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HYPERIDENTITIES IN $(xy)x \approx x(yy)$ GRAPH ALGEBRAS OF TYPE $(2,0)$

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ABSTRACT. Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type $(2,0)$. We say that a graph G satisfies an identity $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A graph $G = (V, E)$ is called an $(xy)x \approx x(yy)$ graph if the graph algebra $\underline{A(G)}$ satisfies the equation $(xy)x \approx x(yy)$. An identity $s \approx t$ of terms s and t of any type τ is called a hyperidentity of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identities hold in $\underline{A}$.

In this paper we characterize $(xy)x \approx x(yy)$ graph algebras, identities and hyperidentities in $(xy)x \approx x(yy)$ graph algebras.

1. Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

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(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and
(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.
Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [11] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if } (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term t, where the vertices correspond to the variables occurring in t. Since on a graph algebra we have one nullary and one binary operation, $\sigma(f)$ in this case is a binary term in $W_\tau(X)$, i.e., a term built up from variables of a two-element alphabet and a binary operation symbol f corresponding to the binary operation of the graph algebra.

In [9] R. Pöschel has shown that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

In [1] K. Denecke and T. Poomsa-ard characterized graph hyperidentities by using normal form graph hypersubstitutions.

In [6] T. Poomsa-ard characterized associative graph hyperidentities by using normal form graph hypersubstitutions.

In [7] T. Poomsa-ard, J. Wetweerapong, and C. Samartkoon characterized idempotent graph hyperidentities by using normal form graph hypersubstitutions.

In [8] T. Poomsa-ard, J. Wetweerapong, and C. Samartkoon characterized transitive graph hyperidentities by using normal form graph hypersubstitutions.

We say that a graph $G = (V, E)$ is $(xy)x \approx x(yy)$ if the corresponding graph algebra $\underline{A(G)}$ satisfies the equation $(xy)x \approx x(yy)$. In this paper we characterize $(xy)x \approx x(yy)$ graph algebras, identities and hyperidentities in $(xy)x \approx x(yy)$ graph algebras.

2. $(xy)x \approx x(yy)$ graph algebras

We begin with one more precise definition of terms of the type of graph algebras.

Definition 2.1. The set $W_\tau(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, \ldots\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, \ldots,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $f(t_1, t_2)$ is a term, we will shortly write t_1t_2 for it;

(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$E(t) = \phi$ if t is a variable and $E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$,

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$, i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\}$, and in this case, we write $G \models s \approx t$.

Definition 2.4. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V'$ carrying edges to edges, that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E'$.

In [3] it was proved:

Proposition 2.1. *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, \ldots, x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

Proposition 2.1 gives a method to check whether a graph $G = (V, E)$ satisfies the equation $s \approx t$. Hence, we can check whether a graph $G = (V, E)$ has an $(xy)x \approx x(yy)$ graph algebra by the following proposition.

Proposition 2.2. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

(i) *G has an $(xy)x \approx x(yy)$ graph algebra,*

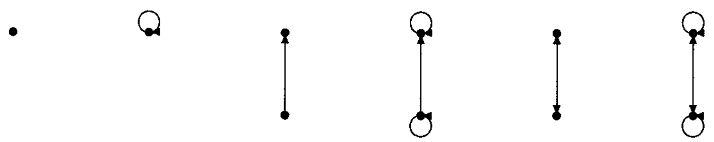
(ii) *$(a, b) \in E$ implies $(a, a) \in E$ if and only if $(b, b) \in E$.*

Proof. Suppose $G = (V, E)$ has an $(xy)x \approx x(yy)$ graph algebra. Let s and t be terms such that $s = (xy)x$ and $t = x(yy)$. Let $(a, b), (a, a) \in E$ and

$h : V(t) \to V$ be a function such that $h(x) = a,\ h(y) = b$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 2.1, we have that h is a homomorphism from $G(t)$ into G. Since $(y, y) \in E(t)$, $(h(y), h(y)) = (b, b) \in E$. Similarly, we prove if $(a, b), (b, b) \in E$, then $(a, a) \in E$.

