Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 641–649

ON Π-ARMENDARIZ RINGS

Chan Huh, Chang Ik Lee, Kwang Sug Park, and Sung Ju Ryu

Abstract. We in this note introduce a concept, so called *π-Armendariz ring*, that is a generalization of both Armendariz rings and 2-primal rings. We first observe the basic properties of π-Armendariz rings, constructing typical examples. We next extend the class of π-Armendariz rings, through various ring extensions.

1. Introduction

Throughout this note all rings are associative with identity unless otherwise stated. Let R be a ring. The polynomial ring with an indeterminate x over R and the n by n matrix ring over R are denoted by $R[x]$ and $Mat_n(R)$, respectively. The prime radical (i.e., the intersection of all prime ideals) of R and the set of all nilpotent elements in R are denoted by $P(R)$ and $N(R)$, respectively. $\mathbb{Z}$ denotes the ring of integers.

A ring is called *reduced* if it has no nonzero nilpotent elements. Over a reduced ring R, Armendariz [2, Lemma 1] proved that $a_ib_j = 0$ for all i, j whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$. Due to Rege et al. [11], such rings (possibly not reduced), that satisfy Armendariz's result, are called *Armendariz*. Reduced rings are Armendariz by [2, Lemma 1]. The structure of the class of non-reduced Armendariz rings was observed by many authors containing Anderson et al. [1], Hirano [5], Huh et al. [6], Kim et al. [7], Lee et al. [8], Rege et al. [11], and so on.

We call a ring R *π-Armendariz* provided that whenever $f(x)g(x) \in N(R[x])$ for $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$ we get $a_ib_j \in N(R)$ for all i, j.

Lemma 1.1. (1) [1, Proposition 1] *Let R be an Armendariz ring. If $f_1, \ldots, f_n \in R[x]$ are such that $f_1 \cdots f_n = 0$, then $a_1 \cdots a_n = 0$ where a_i is a coefficient of f_i.*

(2) *Armendariz rings are π-Armendariz.*

(3) *Subrings of (π-)Armendariz rings are (π-)Armendariz.*

Received June 12, 2006; Revised August 31, 2007.

2000 *Mathematics Subject Classification.* 16D25, 16N40, 16S36.

Key words and phrases. π-Armendariz ring, 2-primal ring, Armendariz ring, nilpotent element.

The first author was supported by Busan National University Research Grant.

Proof. (2) is proved by (1) and (3) is trivial. □

The converse of Lemma 1.1(2) need not hold by the following.

Example 1.2. Let S be a reduced ring and

$$R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \in Mat_4(S) \right\}.$$

Then R is π-Armendariz by Theorem 2.4 below, but R is not Armendariz by [7, Example 3].

Due to Birkenmeier et al. [3], a ring R is called *2-primal* if $P(R) = N(R)$. It is obvious that R is 2-primal if and only if $R/P(R)$ is reduced. A prime ideal P of a ring R is called *completely prime* if R/P is a domain. Shin [12, Proposition 1.11] proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime, and furthermore 2-primal rings were almost completely characterized by Marks [10].

Proposition 1.3. *2-primal rings are π-Armendariz.*

Proof. Let R be a 2-primal ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be in $R[x]$ such that $f(x)g(x) \in N(R[x])$. Since R is 2-primal, $\frac{R}{P(R)}[x] \cong \frac{R[x]}{P(R)[x]}$ is reduced (hence Armendariz) and so we get $a_i b_j \in N(R)$ for all i, j with the help of Lemma 1.1(1). □

As we see in the following the converse of Proposition 1.3 need not be true by Birkenmeier et al. [4, Example 3.3] or Marks [9, Example 2.2].

Example 1.4. (1) Let G be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $\{b(0), b(1), b(-1), \ldots, b(i), b(-i), \ldots\}$ a basis of G. Then there exists one and only one homomorphism $u(i)$ of G, for $i = 1, 2, \ldots$ such that $u(i)(b(j)) = 0$ if $j \equiv 0 \pmod{2^i}$ and $u(i)(b(j)) = b(j-1)$ if $j \not\equiv 0 \pmod{2^i}$. Denote U the ring of endomorphisms of G generated by the endomorphisms $u(1), u(2), \ldots$. Now let A be the ring obtained from U by adjoining the identity map of G and let $R = A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}$ the field of rationals. Then we have $P(R) = 0$ and $0 \neq N(R) = J(R)$ by the argument in [4, Example 3.3], where $J(R)$ is the Jacobson radical of R. Thus R is not 2-primal. Now since $\frac{R}{J(R)}[x] \cong \frac{R[x]}{J(R)[x]}$ is reduced and $J(R)$ is nil, R is π-Armendariz with the help of Lemma 1.1(1).

