space X.

(2) In Figure 1(b), $C_8^{2,12}$ is 8-thinned to be $SC_8^{2,11} = C_8^{2,12} - \{c_5\}$ by deleting the point c_5.

Remark 4.11. For C_k^{n,l_1} not k-contractible, the complete k-thinning space can be k-isomorphic to the k-homotopic thinning space (*see* Remarks *4.12 and 5.6*).

We observe the difference between a k-thinning and a k-homotopic thinning as follows.

Remark 4.12. By Example 4.10(1) we see that the space X in Figure 1(a) can be 8-thinned to be a space $X - \{c_2, c_4, c_{10}\}$.

Meanwhile, doing an 8-homotopic thinning of the space X in Figure 1(a), we obtain a space $A := \{c_0, c_1, c_7, c_8, c_9, c_{11}\} \approx_8 MSC_8$: Consider a map $H : X \times [0,3]_{\mathbf{Z}} \to X$ in such a way:

First, $H(c_i, 0) = c_i$, for any $c_i \in X, i \in [0,11]_{\mathbf{Z}}$;

Second, $H(c_{10}, 1) = c_9, H(c_4, 1) = c_5, H(c_2, 1) = c_7, H(c_i, 1) = c_i, i \in [0,11]_{\mathbf{Z}} - \{2,4,10\}$;

Third, $H(c_{10}, 2) = c_9$, $H(c_4, 2) = H(c_5, 2) = c_6$, $H(c_2, 2) = H(c_3, 2) = c_7$, $H(c_i, 2) = c_i$, $i \in [0,11]_{\mathbf{Z}} - \{2,3,4,5,10\}$;

Finally, $H(c_i, 3) = c_7, i \in \{2,3,4,5,6,7\}, H(c_{10}, 3) = c_9, H(c_j, 3) = c_j, j \in \{0,1,8,9,11\}$.

Then, we see the map H is an (8,8)-homotopy relative to A, where $A = \{c_0, c_1, c_7, c_8, c_9, c_{11}\} \approx_8 MSC_8$. Concretely, we see that A is a strong 8-deformation retract of X. Consequently, doing an 8-homotopic thinning of X in Figure 1(a), we obtain the space $A \approx_8 MSC_8$. Fourth, we see that $\pi^8(X, c_7)$ is isomorphic to $\pi^8(A, c_7)$ by Theorem 4.6. Meanwhile, for the space in Figure 1(b), an 8-thinning space can be equal to an 8-homotopic thinning space, which is related to the algorithm for the calculation of the k-fundamental group *of* $C_k^{n,l}$ *in* Section *9*.

For a space (X, k), proceeding X with a k-homotopic thinning, we calculate the k-fundamental group of a space (X, k). We now recall some properties of a (k_0, k_1)-isomorphism as follows.

Theorem 4.13 ([17]). (1) *If* $h : X \to Y$ *is a* (k_0, k_1)*-isomorphism, then* $X - \{p\} \approx_{(k_0,k_1)} Y - \{h(p)\}$ *for any point* $p \in X$.
(2) *Let* $h : X \to Y$ *be a* (k_0, k_1)*-isomorphism. For any subspace of* $X_0 \subset X$*, the restriction map* h *on* X_0*, briefly* $h|_{X_0} : X_0 \to h(X_0)$*, is a* (k_0, k_1)*-isomorphism.*

5. Strongly local (k_0, k_1)-isomorphism in relation to the k-fundamental group

In this section, motivated by a local (k_0, k_1)-homeomorphism in [6], a strongly local (k_0, k_1)-isomorphism is presented from a digital k-graph theoretical point of view and studied in relation to the digital k-fundamental group.

Definition 11 ([6], see also [13]). For two spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a (k_0, k_1)-continuous map $h : X \to Y$ is called a strongly local (k_0, k_1)-isomorphism if for any $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1) \subset Y$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a strongly local k_0-isomorphism.

Example 5.1 ([6]). A map $f : \mathbf{N} \cup \{0\} \to MSC_8$ defined by $f(i) = b_{i(\text{ mod } 6)} \in MSC_8$ in (4.3) is not a strongly local $(2, 8)$-isomorphism.

Proof. At the point $0 \in \mathbf{N} \cup \{0\}$, a strongly local $(2, 8)$-isomorphism is invalid. □

By Example 5.1, we obtain the following: For spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, if a map $h : X \to Y$ is a (k_0, k_1)-isomorphism, then the map h is a strongly local (k_0, k_1)-isomorphism. But the converse does not hold.

Theorem 5.2 ([6]). *Let (X, k_0) and (Y, k_1) be spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A map $f : X \to Y$ is a strongly local (k_0, k_1)-isomorphic bijection if and only if f is a (k_0, k_1)-isomorphism.*

Theorem 5.3. *The composition of strongly local isomorphisms is also a strongly local isomorphism.*

Proof. If $f : X \to Y$ is a strongly local (k_0, k_1)-isomorphism and $g : Y \to Z$ is a local (k_1, k_2)-isomorphism, then $g \circ f : X \to Z$ is a strongly local (k_0, k_2)-isomorphism. □

A strongly local (k_0, k_1)-isomorphism can be used to investigate some local properties of a space in relation to the preservation of a simple k_0-curve point into a simple k_1-curve point and a generalized simple closed k_i-curve, $i \in \{0, 1\}$.

Example 5.4. Consider a map $f : [0, 3]_{\mathbf{Z}} \to MSC'_8$ defined by $f(i) = c_i \in MSC'_8$ in (4.4). Then, while the map f is a $(2, 8)$-continuous injection, it is not a strongly local $(2, 8)$-isomorphism owing to the invalidity of a strongly local $(2, 8)$-isomorphism at the points 0 and 3.