Conversely, suppose $G = (V, E)$ is a graph which satisfies (ii). Let s and t be non-trivial terms such that $s = (xy)x$ and $t = x(yy)$. Suppose that $h : V(t) \to V$ is a homomorphism from $G(s)$ into G. Since $(x, y), (x, x) \in E(s)$, we have $(h(x), h(y)), (h(x), h(x)) \in E$. By assumption, we get $(h(y), h(y)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. Similarly, we prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence, by Proposition 2.1, we get that $\underline{A(G)}$ satisfies $(xy)x \approx x(yy)$. □

From Proposition 2.2, we see that all graphs which have $(xy)x \approx x(yy)$ graph algebras are the following graphs:



G_1 G_2 G_3 G_4 G_5 G_6

and all graphs such that each component of every subgraph induced by at most two vertices is one of these graphs.

3. Identities in $(xy)x \approx x(yy)$ graph algebras

Graph identities were characterized in [3] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras if and only if either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further it was proved.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

In [6] the following lemma was proved:

Lemma 3.1. *Let $G = (V, E)$ be a graph, t a term and*

$$h : X \longrightarrow V \cup \{\infty\}$$

an evaluation of the variables. Then:

(i) *If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is complete, then $h(t) = h(L(t))$.*

(ii) *If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is disconnected, then $h(t) = \infty$.*

Now we apply our results to characterize all identities in the class of all $(xy)x \approx x(yy)$ graph algebras. Clearly, if s and t are trivial, then $s \approx t$ is an identity in the class of all $(xy)x \approx x(yy)$ graph algebras and $x \approx x$ $(x \in X)$ is an identity in the class of all $(xy)x \approx x(yy)$ graph algebras, too. So we consider the case that s and t are non-trivial and different from variables. Then all identities in the class of all $(xy)x \approx x(yy)$ graph algebras are characterized by the following theorem:

Theorem 3.1. *Let s and t be non-trivial terms and let $x_0 = L(s)$. Then $s \approx t$ is an identity in the class of all $(xy)x \approx x(yy)$ graph algebras if and only if the following conditions are satisfied:*

(i) $L(s) = L(t)$,

(ii) $V(s) = V(t)$,

(iii) *for any x, y in $V(s)$, $x \neq y$, $(x, y) \in E(s)$ if and only if $(x, y) \in E(t)$,*

(iv) *there exists $x \in V(s)$ such that $(x, x) \in E(s)$ if and only if there exists $y \in V(t)$ such that $(y, y) \in E(t)$.*

Proof. Suppose that $s \approx t$ is an identity in the class $(xy)x \approx x(yy)$ graph algebras. Since any complete graph has an $(xy)x \approx x(yy)$ graph algebra, it follows that $L(s) = L(t)$ and $V(s) = V(t)$.

Suppose that there exist $x, y \in V(s)$ with $x \neq y$ such that $(x, y) \in E(s)$ but $(x, y) \notin E(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0,0), (0,1), (1,1), (1,2), (2,1), (2,2), (2,0), (0,2)\}$. Then by Proposition 2.2, $\underline{A(G)}$ has an $(xy)x \approx x(yy)$ graph algebra. Let $h : V(s) \to V \cup \{\infty\}$ be the restriction of an evaluation of the variables such that $h(y) = 0$, $h(x) = 1$ and $h(w) = 2$ for all $w \in V(s)$ such that $w \neq x$ and $w \neq y$. We see that $h(s) = \infty$ and $h(t) = h(L(t))$. Hence, $\underline{A(G)}$ does not satisfy $s \approx t$.

Suppose that there exists $x \in V(s)$ such that $(x, x) \in E(s)$ but there is no $y \in V(t)$ such that $(y, y) \in E(t)$. Consider the graph $G = (V, E)$ such that $V = V(t)$, $E = E(t)$. By Proposition 2.2, $\underline{A(G)}$ has an $(xy)x \approx x(yy)$ graph algebra. Let $h : V(s) \to V$ be the identity function. We see that $h(s) = \infty$ and $h(t) = h(L(t))$. Hence, $\underline{A(G)}$ does not satisfy $s \approx t$.