(2) Let K be a field and let $S = K[\{t_i\}_{i \in \mathbb{Z}}]/(\{t_{n_1} t_{n_2} t_{n_3} | n_3 - n_2 = n_2 - n_1 > 0\})$, and let $R = S[x; \sigma]$ where σ is the K-homomorphism of S satisfying $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$. Then we have $P(R) = 0$ and $0 \neq N(R) = N^*(R)$ by

the computation in [9, Example 2.2] where $N^*(R)$ is the sum of all nil ideals in R. Thus R is not 2-primal. Now since $\frac{R}{N^*(R)}[x] \cong \frac{R[x]}{N^*(R)[x]}$ is reduced, R is π-Armendariz with the help of Lemma 1.1(1).

2. Basic structure of π-Armendariz rings

In this section we study the properties of π-Armendariz rings and construct examples which are necessary in the process. $\prod$ denotes the direct product.

Lemma 2.1. (1) *A finite direct product of π-Armendariz rings is π-Armendariz.*

(2) *A finite subdirect product of π-Armendariz rings is π-Armendariz.*

Proof. (1) Let $R_1, R_2, \ldots, R_n$ be π-Armendariz rings and let $R = \prod_{k=1}^n R_k$. Consider $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x]$ such that $fg \in N(R[x])$, where $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $b_j = (b_{j1}, b_{j2}, \ldots, b_{jn})$ in R. For each $k = 1, 2, \ldots, n$, we put $f_k(x) = \sum_{i=0}^m a_{ik} x^i$, $g_k(x) = \sum_{j=0}^n b_{jk} x^j$ in $R_k[x]$. Then $f_k g_k \in N(R_k[x])$. So by π-Armendarizness of R_k, $a_{ik} b_{jk} \in N(R_k)$ for all i, j. Thus for each i, j, there exists positive integer m_{ijk} such that $(a_{ik} b_{jk})^{m_{ijk}} = 0$. Take $m_{ij} = \max\{m_{ijk} \mid k = 1, 2, \ldots, n\}$, then $(a_i b_j)^{m_{ij}} = ((a_{ik} b_{jk})^{m_{ij}}) = 0$. Thus $a_i b_j \in N(R)$ for all i, j. Therefore R is π-Armendariz.

(2) is obtained from (1) and Lemma 1.1 (3). □

Lemma 2.2. *For a ring R suppose that R/I is π-Armendariz for some ideal I of R. If I is nil then R is π-Armendariz.*

Proof. Suppose that $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ are such that $f(x)g(x) \in N(R[x])$. Write $\bar{R} = R/I$ and $\bar{r} = r + I$. Then $\bar{f}(x)\bar{g}(x) \in N(\bar{R}[x])$. Since $\bar{R}$ is π-Armendariz, $\bar{a}_i \bar{b}_j \in N(\bar{R})$ for each i, j. But I is nil, $a_i b_j \in N(R)$ for each i, j. □

In Lemma 2.2 the condition "I is nil" is not superfluous by the following. Let R be an algebra over a commutative ring S. The *Dorroh extension* of R by S, written by $R \oplus_D S$, is the ring $R \oplus S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$, where $r_i \in R$ and $s_i \in S$.

Example 2.3. Let A be an algebra over $\mathbb{Z}$ such that $A^2 = 0$. Then $Mat_2(A)$ is nilpotent. So by Proposition 3.4 below, $Mat_2(A) \oplus_D \mathbb{Z}$ is π-Armendariz. Next consider $R = Mat_2(\mathbb{Z} \oplus A) \oplus_D \mathbb{Z}$ and an ideal $I = Mat_2(\mathbb{Z} \times 0) \oplus_D 0$ of R. Then $I \cong Mat_2(\mathbb{Z})$ and $\frac{R}{I} \cong Mat_2(A) \oplus_D \mathbb{Z}$ is π-Armendariz. Note that I is not nil and is not π-Armendariz (as a ring without identity) by the computation in Example 2.5 below. Thus R is not π-Armendariz.