Example 5.5. Consider a map $g : MSC_4 \to MSC'_8$ given by $g(a_i) = c_{i(\text{ mod } 4)} \in MSC'_8$ in (4.2) and (4.4). Then, the map g is a strongly local $(4, 8)$-isomorphism. While the map g is $(4, 8)$-continuous, it is not injective.

By Examples 5.4 and 5.5 we can observe that the comparison between a strongly local (k_0, k_1)-isomorphism and a (k_0, k_1)-continuous injection cannot be successful.

While a (k_0, k_1)-isomorphism $h : (X, x_0) \to (Y, y_0)$ preserves the k_0-fundamental group into the k_1-fundamental group under an isomorphism [3, 18], a strongly local (k_0, k_1)-isomorphism does not have the property. Precisely, a strongly local (k_0, k_1)-isomorphism need not preserve the k_0-fundamental group into the k_1-fundamental group under an isomorphism. To be specific, let us consider the following: For the unit lattice square X_1 in $\mathbf{Z}^2$, i.e., $X_1 = \{b_0, b_1, b_2, b_3\}$ and each b_i is 4-adjacent just only to $b_{(i+1) \bmod 4}$ and $b_{(i-1) \bmod 4}$ and further, assume $MSC_4 = (a_i)_{i \in [0,7]_{\mathbf{Z}}}$ in (4.2). Then, a map $p : MSC_4 \to X_1$ is assumed in the counter-clockwise direction with an initial point b_0 such that $p(a_i) = b_{i(\bmod 4)}$ is a strongly local 4-isomorphism. While $\pi^4(MSC_4, a_0)$ is not trivial [18], $\pi^4(X_1, b_0)$ is trivial. More precisely, there is the 4-homotopy on X_1 such that $1_{X_1} \simeq_4 c_{\{b_0\}}$:

Consider the map $H : X_1 \times [0, 2]_{\mathbf{Z}} \to X_1$ in such a way:

First, $H(b_i, 0) = b_i$, for any $b_i \in X_1, i \in [0, 3]_{\mathbf{Z}}$;

Second, $H(b_2, 1) = H(b_1, 1) = b_1, H(b_3, 1) = H(b_0, 1) = b_0$; Finally, $H(b_i, 2) = b_0$, for any $b_i \in X_1, i \in [0, 3]_{\mathbf{Z}}$.

Then, we see the map H is a (4, 4)-homotopy relative to A, where $A = \{b_0\}$. Thus, it turns out that X_1 is 4-contractible so that $\pi^4(X_1, b_0)$ is isomorphic to $\pi^4(\{b_0\}, b_0)$ which is trivial, as required.

Remark 5.6. (1) It turns out that the above space X_1 is 4-homotopically thinned to be a singleton $\{b_0\}$. Meanwhile, doing a 4-thinning of the space X_1, we still have X_1 itself because there is no simple 4-curve point on X_1.
(2) While MSC' is 8-homotopically thinned to be a singleton $\{c_0\}$, MSC' is 8-thinned to be the space MSC' itself because there is no strong 8-deformation retractable point in MSC'.

6. Some properties of a digital (k_0, k_1)-covering

A k-fundamental group of some space has been used to classify spaces in relation to the k-connectivity and a k-homotopic invariant. Furthermore, the notion of digital (k_0, k_1)-covering is also helpful to discriminate spaces and to calculate the digital fundamental group of some space. A digital version of a covering space in algebraic topology in [30] was shown in [10, 18].

By Theorem 3.3 and some properties of a (k_0, k_1)-isomorphism, we obtain the following.

Lemma 6.1. *For two spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, if a (k_0, k_1)-continuous map $h : X \to Y$ has $f(N_{k_0}(x, \delta)) \approx_{k_1} N_{k_1}(h(x), \varepsilon)$, then we see that $\delta = \varepsilon$.*

By the axiom for the (k_0, k_1)-covering in [10, 18], Remark 2.2 and Lemma 6.1, we obtain the following which is equivalent version of a digital (k_0, k_1)-covering [10, 18] and is used later in this paper. Each space (X, k) in Definition 12 is assumed to be k-connected.

Definition 12 ([12], see also [12, 13, 16, 19, 21]). Let (E, k_0) and (B, k_1) be spaces. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose, for any $b \in B$, there exists $\varepsilon \in \mathbf{N}$ such that

(C1) for some index set M,
$p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in p^{-1}(b)$;

(C2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon) = \emptyset$; and

(C3) the restriction map $p|_{N_{k_0}(e_i,\varepsilon)} : N_{k_0}(e_i, \varepsilon) \to N_{k_1}(b, \varepsilon)$ is a (k_0, k_1)-isomorphism for all $i \in M$.

Then, we call the map $p : E \to B$ a (k_0, k_1)-covering map and (E, p, B) is said to be a (k_0, k_1)-covering. Furthermore, a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$ is called a pointed one if $p(e_0) = b_0$.

The collection $\{N_{k_0}(e_i, \varepsilon) | i \in M\}$ is called a partition of $p^{-1}(N_{k_1}(b, \varepsilon))$ into slices. Furthermore, the above k_1-neighborhood $N_{k_1}(b, \varepsilon)$ is called an *elementary k_1-neighborhood of b with radius ε* [12, 13, 19, 21]. Hereafter, we briefly use the terminology *(k_0, k_1)-covering* instead of *digital (k_0, k_1)-covering.*

For example, for a simple closed k-curve with l elements

$$SC_k^{n,l} := (c_t)_{t \in [0, l-1]_{\mathbf{Z}}},$$

$(\mathbf{Z}, p, SC_k^{n,l})$ is a $(2, k)$-covering, where the map $p : \mathbf{Z} \to SC_k^{n,l}$ is given by $p(t) = c_{t(mod\, l)}$ for any $t \in \mathbf{Z}$ [10, 12, 13].