Conversely, suppose that s and t are non-trivial terms satisfying (i), (ii), (iii) and (iv). Let $G = (V, E)$ be an $(xy)x \approx x(yy)$ graph and let $h : V(s) \to V$ be a mapping. Suppose that h is a homomorphism from $G(s)$ into G and $(x, y) \in E(t)$. If $x = y$, then $(x, x) \in E(t)$. By (iii), there exists $z \in V(s)$ such that $(z, z) \in E(s)$. So $(h(z), h(z)) \in E$. By Proposition 2.2, $(a, a) \in E$ for all $a \in h(V(s))$. Hence $(h(x), h(x)) \in E$. If $x \neq y$, then by (iii), we have $(x, y) \in E(s)$, thus $(h(x), h(y)) \in E$. Therefore h is a homomorphism from

$G(t)$ into G. By the same way, if h is a homomorphism from $G(t)$ into G, then we prove that it is a homomorphism from $G(s)$ into G. By Proposition 2.1, we get that $\underline{A(G)}$ satisfies $s \approx t$. □

4. Hyperidentities in $(xy)x \approx x(yy)$ graph algebras

Let $\mathcal{K}'$ be the class of all $(xy)x \approx x(yy)$ graph algebras and let $Id\mathcal{K}'$ be the set of all identities satisfied in the class $\mathcal{K}'$. Now we want to make precise the concept of a hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$, where f is the operation symbol corresponding to the binary operation of a graph algebra, is called a *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is an $(xy)x \approx x(yy)$ *graph hyperidentity* if and only if for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in the class $\mathcal{K}'$.

If we want to check that an identity $s \approx t$ is a hyperidentity in the class $\mathcal{K}'$ we can restrict ourselves to a (small) subset of the set of all graph hypersubstitutions (we will write $Hyp\mathcal{G}$).

In [4] the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called *$\mathcal{K}'$-equivalent* if and only if $\sigma_1(f) \approx \sigma_2(f)$ is an identity in the class $\mathcal{K}'$. In this case we write $\sigma_1 \sim_{\mathcal{K}'} \sigma_2$.

In [2] (see also [4]) the following lemma was proved:

Lemma 4.1. *If* $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}'$ *and* $\sigma_1 \sim_{\mathcal{K}'} \sigma_2$, *then* $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}'$.

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{K}'}$.

In [9] it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term of* t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, \ldots, x_{i_{k(x)}})$ of all out-neighbors (i.e., $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq \cdots \leq i_{k(x)}$ and let s_x be the term $(\cdots((xx_{i_1})x_{i_2})\cdots x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. Continue this procedure while $Z \neq \phi$. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

In [1] the following definition was given:

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$ is called a *normal form graph hypersubstitution* where $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}'$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}'} \sigma_{NF(t)}$.

In [1] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.

normal form term	graph hypers.	normal form term	graph hypers.
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$x_2((x_1x_1)x_2)$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$(x_2(x_1x_2))x_2$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

By Theorem 3.1, we have the following relations:
(i) $\sigma_6 \sim_{\mathcal{K}'} \sigma_8 \sim_{\mathcal{K}'} \sigma_{10}$,
(ii) $\sigma_7 \sim_{\mathcal{K}'} \sigma_9 \sim_{\mathcal{K}'} \sigma_{11}$,
(iii) $\sigma_{14} \sim_{\mathcal{K}'} \sigma_{16} \sim_{\mathcal{K}'} \sigma_{18}$,
(iv) $\sigma_{15} \sim_{\mathcal{K}'} \sigma_{17} \sim_{\mathcal{K}'} \sigma_{19}$.
Let $M_{\mathcal{K}'}$ be the set of all normal form graph hypersubstitutions in the class $\mathcal{K}'$.

Then we get

$$M_{\mathcal{K}'} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

We defined the product of two normal form graph hypersubstitutions in $M_{\mathcal{K}'}$ as follows.

Definition 4.5. The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ *of two normal form graph hypersubstitutions* is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The following table gives the multiplication of elements in $M_{\mathcal{K}'}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_{12}	σ_{13}	σ_{14}	σ_{15}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_{12}	σ_{13}	σ_{14}	σ_{15}
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2
σ_3	σ_3	σ_1	σ_2	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4	σ_3	σ_4
σ_4	σ_4	σ_1	σ_2	σ_3	σ_4	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4
σ_5	σ_5	σ_1	σ_2	σ_3	σ_4	σ_0	σ_7	σ_{13}	σ_6	σ_7	σ_6	σ_7
σ_6	σ_6	σ_1	σ_2	σ_3	σ_4	σ_7	σ_6	σ_7	σ_{14}	σ_{15}	σ_{14}	σ_{15}
σ_7	σ_7	σ_1	σ_2	σ_3	σ_4	σ_6	σ_7	σ_{15}	σ_6	σ_7	σ_6	σ_7
σ_{12}	σ_{12}	σ_1	σ_2	σ_3	σ_4	σ_{13}	σ_{14}	σ_{15}	σ_{12}	σ_{13}	σ_{14}	σ_{15}
σ_{13}	σ_{13}	σ_1	σ_2	σ_3	σ_4	σ_{12}	σ_{15}	σ_{13}	σ_7	σ_{15}	σ_{14}	σ_{15}
σ_{14}	σ_{14}	σ_1	σ_2	σ_3	σ_4	σ_{15}	σ_7	σ_{15}	σ_{14}	σ_{15}	σ_{14}	σ_{15}
σ_{15}	σ_{15}	σ_1	σ_2	σ_3	σ_4	σ_{14}	σ_{15}	σ_{15}	σ_{14}	σ_{15}	σ_{14}	σ_{15}