Let $UTM_n(R)$ (resp. $LTM_n(R)$) denotes the n by n upper (resp. lower) triangular matrix ring over a ring R. $\oplus$ denotes the direct sum. The following is one of our main results.

Theorem 2.4. *Let R be a ring. Then the following conditions are equivalent:*

(1) *R is π-Armendariz;*

(2) *$UTM_n(R)$ is π-Armendariz for each $n \geq 1$;*

(3) $\left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in Mat_n(R) \right\}$ *is π-Armendariz;*

(4) *$LTM_n(R)$ is π-Armendariz for each $n \geq 1$;*

(5) $\left\{ \begin{pmatrix} b & 0 & \cdots & 0 \\ b_{21} & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b \end{pmatrix} \in Mat_n(R) \right\}$ *is π-Armendariz.*

Proof. (1) $\Rightarrow$ (2): Let $I = \{A \in U \mid \text{each diagonal entry of } A \text{ is zero}\}$, where $U = UTM_n(R)$. Then I is nilpotent ideal of U and $U/I \cong R \oplus R \oplus \cdots \oplus R$. So U/I is π-Armendariz by Lemma 2.1. Thus, by Lemma 2.2, U is also π-Armendariz.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial. The proof of (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) is similar to (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). □

From Theorem 2.4, one may suspect that if R is π-Armendariz then $Mat_n(R)$ is π-Armendariz for $n \geq 2$. But the following example erases the possibility.

Example 2.5. Let R be a ring and let $S = Mat_2(R)$. Let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}x$ be polynomials in $S[x]$. Then $f(x)g(x) = 0$. But $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nilpotent. Thus S is not π-Armendariz.

Proposition 2.6. *Let $\{R_\alpha | \alpha \in \Lambda \text{ an index set}\}$ be a family of π-Armendariz rings. If $R = \prod_{\alpha \in \Lambda} R_\alpha$ is of bounded index of nilpotency, then R is π-Armendariz.*

Proof. We put $N(\geq 1)$ as the index of nilpotency of R. Then R_α is of bound index of nilpotency $\leq N$, for each $\alpha \in \Lambda$. Consider $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ such that $fg \in N(R[x])$, where $a_i = (a_{i\alpha})_{\alpha \in \Lambda}$, $b_j = (b_{j\alpha})_{\alpha \in \Lambda}$ $\in R$. For $\alpha \in \Lambda$, we put $f_\alpha(x) = \sum_{i=0}^m a_{i\alpha} x^i$, $g_\alpha(x) = \sum_{j=0}^n b_{j\alpha} x^j \in R_\alpha[x]$, then $f_\alpha g_\alpha \in N(R_\alpha[x])$. Since R_α is a π-Armendariz ring of bounded index of nilpotency $\leq N$, $(a_{i\alpha} b_{j\alpha})^N = 0$ for all i, j. Thus $(a_i b_j)^N = ((a_{i\alpha})_{\alpha \in \Lambda} (b_{j\alpha})_{\alpha \in \Lambda})^N = (a_{i\alpha} b_{j\alpha})^N_{\alpha \in \Lambda} = \left((a_{i\alpha} b_{j\alpha})^N\right)_{\alpha \in \Lambda} = 0$ for each i, j. Therefore $(a_i b_j) \in N(R)$ and so R is π-Armendariz. □

In Proposition 2.6, the condition "of bounded index of nilpotency" is not superfluous by the following.

Example 2.7. Let $R_n = UTM_{2^n}(\mathbb{Z})$ $(n = 1, 2, \ldots)$. Then by Theorem 2.4, R_n is a π-Armendariz ring. But their direct product $R = \prod_{n\geq 1} R_n$ is not π-Armendariz.