Definition 13 ([10]). A (k_0, k_1)-covering (E, p, B) is called a radius n-(k_0, k_1)-covering if $\varepsilon \geq n$ in Definition 12.

By Remark 2.2 and Theorem 5.2, we obtain the following.

Remark 6.2 ([12], see also [13, 21]). In the (k_0, k_1)-covering of Definition 12, we may take $\varepsilon = 1$. But, for the study of Lemma 6.4 and Theorem 7.1 and various digital topological properties in Sections 6, 7, and 8, we take $\varepsilon \in \mathbf{N}$ in Definition 12 according to the situation.

Definition 14 ([10, 12, 13, 18]). For three spaces (E, k_0) in $\mathbf{Z}^{n_0}$, (B, k_1) in $\mathbf{Z}^{n_1}$, and (X, k_2) in $\mathbf{Z}^{n_2}$, let $p : E \to B$ be a (k_0, k_1)-continuous map. For a (k_2, k_1)-continuous map f from X into B, we say that a digital lifting of f is a (k_2, k_0)-continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

Lemma 6.3 ([10, 18], see also [12, 13, 21]). *For pointed spaces $((E, e_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((B, b_0), k_1)$ in $\mathbf{Z}^{n_1}$, let $p : (E, e_0) \to (B, b_0)$ be a pointed (k_0, k_1)-covering map. Every k_1-path $f : [0, m]_{\mathbf{Z}} \to B$ beginning at b_0 has a unique digital lifting to a k_0-path $\tilde{f}$ in E beginning at e_0.*

The following *digital homotopy lifting theorem* was introduced in [10].

Lemma 6.4 ([10], see also [10, 13, 21]). *Let (E, k_0) be a space and $e_0 \in E$. Let (B, k_1) be a space and $b_0 \in B$. Let $p : (E, e_0) \to (B, b_0)$ be a pointed radius 2-(k_0, k_1)-covering map. For k_0-paths g_0, g_1 in (E, e_0) that start at e_0, if there is a k_1-homotopy in B from $p \circ g_0$ to $p \circ g_1$ that holds the endpoints fixed, then*

g_0 and g_1 have the same terminal point, and there is a k_0-homotopy in E from g_0 to g_1 that holds the endpoints fixed.

Lemma 6.4 will be often used in Sections 7, 8, and 9.

Definition 15 ([18]). A pointed k-connected space (X, x_0) is called *simply k-connected* if $\pi^k(X, x_0)$ is a trivial group.

Theorem 6.5. *Let $p_1 : (E_1, e_1) \to (B, b_0)$ be a pointed (k_1, k_0)-covering map and $p_2 : (E_2, e_2) \to (B, b_0)$ be a pointed (k_2, k_0)-covering map. If there is a pointed (k_1, k_2)-continuous map $\phi : (E_1, e_1) \to (E_2, e_2)$ such that $p_2 \circ \phi = p_1$, then the map ϕ is a pointed (k_1, k_2)-covering map.*

Proof. (Step 1): We prove that the map ϕ is a (k_1, k_2)-continuous surjection. For any $y \in E_2$, we must show that there is a point $x \in E_1$ such that $\phi(x) = y$. Choose a k_2-path $f : [0, m_1]_{\mathbf{Z}} \to E_2$ such that $f(0) = e_2$ and $f(m_1) = y$ and let $g = p_2 \circ f$ in B with the initial point b_0. By Lemma 6.3, there is a unique k_1-path $h : [0, m_1]_{\mathbf{Z}} \to E_1$ with $h(0) = e_1$ as the lifting of g such that $p_1 \circ h = g$. Let x be the terminal point of h. Then, both the k_2-paths $\phi \circ h$ and f have the same initial point e_2 by the hypothesis of ϕ, and $p_2 \circ \phi \circ h = g = p_2 \circ f$. Hence, by Lemma 6.3 $\phi \circ h = f$, we obtain $\phi(x) = y$.

Next, we remind the (k_1, k_2)-continuity of ϕ. By Remark 6.2, for any point $e'_2 \in E_2$ and $N_{k_2}(e'_2, 1) \subset E_2$, we now find some k_1-neighborhood $N_{k_1}(e'_1, 1) \subset E_1$ such that $\phi(N_{k_1}(e'_1, 1)) \subset N_{k_2}(\phi(e'_1), 1)$, where $e'_2 = \phi(e'_1)$. Precisely, for any element $e'_2 \in E_2$, consider $p_2(e'_2) := b \in B$. Furthermore, take $p_1^{-1}(b) := e'_1 \in E_1$ and $N_{k_0}(b, 1)$ such that

$$p_2(N_{k_2}(e'_2, 1)) \approx_{k_0} N_{k_0}(b, 1)$$

which is an elementary k_0-neighborhood of b by the (k_2, k_0)-covering map p_2. Thus, for $N_{k_2}(e'_2, 1) \in p_2^{-1}(N_{k_0}(b, 1))$, since $p_2 \circ \phi = p_1$, we take some $N_{k_1}(e'_1, 1) \in p_1^{-1}(N_{k_0}(b, 1))$ such that $\phi(N_{k_1}(e'_1, 1)) = N_{k_2}(e'_2, 1)$, as required.