In [1] the concept of the leftmost normal form graph hypersubstitution was defined.

Definition 4.6. A graph hypersubstitution σ is called *leftmost* if $L(\sigma(f)) = x_1$.

The set $M_{L(\mathcal{K}')}$ of all leftmost normal form graph hypersubstitutions in $M_{\mathcal{K}'}$ is

$$M_{L(\mathcal{K}')} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_{12}, \sigma_{14}\}.$$

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

Definition 4.7. A hypersubstitution σ is called *proper with respect to the class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called *regular.* It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

We want to prove that $\{\sigma_0, \sigma_6, \sigma_{12}, \sigma_{14}\}$ is the set of all proper graph hypersubstitutions with respect to the class $\mathcal{K}'$.

In [1] the following lemma was proved.

Lemma 4.2. *For each non-trivial term* $s, (s \neq x \in X)$ *and for all* $u, v \in X$, *we have*

$$E(\hat{\sigma_6}[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\}$$

and

$$E(\hat{\sigma_{12}}[s]) = E(s) \cup \{(v,u)|(u,v) \in E(s)\}.$$

Then we obtain:

Theorem 4.1. *$\{\sigma_0, \sigma_6, \sigma_{12}, \sigma_{14}\}$ is the set of all proper graph hypersubstitutions with respect to the class $\mathcal{K}'$ of $(xy)x \approx x(yy)$ graph algebras.*

Proof. For $s \approx t \in Id\mathcal{K}'$, if s, t are trivial terms, then $\hat{\sigma_6}[s], \hat{\sigma_{12}}[s], \hat{\sigma_6}[t]$ and $\hat{\sigma_{12}}[t]$ are also trivial terms and thus $\hat{\sigma_6}[s] \approx \hat{\sigma_6}[t] \in Id\mathcal{K}'$ and $\hat{\sigma_{12}}[s] \approx \hat{\sigma_{12}}[t] \in Id\mathcal{K}'$. If $s = t = x$, we have the same result in the same manner.

Now, assume that s and t are non-trivial terms, different from variables, and $s \approx t \in Id\mathcal{K}'$. Then (i), (ii), (iii) and (iv) of Theorem 3.1 hold.

For σ_6 and σ_{12}, we obtain:

$$L(\hat{\sigma}_6[s]) = L(s) = L(t) = L(\hat{\sigma}_6[t])$$

and

$$L(\hat{\sigma}_{12}[s]) = L(s) = L(t) = L(\hat{\sigma}_{12}[t]).$$

Since σ_6, σ_{12} are regulars, we have the following:

$$V(\hat{\sigma}_6[s]) = V(s) = V(t) = V(\hat{\sigma}_6[t])$$

and

$$V(\hat{\sigma}_{12}[s]) = V(s) = V(t) = V(\hat{\sigma}_{12}[t]).$$

By Lemma 4.2, we get that:

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) \mid (u,v) \in E(s)\},$$

$$E(\hat{\sigma}_6[t]) = E(t) \cup \{(u,u) \mid (u,v) \in E(t)\},$$

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) \mid (u,v) \in E(s)\}$$

and

$$E(\hat{\sigma}_{12}[t]) = E(t) \cup \{(v,u) \mid (u,v) \in E(t)\}.$$

For any x, y with $x \neq y$, suppose that $(x,y) \in E(\hat{\sigma}_6[s])$, we have $(x,y) \in E(s)$. Then by Theorem 3.1 (iii) $(x,y) \in E(t)$. Hence $(x,y) \in E(\hat{\sigma}_6[t])$. Conversely, we prove that if $(x,y) \in E(\hat{\sigma}_6[t])$, then $(x,y) \in E(\hat{\sigma}_6[s])$.