Consider two polynomials $f(x) = A_0 + A_1x$, $g(x) = B_0 + B_1x$ in $R[x]$ where

$$A_k = \left(A_{k1} = \begin{pmatrix} 1 & (-1)^k \\ 0 & 0 \end{pmatrix}, \ldots, A_{kn} = \left(\begin{array}{c|c} A_{k(n-1)} & C_{k(n-1)} \\ \hline 0 & A_{k(n-1)} \end{array}\right), \ldots\right),$$

$$B_k = \left(B_{k1} = \begin{pmatrix} 0 & (-1)^{k+1} \\ 0 & 1 \end{pmatrix}, \ldots, B_{kn} = \left(\begin{array}{c|c} B_{k(n-1)} & D_{k(n-1)} \\ \hline 0 & B_{k(n-1)} \end{array}\right), \ldots\right),$$

and $C_{kn} = \left((-1)^{k(j+1)}\right)_{2^n\times 2^n}$, $D_{kn} = \left((-1)^{(k+1)i}\right)_{2^n\times 2^n} \in \mathrm{Mat}_{2^n\times 2^n}(\mathbb{Z})$ for each $k = 0, 1$ and $n = 1, 2, \ldots$. Now we will show that $fg = 0 \in N(R[x])$, but $A_0B_1 \notin N(R)$, that is R is not π-Armendariz. To complete our result, we prove the following claim by induction on n.

Claim. 1. $A_{kn}B_{kn} = 0$ for $k = 0, 1$ and $n = 1, 2, \ldots$.

2. $A_{0n}B_{1n} = -A_{1n}B_{0n}$ such that $A_{0n}B_{1n} \in N(R_n)\backslash(0)$ for $n = 1, 2, \ldots$.

If $n = 1$ then $A_{k1}B_{k1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $k = 0, 1$. $A_{01}B_{11} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = -A_{11}B_{01} \neq 0$ and $(A_{01}B_{11})^2 = 0$. We are done. Now suppose that it holds for $n < l$ $(l \geq 2)$ and let $n = l$. Note that $A_{kn}D_{kn} = C_{kn}B_{kn} = 0$ for $k = 0, 1$ and

$$\begin{aligned} &A_{0n}D_{1n} + C_{0n}B_{1n} \\ &= \begin{pmatrix} 2^n & 2^n+2 & 2^n+2 & \cdots & 2^{n+1}-2 & 2^{n+1}-2 & 2^{n+1} \\ 2^n-2 & 2^n & 2^n & \cdots & 2^{n+1}-4 & 2^{n+1}-4 & 2^{n+1}-2 \\ 2^n-2 & 2^n & 2^n & \cdots & 2^{n+1}-4 & 2^{n+1}-4 & 2^{n+1}-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 4 & \cdots & 2^n & 2^n & 2^n+2 \\ 2 & 4 & 4 & \cdots & 2^n & 2^n & 2^n+2 \\ 0 & 2 & 2 & \cdots & 2^n-2 & 2^n-2 & 2^n \end{pmatrix} \\ &= -\left(A_{1n}D_{0n} + C_{1n}B_{0n}\right), \end{aligned}$$

that is $A_{0n}D_{1n} + C_{0n}B_{1n} = -(A_{1n}D_{0n} + C_{1n}B_{0n}) = 2^nI_{2^n}$+(some matrix whose entries are all non-negative), where I_{2^n} is the $2^n \times 2^n$ identity matrix.

By notes and inductive hypothesis,

$$A_{kl}B_{kl} = \left(\begin{array}{c|c} A_{k(l-1)} & C_{k(l-1)} \\ \hline 0 & A_{k(l-1)} \end{array}\right) \left(\begin{array}{c|c} B_{k(l-1)} & D_{k(l-1)} \\ \hline 0 & B_{k(l-1)} \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_{k(l-1)}B_{k(l-1)} & A_{k(l-1)}D_{k(l-1)} + C_{k(l-1)}B_{k(l-1)} \\ \hline 0 & A_{k(l-1)}B_{k(l-1)} \end{array}\right) = 0.$$