(Step 2): For each $e'_2 \in E_2$, we prove the existence of an elementary k_2-neighborhood of e'_2 for the (k_1, k_2)-covering (E_1, ϕ, E_2). Precisely, for the point $e'_2 \in E_2$, take $p_2(e'_2) = b$. Then, there is an elementary k_0-neighborhood of b, $N_{k_0}(b, 1)$, of both the (k_2, k_0)-covering map p_2 and the (k_1, k_0)-covering map p_1. Thus, there is the set $\{E^2_{\alpha_i} | \alpha_i \in M\}$ as a partition of $p_2^{-1}(N_{k_0}(b, 1))$ such that $E^2_{\alpha_i} \cap E^2_{\alpha_j} = \emptyset$ if $\alpha_i \neq \alpha_j \in M$ and $p_2|_{E^2_{\alpha_i}} : E^2_{\alpha_i} \to N_{k_0}(b, 1)$ is a (k_2, k_0)-isomorphism. Similarly, we can take a partition of $p_1^{-1}(N_{k_0}(b, 1))$, $\{E^1_{\alpha_i} | \alpha_i \in M\}$. Then, choose the k_0-neighborhood $N_{k_0}(b, 1)$ and further, we obtain the set $E^2_{\alpha_i}$ containing e'_2 and put

$$E^2_{\alpha_i} \cap p_2^{-1}(N_{k_0}(b, 1)) = U_{\alpha_i}.$$

Then, U_{α_i} is obviously an elementary k_2-neighborhood of e'_2 for the (k_1, k_2)-covering map ϕ with the partition of $\phi^{-1}(U_{\alpha_i})$, as required. □

The paper [13] established the generalization of the *digital lifting theorem* for a pointed (k_2, k_1)-continuous map, $f : (X, x_0) \to (B, b_0)$ as follows.

Theorem 6.6 ([13], Generalized digital lifting theorem). *Let $((E,e_0),k_0)$ and $((B,b_0),k_1)$ be pointed spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively, and let $((X,x_0),k_2)$ be a pointed k_2-connected space in $\mathbf{Z}^{n_2}$. Let $p:(E,e_0)\to(B,b_0)$ be a pointed radius 2-(k_0,k_1)-covering map. Given a pointed (k_2,k_1)-continuous map $f:(X,x_0)\to(B,b_0)$, there is a pointed digital lifting $\tilde{f}:(X,x_0)\to(E,e_0)$ if and only if $f_*(\pi^{k_2}(X,x_0))\subset p_*(\pi^{k_0}(E,e_0))$.*

By Theorems 6.5 and 6.6, we obtain the following.

Corollary 6.7. *Let $p_1:(E_1,e_1)\to(B,b_0)$ be a pointed (k_1,k_0)-covering map and $p_2:(E_2,e_2)\to(B,b_0)$ be a pointed radius 2-(k_2,k_0)-covering map, If $(p_1)_*(\pi^{k_1}(E_1,e_1))\subset(p_2)*(\pi^{k_2}(E_2,e_2))$, then there is a pointed (k_1,k_2)-covering map $\phi:(E_1,e_1)\to(E_2,e_2)$ such that $p_2\circ\phi=p_1$.*

7. Calculation of the k-fundamental group of a closed k-curve in $\mathbf{Z}^n$

In this section, we calculate the k-fundamental group of C_k^{n,l_1} not k-contractible in terms of a strong k-deformation retract and some properties of a digital covering. In [18], we proved that $\pi^8(MSC_8)$ is isomorphic to an infinite cyclic group, precisely $(6\mathbf{Z},+)$ (although, it should be noted the usage of Lemma 6.4 for the proof in [18]). Thus, we need to prove the general case of $\pi^8(MSC_8)$ as follows.

Theorem 7.1. *For a closed k-curve $C_k^{n,l_1}=(c_t)_{t\in[0,l-1]_\mathbf{Z}}$ not k-contractible, $\pi^k(C_k^{n,l_1},c_0)$ is isomorphic to an infinite cyclic group, precisely $(l\mathbf{Z},+)$, where $l=l_1-$ the cardinal number of the set of simple k-curve points in C_k^{n,l_1} and the point c_0 is not a simple k-curve point.*

Before we prove Theorem 7.1, we show the need for the assumption of the non-k-contractibility of C_k^{n,l_1}. Let us consider the space MSC_8' in (4.4). Since $\pi^8(MSC_8')$ is trivial by (4.5), MSC_8' does not satisfy the hypothesis of the non-k-contractibility of C_k^{n,l_1}. Besides, since a $(2,8)$-covering map $p:\mathbf{Z}\to MSC_8'$ given by $p(t)=c_{t(\text{mod } 4)}$ is not a radius 2-$(2,8)$-covering map, which cannot use Lemma 6.4 to assert Theorem 7.1. Thus, the hypothesis of the non-k-contractibility of C_k^{n,l_1} should be required.

Proof. With the hypothesis, by Theorem 4.9, the space $SC_k^{n,l}$ is taken from C_k^{n,l_1} by the strong k-deformation retract of C_k^{n,l_1} with $l\leq l_1$. Furthermore, by Theorem 4.6, $\pi^k(C_k^{n,l_1},c_0)$ is isomorphic to $\pi^k(SC_k^{n,l_1},c_0)$. Let us now prove this theorem more precisely.