Now suppose that $(x,x) \in E(\hat{\sigma}_6[s])$. If $(x,x) \in E(s)$, then by Theorem 3.1 (iv) there exists $y \in V(t)$ such that $(y,y) \in E(t)$. Hence there exists $y \in V(t)$ such that $(y,y) \in E(\hat{\sigma}_6[t])$. If $(x,x) \notin E(s)$, then there exists $y \in V(s)$ such that $(x,y) \in E(s)$. By Theorem 3.1 (iii), we get $(x,y) \in E(t)$ and thus $(x,x) \in E(\hat{\sigma}_6[t])$. Conversely, we prove that if $(x,x) \in E(\hat{\sigma}_6[t])$, then there exists $y \in V(s)$ such that $(y,y) \in E(s)$. Hence by Theorem 3.1, we get that $\hat{\sigma_6}[s] \approx \hat{\sigma_6}[t] \in Id\mathcal{K}'$.

Similarly, we get that $\hat{\sigma_{12}}[s] \approx \hat{\sigma_{12}}[t] \in Id\mathcal{K}'$.

Since $\sigma_{14} = \sigma_6 \circ_N \sigma_{12}$ and σ_6 and σ_{12} are proper graph hypersubstitutions, we have σ_{14} is a proper graph hypersubstitution.

For any $\sigma \notin \{\sigma_0, \sigma_6, \sigma_{12}, \sigma_{14}\}$, we give an identity $s \approx t$ in $Id\mathcal{K}'$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}'$. Clearly, if s and t are trivial terms with different leftmost and different rightmost, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \notin Id\mathcal{K}'$, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \notin Id\mathcal{K}'$, $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \notin Id\mathcal{K}'$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \notin Id\mathcal{K}'$.

Let $s = (x_1x_1)(x_2x_1)$ and $t = x_1((x_2x_1)x_2)$. By Theorem 3.1, we get $s \approx t \in Id\mathcal{K}'$. If $\sigma \in \{\sigma_5, \sigma_7, \sigma_{13}, \sigma_{15}\}$, then $L(\sigma(f)) = x_2$. We see that $L(\hat{\sigma}[s]) = x_1$ and $L(\hat{\sigma}[t]) = x_2$. Thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}'$. □

Now, we apply our results to characterize all hyperidentities in the class of all $(xy)x \approx x(yy)$ graph algebras. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in the class $\mathcal{K}'$ if and only if they have the same leftmost and the same rightmost and $x \approx x, x \in X$ is a hyperidentity in the class $\mathcal{K}'$, too. So we consider the case that s and t are non-trivial and different from variables. In [1] the concept of a *dual term* s^d of the non-trivial term s was defined in the following way:

If $s = x \in X$, then $x^d = x$, if $s = t_1t_2$, then $s^d = t_2^dt_1^d$. The dual term s^d can be obtained by application of the graph hypersubstitution σ_5, namely $\hat{\sigma}_5[s] = s^d$.

Theorem 4.2. *An identity $s \approx t$ in the class $\mathcal{K}'$, where s and t are non-trivial and $s \neq x, t \neq x$, is a hyperidentity in the class $\mathcal{K}'$ if and only if the dual equation $s^d \approx t^d$ is also an identity in the class $\mathcal{K}'$.*

Proof. If $s \approx t$ is a hyperidentity in the class $\mathcal{K}'$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in the class $\mathcal{K}'$, i.e., $s^d \approx t^d$ is an identity in the class $\mathcal{K}'$. Conversely, assume that $s \approx t$ is an identity in the class $\mathcal{K}'$ and that $s^d \approx t^d$ is an identity in the class $\mathcal{K}'$, too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{K}'}$.

If $\sigma \in \{\sigma_0, \sigma_6, \sigma_{12}, \sigma_{14}\}$, then σ is a proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}'$. By assumption, $\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]$ is an identity in the class $\mathcal{K}'$.

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$.

Because of $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$, $\sigma_{14} \circ_N \sigma_5 = \sigma_{15}$ and $\hat{\sigma}[\hat{\sigma}_5[t']] = \hat{\sigma}[t'^d]$ for all $\sigma \in M_{\mathcal{K}'}$, $t' \in W_\tau(X)$, we have that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$, $\hat{\sigma}_{15}[s] \approx \hat{\sigma}_{15}[t]$ are the identities in the class $\mathcal{K}'$. □

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