Also

$$A_{0l}B_{1l} = \left(\begin{array}{c|c} A_{0(l-1)}B_{1(l-1)} & A_{0(l-1)}D_{1(l-1)} + C_{0(l-1)}B_{1(l-1)} \\ \hline 0 & A_{0(l-1)}B_{1(l-1)} \end{array}\right)$$

$$= \left(\begin{array}{c|c} -A_{1(l-1)}B_{0(l-1)} & -(A_{1(l-1)}D_{0(l-1)} + C_{1(l-1)}B_{0(l-1)}) \\ \hline 0 & -A_{1(l-1)}B_{0(l-1)} \end{array}\right)$$

$$= -A_{1l}B_{0l}.$$

Since $A_{0(l-1)}B_{1(l-1)}$ is in $N(R_{(l-1)})\backslash(0)$, by the inductional hypothesis, there exists m in positive integers such that $(A_{0(l-1)}B_{1(l-1)})^m \neq 0$ and

$$(A_{0(l-1)}B_{1(l-1)})^{m+1} = 0.$$

So by notes, $(A_{0l}B_{1l})^{m+1} = \left(\begin{array}{c|c} 0 & (*) \\ \hline 0 & 0 \end{array}\right)$ is not a zero matrix and $(A_{0l}B_{1l})^{2(m+1)} = 0$, that is $A_{0l}B_{1l} \in N(R_l)\backslash(0)$. Therefore our claim is proved by the induction and so $f(x)g(x) = 0$. Furthermore the sequence of index of $A_{0n}B_{1n}$ is increasing. (In fact, the index of $A_{0n}B_{1n}$ is equal to $2^{n-1} + 1$, by using another method.) Thus A_0B_1 is not a nilpotent element of R.

3. More examples of π-Armendariz rings

In this section we extend the class of π-Armendariz rings through various extensions. We first consider the case of direct limits of direct systems of π-Armendariz rings, comparing with Lemma 2.1.

Proposition 3.1. *The direct limit of a direct system of π-Armendariz rings is also π-Armendariz.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of π-Armendariz rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. We will prove that R is a π-Armendariz ring. Take $x, y \in R$. Then $x = \iota_i(x_i)$, $y = \iota_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k$, $j \leq k$. Define

$$x + y = \iota_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j)) \text{ and } xy = \iota_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j)),$$

where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Now suppose $f(x)g(x) \in N(R[x])$ for $f(x) = \sum_{s=0}^{m} a_s x^s$, $g(x) = \sum_{t=0}^{n} b_t x^t$ in $R[x]$. There are $i_s, j_t, k \in I$ such that $a_s = \iota_{i_s}(a_{i_s})$, $b_t = \iota_{j_t}(b_{j_t})$, $i_s \leq k$, $j_t \leq k$. So

$$a_s b_t = \iota_k(\alpha_{i_s k}(a_{i_s})\alpha_{j_t k}(b_{j_t})),$$

and from $f(x)g(x) \in N(R[x])$ we have

$$f(x)g(x) = (\sum_{s=0}^{m} \iota_k(\alpha_{i_s k}(a_{i_s}))x^s)(\sum_{t=0}^{n} \iota_k(\alpha_{j_t k}(b_{j_t}))x^t) \in N(R_k[x]).$$

But R_k is π-Armendariz and so $\iota_k(\alpha_{i_s k}(a_{i_s})\alpha_{j_t k}(b_{j_t})) \in N(R_k)$. Thus $a_s b_t \in N(R)$ and R is π-Armendariz. □

Proposition 3.2. *Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements. Then R is π-Armendariz if and only if so is $\Delta^{-1}R$.*

Proof. (⇐) is obtained from Lemma 1.1(3). (⇒) Let R be a π-Armendariz ring and let $S = \Delta^{-1}R$, where is a multiplicative monoid in R consisting central regular elements of R. Note that if $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ are in $S[x](\alpha_i, \beta_j \in S)$, then we can assume that $\alpha_i = a_i u^{-1}$ and $\beta_j = b_j v^{-1}$ for some $a_i, b_j \in R$, $u, v \in \Delta$ for all i, j. Now suppose that $f(x)g(x) \in N(S[x])$ then there exist a positive integer k such that

$$\begin{aligned} 0 = (fg)^k &= (\sum_{i,j} \alpha_i \beta_j x^{i+j})^k \\ &= (\sum_{i,j} a_i u^{-1} b_j v^{-1} x^{i+j})^k = (\sum_{i,j} a_i b_j x^{i+j})^k \left((uv)^k\right)^{-1}. \end{aligned}$$

Since $(uv)^k \in \Delta$, $(\sum_{i,j} a_i b_j x^{i+j})^k = 0$ and so that $\sum_{i,j} a_i b_j x^{i+j} \in N(R[x])$. By the hypothesis, $a_i b_j \in N(R)$ for all i, j. Immediately, we can show that $\alpha_i \beta_j = (a_i u^{-1} b_j v^{-1})$ is also a nilpotent element of S for all i, j. Therefore S is a π-Armendariz ring. □

The ring of *Laurent* polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denotes it by $R[x; x^{-1}]$.