(Step 1) Let $H:C_k^{n,l_1}\times[0,m]_\mathbf{Z}\to C_k^{n,l_1}$ be a strong k-deformation of C_k^{n,l_1} onto SC_k^{n,l_1}. In other words, the k-homotopy H is considered in such a way (see also Theorem 3.2 in [21]):

(1) $H(x,0)=x$ for all $x\in C_k^{n,l_1}$

(2) $H(x,m)\in SC_k^{n,l}$ for all $x\in C_k^{n,l_1}$, and

(3) $H_t:X\to X, t\in[0,m]_\mathbf{Z}$, is (k,k)-continuous

(4) for all $x \in C_k^{n,l_1}$, $H_t(x) : [0,m]_{\mathbf{Z}} \to X$ is $(2,k)$-continuous map, and

(5) $H(a,t) = a$ for all $a \in SC_k^{n,l}$ and for all $t \in [0,m]_{\mathbf{Z}}$.

Then, let $r : C_k^{n,l_1} \to SC_k^{n,l}$ be defined in such a way that $(i \circ r)(x) = H(x,m)$, $x \in C_k^{n,l_1}$. Then, H makes $1_{C_k^{n,l_1}}$ be k-homotopic relative to $SC_k^{n,l}$ to $i \circ r$ (see Theorem 4.9). Thus, for any $[g] \in \pi^k(C_k^{n,l_1}, c_0)$, there is a k-path $f \in F^k(SC_k^{n,l}, c_0)$ and a set of k-paths $\{g_1, g_2, \dots, g_m\} \subset F^k(C_k^{n,l_1}, c_0)$ such that $f \simeq_k g_1, g_i \simeq_k g_{i+1}$ for $i \in \{1,2,\dots,m-1\}$ and $g_m \simeq_k g$, where '$\simeq_k$' means a k-homotopy relative to $SC_k^{n,l}$. Consequently, $\pi^k(r)([g]) = [f]$. By the same method above, for any $\pi^k(r)([g_1]) = \pi^k(r)([g_2]) \in \pi^k(SC_k^{n,l_1}, c_0)$, we obtain that $g_1 \simeq_{k\mathrm{rel}.SC_k^{n,l}} g_2$ and finally, $[g_1] = [g_2]$, which $\pi^k(r)$ is injective.

Next, for the k-retraction $r : (C_k^{n,l_1}, c_0) \to (SC_k^{n,l}, c_0)$ such that $r \circ i = 1_{(SC_k^{n,l}, c_0)}$ and the inclusion map $i : (SC_k^{n,l}, c_0) \to (C_k^{n,l_1}, c_0)$ (see Theorem 4.9), $\pi^k(r)$ and $\pi^k(i)$ are homomorphisms. Then, $\pi^k(r \circ i) = \pi^k(r) \circ \pi^k(i) = 1_{\pi^k(SC_k^{n,l}, c_0)}$. Thus, $\pi^k(r)$ is surjective. Therefore, $\pi^k(r)$ is an group isomorphism.

(Step 2) For the space $SC_k^{n,l} := (c_t)_{t \in [0,l-1]_{\mathbf{Z}}}$ not k-contractible, we may choose a base point $c_0 \in SC_k^{n,l}$, and a pointed $(2,k)$-covering map $p : (\mathbf{Z}, 0) \to (SC_k^{n,l}, c_0)$ with $p(t) = c_{t(mod\, l)}$. As a generalization of $\pi^4(MSC_4) \simeq (8\mathbf{Z}, +)$ in [18], we obtain that $\pi^k(SC_k^{n,l}, c_0)$ is isomorphic to a cyclic group, i.e., $(l\mathbf{Z}, +)$. Precisely, for $[f] \in \pi^k(SC_k^{n,l}, c_0)$, take $f \in F^k(SC_k^{n,l}, c_0)$ such that $f : [0, m_f]_{\mathbf{Z}} \to (SC_k^{n,l}, c_0)$ with $f(0) = f(m_f) = c_0$, where the number m_f is some natural number. For any $f, f_1 \in F^k(SC_k^{n,l}, c_0)$ such that $f \simeq_k f_1$, by Lemma 6.3 there are uniquely liftings $\tilde{f}$ of f and $\tilde{f}_1$ of f_1 such that $p\tilde{f} = f, p\tilde{f}_1 = f_1$, respectively. Furthermore, for the 2-paths $\tilde{f}, \tilde{f}_1$ in $\mathbf{Z}$ which have the same initial point 0, if $p\tilde{f} \simeq_k p\tilde{f}_1$ in $(SC_k^{n,l}, c_0)$ and $p\tilde{f}$ and $p\tilde{f}_1$ end at the same point c_0, then $\tilde{f} \simeq_2 \tilde{f}_1$ by Lemma 6.4 and further, f and $\tilde{f}_1$ have the same terminal by Lemma 6.4. Consequently, any k-path f in $(SC_k^{n,l}, c_0)$ beginning at c_0 has a unique digital lifting to 2-path $\tilde{f} : [0, m_f]_{\mathbf{Z}} \to \mathbf{Z}$ beginning at 0. Moreover, the point $\tilde{f}(m_f)$ must be a point of the set $p^{-1}(c_0) = l\mathbf{Z}$ by Lemma 6.4. This integer depends on the k-homotopy class of f. Therefore, we can define $\Phi : \pi^k(SC_k^{n,l}, c_0) \to (l\mathbf{Z}, +)$ by letting $\Phi([f]) = \tilde{f}(m_f) \in p^{-1}(c_0) = l\mathbf{Z}$ and it is well-defined.

We now assert that Φ is an isomorphism. We prove that the map Φ is surjective. For any $ln \in \mathbf{Z}$, because $\mathbf{Z}$ is 2-connected [18], we can choose a 2-path $\tilde{f} : [0, m_f]_{\mathbf{Z}} \to \mathbf{Z}$ from 0 to ln. Define $f = p \circ \tilde{f}$. Then, $f \in F^k(SC_k^{n,l}, c_0)$ and $\tilde{f}$ is its lifting in $\mathbf{Z}$ beginning at 0 and ending at ln. Then, $\tilde{f}(m_f) \in p^{-1}(c_0)$. Thus, by Lemma 6.4, we get $\Phi[f] \in l\mathbf{Z}$.