Corollary 3.3. (1) *A commutative ring R is π-Armendariz if and only if so is the total quotient ring of R.*

(2) *Let R be a ring. $R[x]$ is π-Armendariz if and only if so is $R[x;x^{-1}]$.*

Proof. It suffices to show the necessity by Lemma 1.1(3).

(1) Let Δ be the set of all regular elements of R. Then Δ satisfies the condition of Proposition 3.2 and $\Delta^{-1}R$ is the total quotient ring of R. Thus the total quotient ring of R is π-Armendariz.

(2) Let $\Delta = \{1, x, x^2, \ldots\} \subset R[x]$. Then Δ satisfies the condition of Proposition 3.2 and so $R[x;x^{-1}] \cong \Delta^{-1}R$ is π-Armendariz. □

Proposition 3.4. *Let A be a nil algebra over $\mathbb{Z}$. Then $A \oplus_D \mathbb{Z}$ of A by $\mathbb{Z}$ is π-Armendariz.*

Proof. Since A is nil, $A \oplus_D 0$ is a nil ideal of $A \oplus_D \mathbb{Z}$. Thus $A \oplus_D \mathbb{Z}$ is a ring with a nil ideal $A \oplus_D 0$ such that $\frac{A \oplus_D \mathbb{Z}}{A \oplus_D 0} \cong \mathbb{Z}$ is a π-Armendariz ring. So by Lemma 2.2, $A \oplus_D \mathbb{Z}$ is π-Armendariz. □

Proposition 3.5. *A ring R is π-Armendariz if and only if $R[x]/\langle x^n \rangle$ is π-Armendariz for any positive integer n, where $\langle x^n \rangle$ is the ideal of $R[x]$ generated by x^n.*

Proof. It suffices to show the necessity by Lemma 1.1(3). Let R be a π-Armendariz ring and n be a positive integer. Put $S = R[x]/\langle x^n \rangle$ and $\bar{x} = x + \langle x^n \rangle$. Then $\frac{S}{S\bar{x}} \cong R$ and so $S/S\bar{x}$ is π-Armendariz. Since $S\bar{x}$ is a nil ideal of S, S is π-Armendariz by Lemma 2.2. □

References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[2] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[3] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, *Completely prime ideals and associated radicals*, Ring theory (Granville, OH, 1992), 102–129, World Sci. Publ., River Edge, NJ, 1993.

[4] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, *Regularity conditions and the simplicity of prime factor rings*, J. Pure Appl. Algebra **115** (1997), no. 3, 213–230.

[5] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring*, J. Pure Appl. Algebra **168** (2002), no. 1, 45–52.

[6] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[7] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488.

[8] T.-K. Lee and T.-L. Wong, *On Armendariz rings*, Houston J. Math. **29** (2003), no. 3, 583–593.

[9] G. Marks, *On 2-primal Ore extensions*, Comm. Algebra **29** (2001), no. 5, 2113–2123.

[10] ———, *A taxonomy of 2-primal rings*, J. Algebra **266** (2003), no. 2, 494–520.

[11] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[12] G. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.

Chan Huh
Department of Mathematics
Busan National University
Busan 609-735, Korea
E-mail address: chuh@pusan.ac.kr

Chang Ik Lee
Department of Mathematics
Busan National University
Busan 609-735, Korea
E-mail address: cilee@pusan.ac.kr

Kwang Sug Park
Department of Mathematics
Busan National University
Busan 609-735, Korea
E-mail address: manchun76@chollian.net

Sung Ju Ryu
Department of Mathematics
Busan National University
Busan 609-735, Korea
E-mail address: sjryu@pusan.ac.kr