Next, we prove that the map Φ is injective. Assume that $\Phi([f]) = \Phi([g]) \in l\mathbf{Z}$; let us verify that $[f] = [g] \in \pi^k(SC_k^{n,l}, c_0)$. For a $(2,k)$-covering map $p : (\mathbf{Z}, 0) \to (SC_k^{n,l}, c_0)$, by Lemma 6.3, let $\tilde{f}$, $\tilde{g}$ be the liftings of f, g, respectively.

Then, the two 2-paths $\tilde{f}, \tilde{g}$ on $\mathbf{Z}$ begin at 0, and both $\tilde{f}$ and $\tilde{g}$ end at $lm \in l\mathbf{Z}$. Because $\mathbf{Z}$ is simply 2-connected and $\tilde{f}$ and $\tilde{g}$ are 2-path homotopic keeping the end points fixed by Lemma 6.4. Let $\tilde{F} : [0, lm]_{\mathbf{Z}} \times [0, m_1]_{\mathbf{Z}}$ be the 2-homotopy between $\tilde{f}$ and $\tilde{g}$ for some $m, m_1 \in \mathbf{Z}$. Then, the map $F = p \circ \tilde{F}$ will be a k-homotopy between f and g keeping the endpoints fixed. Thus, $[f] = [g]$, as required.

Let us prove that the map Φ is a homomorphism. For any $[f]$, $[g] \in \pi^k(SC_k^{n,l}, c_0)$, let f, $g \in F^k(SC_k^{n,l}, c_0)$ such that

$f : [0, m_f]_{\mathbf{Z}} \to SC_k^{n,l}$ with $f(0) = c_0 = f(m_f)$,

$g : [0, m_g]_{\mathbf{Z}} \to SC_k^{n,l}$ with $g(0) = c_0 = g(m_g)$.

Let $\tilde{f}$ and $\tilde{g}$ be the digital liftings of f and g to 2-paths on $\mathbf{Z}$ beginning at 0, respectively. Furthermore, $\tilde{f}(m_f) = lm$ and $\tilde{g}(m_g) = ln$. Let us define a path h on $\mathbf{Z}$ by the equations:

$$h(s) = \begin{cases} \tilde{f}(s), & 0 \le s \le m_f, \\ lm + \tilde{g}(s - m_f), & m_f \le s \le m_f + m_g. \end{cases}$$

Then, h is also a 2-path on $\mathbf{Z}$ beginning at 0. We assert that h is a digital lifting of $f * g$, where $*$ means the Khalimsky operation. For any s, we have $p(lm + s) = p(s)$. Besides,

$$p(h(s)) = \begin{cases} p(\tilde{f}(s)) = f(s), & 0 \le s \le m_f, \\ p(lm + \tilde{g}(s - m_f)) & \\ = p(\tilde{g}(s - m_f)) = g(s - m_f), & m_f \le s \le m_f + m_g. \end{cases}$$

Thus, $p \circ h = f * g$ such that h is the digital lifting of $f * g$ which begins at 0. $\Phi([f * g])$ is $h(m_f + m_g)$ which equals $lm + ln$. Then, $\Phi([f \cdot g]) = \Phi([f]) + \Phi([g])$.

Therefore, the proof is completed via Steps 1 and 2. □

As a special case of Theorem 7.1 and a general case of both $\pi^4(MSC_4)$ and $\pi^8(MSC_8)$ in [18], we obtain the following.

Corollary 7.2 ([12, 13]). *$\pi^k(SC_k^{n,l}, c_0)$ is isomorphic to an infinite cyclic group, precisely $(l\mathbf{Z}, +)$, if $SC_k^{n,l}$ is not k-contractible.*

8. k-fundamental group of a wedge product of closed k-curves

The k-fundamental group of a wedge product of the spaces is calculated in terms of the k-homotopy, the k-contractibility, and the digital covering in [10, 13, 21]. We now recall the wedge product of disjoint discrete spaces (X_i, k_i) in $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$. For spaces (X, k) and (Y, k), a *wedge product* of X and Y is presented as follows: We assume that $X \vee Y$ is the disjoint union of X and Y with only a base point in common and any two elements $x \in X \subset X \vee Y$ and $y \in Y \subset X \vee Y$ are not k-adjacent each other except the only the common point in $X \vee Y$ [18].

Definition 16 ([18]). For two disjoint spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, and a subspace $\{x_0\} \subset X$, let $f : \{x_0\} \to Y$ be a map. Then, we assume that $X \vee Y$, called a wedge product of X and Y, is the disjoint union of X and Y with the only one point $y_0 = f(x_0)$ in common and further, $((X \vee Y, y_0), k)$ assumed with k-adjacency in $\mathbf{Z}^n$, the number k is determined by the number m via $(CON\star)$, where $n = \max\{n_0, n_1\}$, $m = \max\{m_0, m_1\}$ and m_i is taken from the k_i-adjacency via $(CON\star)$, $i \in \{0,1\}$. And any two elements $x(\neq x_0) \in X \subset X \vee Y$ and $y(\neq f(x_0)) \in Y \subset X \vee Y$ are not k-adjacent to each other except the only one point $f(x_0) = y_0 \in X \vee Y$.

Remark 8.1. In view of Definition 16, both $(C_{k_1}^{n_1,l_1} \vee C_{k_2}^{n_2,l_2}, c_0)$ and $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, c_0)$ can be considered with the k-adjacency determined by the number $m = \max\{m_0, m_1\}$ and m_i is taken from the k_i-adjacency via $(CON\star)$, $i \in \{0,1\}$. Then, we always assume that for $C_{k_j}^{n_j,l_j} := (c_i)_{i \in [0, l_j - 1]_{\mathbf{Z}}}, j \in \{1,2\}$, each $I_k(i) = \{t | c_t \in N_k(c_i, 1)\}$ is consecutive modulo $l_j, i \in [0, l_j - 1]_{\mathbf{Z}}$.

Example 8.2. $(SC_{18}^{3,l_1} \vee SC_{26}^{3,l_2}, \star)$ should be considered with 26-adjacency in $\mathbf{Z}^3$.

We now study the k-fundamental group of a wedge product. Precisely, for the spaces MSC_4 and MSC_8 in (4.2) and (4.3), we obtain their wedge products as follows. $(MSC_8 \vee MSC_8, (0,0))$ and $(MSC_4 \vee MSC_4, (0,0))$ in [18].

We now recall the notion of the *free group with n generators* [30]. Let $A = \{a_1, a_2, \ldots, a_n\} \cup \{a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\}$ be a set of alphabets with $2n$ distinct letters, and let W_n be the set of all words over the set A. We say that two words $w, w' \in W_n$ are the same up to an *elementary simplification* in [28] if, either w can be obtained from w' by inserting in w' a sequence of the form $a_i a_i^{-1}$, $i \in [1, m]_{\mathbf{Z}}$, or w' can be obtained from w by inserting in w a sequence of the form $a_i^{-1} a_i$ with $i \in [1, n]_{\mathbf{Z}}$. Now two words $w, w' \in W_n$ are said to be *free equivalent* if there is a finite sequence $w = w_1, \ldots, w' = w_n$ of words of W_n such that for $i = 2, \ldots, k$ the word w_{i-1} and w_i are the same to an elementary simplification. This defines an equivalence relation. If $w \in W_n$, we denote by $[w]$ the equivalence class of w under the current equivalence relation on W_n. The concatenation of words defines an operation on $F_n = \{[w] | w \in W_n\}$ which provides F_n with a group structure. The group is called the *free group with n generators* [30].

As a generalization of the 4-fundamental group of the calculation of

$$\pi^4(MSC_4 \vee MSC_4, c_0)$$

which is a free group with order two. Precisely, $\pi^4(MSC_4 \vee MSC_4, c_0)$ is isomorphic to the free group $8\mathbf{Z} * 8\mathbf{Z}$ in [18]. Indeed, $MSC_4 \vee MSC_4$ has countably many $(2n, 4)$-covering spaces in $\mathbf{Z}^n, n \in \mathbf{N}$.

Theorem 8.3. *For two simple closed k-curves $SC_k^{n,l_i}, i \in \{1,2\}$ which is not k-contractible, the group $\pi^k(SC_k^{n,l_1} \vee SC_k^{n,l_2}, c_0)$ is a free group with order two. Precisely, $\pi^k(SC_k^{n,l_1} \vee SC_k^{n,l_2}, c_0)$ is isomorphic to a free group $l_1\mathbf{Z} * l_2\mathbf{Z}$.*

From above, we obtain the following by Theorem 4.9 and an analog of Example 4.10.

Theorem 8.4. *For two closed k-curve C_k^{n,l_i} not k-contractible, $i \in \{1,2\}$, the group $\pi^k(C_k^{n,l_1} \vee C_k^{n,l_2}, c_0)$ is not abelian. To be specific, $\pi^k(C_k^{n,l_1} \vee C_k^{n,l_2}, c_0)$ is a free group with two generators from each of the cyclic groups $l_1'\mathbf{Z}$ and $l_2'\mathbf{Z}$, where $l_i' = l_i -$ the cardinal number of the set of the simple k-curve points in $C_k^{n,l_i}, i \in \{1,2\}$, where c_0 is not a simple k-curve point in C_k^{n,l_i}.*

Before we prove Theorem 8.4, we need to show the assumption that C_k^{n,l_i} not k-contractible, $i \in \{1,2\}$. If not, consider the following wedge product $MSC_8 \vee MSC_8'$. Then, since MSC_8' is 8-contractible in (4.5), MSC_8 is a strong 8-deformation retract of $MSC_8 \vee MSC_8'$. Thus, we obtain that $\pi^8(MSC_8 \vee MSC_8')$ is isomorphic to an infinite cyclic group, precisely $(6\mathbf{Z}, +)$ (see the space Y in Figure 2). Thus, if we omit the assumption that C_k^{n,l_i} is not k-contractible, then Theorem 8.4 may fail.

Proof. Since $C_k^{n,l_1} \vee C_k^{n,l_2}$ can be k-homotopically thinned to be $SC_k^{n,l_1'} \vee SC_k^{n,l_2'}$ by Theorem 4.9 and Remark 4.12, where $l_i' = l_i -$ the number of the simple k-curve points in C_k^{n,l_i} and $i \in \{1,2\}$. By Theorem 7.1 and 8.3, and Corollary 7.2 the proof is completed, as required. □

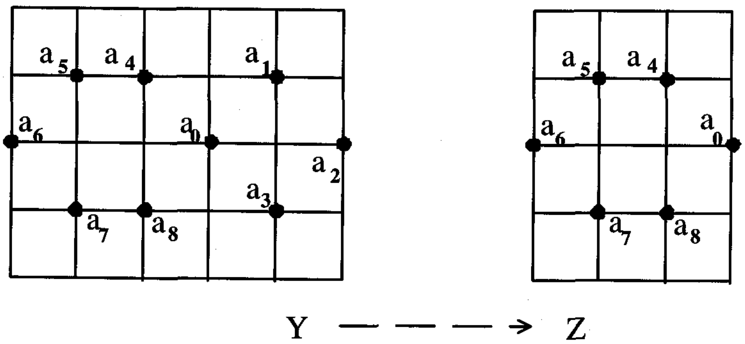

FIGURE 2. 8-homotopic thinning

Theorem 8.5. *Consider $C_{k_1}^{n_1,l_1} \vee C_{k_2}^{n_2,l_2}$ with k-adjacency. Then, $\pi^k(C_{k_1}^{n_1,l_1} \vee C_{k_2}^{n_2,l_2}, c_0)$ is not abelian, where both $\{c_0\} \vee C_{k_2}^{n_2,l_2} \subset (C_{k_1}^{n_1,l_1} \vee C_{k_2}^{n_2,l_2}, c_0)$ and $C_{k_1}^{n_1,l_1} \vee \{c_0\} \subset (C_{k_1}^{n_1,l_1} \vee C_{k_2}^{n_2,l_2}, c_0)$ are not k-contractible. Precisely, $\pi^k(C_{k_1}^{n_1,l_1} \vee$*

$C_{k_2}^{n_2,l_2}, c_0)$ *is a free group with two generators from each of the cyclic groups* $l_1'\mathbf{Z}$ *and* $l_2'\mathbf{Z}$*, where* $l_i' = l_i -$ *the cardinal number of the set of the simple k-curve points in* C_k^{n,l_i}*, where* c_0 *is not a simple k-curve point in* C_k^{n,l_i}.

9. An algorithm for the calculation of the k-fundamental group of a closed k-curve

For $C_k^{n,l_1} := (c_t)_{t\in[0,l_1-1]_{\mathbf{Z}}}$ not k-contractible and satisfying the condition ★ in Section 4, a k-homotopic thinning algorithm for calculating the k-fundamental group of the space C_k^{n,l_1} can be established by Theorems 4.9 and 7.1, and Remarks 4.12 and 5.6, as follows.

In general, a k-thinning algorithm of C_k^{n,l_1} not k-contractible can be proceeded as follows.

(1) **while** n-xels are detected

(2) **detect** the simple k-curve points

(3) **delete** simple k-curve points sequentially from the one near to some non-simple k-curve point.

Consequently, we obtain an algorithm for calculating the k-fundamental group of C_k^{n,l_1} not k-contractible motivated by Remark 4.12 and Theorem 7.1.

[An algorithm for calculating $\pi^k(C_k^{n,l_1}, c_0)$, where the point c_0 is not a simple k-curve point]

For every n-xels in $C_k^{n,l_1} := (c_i)_{i\in[0,l_1-1]_{\mathbf{Z}}}$ not k-contractible {
  detect $N_k(c_i, 1)$
}
let c_0' be one of the elements such that

$$\sharp(N_k(c_{m-1(mod\, l_1)}, 1) \cap N_k(c_{m+1(mod\, l_1)}), 1) = 1.$$

put m the selected number
arrange C_k^{n,l_1} with $\{c_0' = c_{m(mod\, l_1)}, c_1' = c_{m+1(mod\, l_1)}, c_2' = c_{m+2(mod\, l_1)}, \cdots\}$
for $(i = 1; i \leq l_1 - 1; i++)$ {
  detect $N_k(c'_{i-1(mod\, l_1)}, 1)$
  detect $N_k(c'_{i+1(mod\, l_1)}, 1)$
  if $\sharp(N_k(c'_{i-1(mod\, l_1)}, 1) \cap N_k(c'_{i+1(mod\, l_1)}, 1)) \geq 3$
    then delete c_i' in C_k^{n,l_1}
}
let l be the number of the elements in C_k^{n,l_1}.

Then, we consider $l\mathbf{Z}$ to be the k-fundamental group of C_k^{n,l_1} from Theorem 7.1 and Corollary 7.2.

Remark 9.1. Let us examine the above algorithm in terms of the calculation of $\pi^8(C_8^{2,12}, c_0)$ in Figure 1(b). Then, we can arrange the space $C_8^{2,12} := (c_i)_{i\in[0,11]_{\mathbf{Z}}}$ so that it is 8-homotopically thinned to be $SC_8^{2,11}$ by Remark 5.6(2). Consequently, we obtain that $\pi^8(C_8^{2,12}, c_0) \simeq \pi^8(SC_8^{2,11}, c_0) \simeq 11\mathbf{Z}$ by Theorems 4.9 and 7.1, and Corollary 7.2.

10. Summary

We have calculated the k-fundamental groups of a closed k-curves and a wedge product of two closed k_1- and k_2-curves by the use of digital k-homotopy theory and digital covering theory. To be specific, we have investigated some properties of a digital covering that could be used to calculate the k-fundamental group of a closed k-curve not k-contractible. It turns out that a closed k-curve C_k^{n,l_1} is k-homotopically thinned to be a simple closed k-curve $SC_k^{n,l}$ in terms of a strong k-deformation retract. Consequently, $\pi^k(C_k^{n,l_1}, c_0)$ was proved isomorphic to $l\mathbf{Z}$, which is often used for the study of the discrete Deck's transformation group of a space, where $l = l_1 -$ the number of the simple k-curve points in C_k^{n,l_1}. Furthermore, an algorithm for calculating the k-fundamental group of C_k^{n,l_1} has written in terms of the deletion of a simple k-curve point in C_k^{n,l_1} motivated by Remarks 4.12 and 5.6, and Theorems 4.9 and 7.1. Since digital topology can play an important role in computer science, the current results could be used to study 2D-, 3D-digital images and further, hyperspectral images.

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