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ON THE COMMUTANT OF MULTIPLICATION OPERATORS WITH ANALYTIC POLYNOMIAL SYMBOLS

B. Khani Robati

Abstract. Let $\mathcal{B}$ be a certain Banach space consisting of analytic functions defined on a bounded domain G in the complex plane. Let φ be an analytic polynomial or a rational function and let M_φ denote the operator of multiplication by φ. Under certain condition on φ and G, we characterize the commutant of M_φ that is the set of all bounded operators T such that $TM_\varphi = M_\varphi T$. We show that $T = M_\Psi$ for some function Ψ in $\mathcal{B}$.

1. Introduction

Let $\mathcal{B}$ be a Banach space consisting of analytic functions defined on a bounded domain G in the complex plane such that $1 \in \mathcal{B}, z\mathcal{B} \subset \mathcal{B}$, and for every $\lambda \in G$ the evaluation functional at λ, $e_\lambda : \mathcal{B} \to \mathbb{C}$, given by $f \mapsto f(\lambda)$, is bounded. Also assume $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$ for every $\lambda \in G$ and if $f \in \mathcal{B}$ and $|f(\lambda)| > c > 0$ for every $\lambda \in G$, then $\frac{1}{f}$ is a multiplier of $\mathcal{B}$.

Throughout this article unless otherwise is explicitly stated, we assume that G is a bounded domain in the complex plane and by a Banach space of analytic functions $\mathcal{B}$ on G we mean one satisfying the above conditions.

Some examples of such spaces are as follows:

1) The algebra $A(G)$ which is the algebra of all continuous functions on the closure of G that are analytic on G.

2) The Bergman space of analytic functions defined on G, $L_a^p(G)$ for $1 \leq p \leq \infty$.

3) The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $\mathbf{D}$, for which

$$\| f \|_\alpha^2 = \sum (n+1)^\alpha |\hat{f}(n)|^2 < \infty$$

for every $\alpha \geq 1$ or $\alpha \leq 0$.

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4) The analytic Lipschitz spaces $Lip(\alpha, \overline{G})$ for $0 < \alpha < 1$, i.e., the space of all analytic functions defined on G that satisfy a Lipschitz condition of order α.

5) The subspace $lip(\alpha, \overline{G})$ of $Lip(\alpha, \overline{G})$ consisting of functions f in $Lip(\alpha, \overline{G})$ for which

$$\lim_{z \to w} \frac{| f(z) - f(w) |}{| z - w |^{\alpha}} = 0.$$

6) The classical Hardy spaces H^p for $1 \leq p \leq \infty$.

In this article $r(z) = p(z)/q(z)$ is a rational function such that $p(z)$ and $q(z)$ are polynomials without common factors. Also the poles of $r(z)$ which are exactly the zeros of $q(z)$ are off $\overline{G}$. Let r be a rational function and $\lambda \in \overline{G}$. If $r(z)$ has a zero of order one at $\lambda \in \overline{G}$ and $r(z) \neq 0$ for all $z \neq \lambda$ in $\overline{G}$, then we say that r has only a simple zero in $\overline{G}$.

A complex valued function ϕ defined on G is called a multiplier of $\mathcal{B}$ if $\phi\mathcal{B} \subset \mathcal{B}$, i.e., ϕf is in $\mathcal{B}$ for every f in $\mathcal{B}$, and the set of all multipliers of $\mathcal{B}$ is denoted by $\mathcal{M}(\mathcal{B})$. By the Closed Graph Theorem, it is easy to see that every multiplier ϕ defines a bounded linear operator $M_\phi : f \to \phi f$ on $\mathcal{B}$. The algebra of all bounded operators on $\mathcal{B}$ is denoted by $L(\mathcal{B})$. Let $T \in L(\mathcal{B})$ and $TM_z = M_z T$, it is easy to see that $T = M_\varphi$ for some function $\varphi \in \mathcal{M}(\mathcal{B})$. A good source on this topics is [7]. We denote by $\{M_\varphi\}'$ the set of operators $T \in L(\mathcal{B})$ such that $M_\varphi T = TM_\varphi$, i.e., the commutant of M_φ.

The commutant of Toeplitz operator on certain Hilbert spaces of functions was studied in several papers. See, for example, [1-4, 8]. Also the commutant of multiplication operators on Banach spaces of functions were investigated for certain multiplication operators. See for instance [5-7, 9-11]. In section 2 of this article we investigate the commutant of the operator M_φ, when φ is an analytic polynomial or a rational function. By the Implicit Function Theorem under certain condition on φ and G, we characterize the commutant of M_φ. In fact, when p is a polynomial of degree one it is an univalent function and it is well known that $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$. Hence we consider certain polynomials with degree $n \geq 2$. We conclude the introduction with a theorem that will be used in the proof of Proposition 2.2.

Theorem 1.1. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(G)$. If for some $\lambda \in G$, $\phi - \phi(\lambda)$ has only a simple zero in $\overline{G}$, then $T(f)(\lambda) = T(1)(\lambda) f(\lambda)$ for each $f \in \mathcal{B}$ and every $T \in \{M_\phi\}'$.*

Proof. See [6, Theorem 2.1]. □

2. The main results

In [4] Ž. Čučković and Dashan Fan have shown that if $G = \{z \in \mathbb{C} : r < |z| < 1\}$, $\mathcal{B} = L_a^2(G)$ and $p(z) = z + a_2 z^2 + \cdots + a_n z^n$, where $a_i \geq 0$ and $p(z) - p(1)$ has n distinct zeros, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$. In this section we extend the result obtained in [4] to various domains G, to Banach spaces of analytic functions and to certain polynomial or rational symbols. The proof of the next theorem is similar to a part of the proof of Theorem 4 in [4].

Theorem 2.1. *Suppose that G is an open set in $\mathbb{C}$. Let $r(z) = p(z)/q(z)$ be a rational function with poles off $\overline{G}$, let $n = \max\{\deg(p), \deg(q)\} \geq 2$ and let $\lambda \in \overline{G}$. If $r(z) - r(\lambda)$ has $n-1$ distinct zeros outside of $\overline{G}$, then there is an open set $U \subseteq G$ such that for every $w \in U$ the function $r(z) - r(w)$ has only a simple zero in $\overline{G}$.*

Proof. Let A be the set of zeros of q and let $\Omega = \mathbb{C} - A$. Assume that $z_1, z_2, \ldots, z_{n-1}$ are distinct zeros of $r(z) - r(\lambda)$ outside $\overline{G}$. We now choose open subsets $\Omega_1, \Omega_2, \ldots, \Omega_{n-1}$ of Ω such that $z_i \in \Omega_i$ and $\Omega_i \cap \overline{G} = \emptyset$ for every $i = 1, 2, \ldots, n-1$, and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Assume that$F : \Omega \times \Omega \to \mathbb{C}$ defined by $F(z, w) = r(z) - r(w)$. Then $F(\lambda, w)$ has $n-1$ zeros outside $\overline{G}$. Since by the hypothesis the zeros of $F(\lambda, w)$ are simple, we have

$$\frac{\partial F}{\partial w}(\lambda, z_i) = (r(\lambda) - r(w))'(z_i) \neq 0 \quad \text{for } i = 1, 2, \ldots, n-1.$$

Thus by the Implicit Function Theorem, for each i, there exists an open neighborhood V_i and a continuous map $\varphi_i : V_i \to \mathbb{C}$ such that $\lambda \in V_i$, $\varphi_i(\lambda) = z_i$ and $F(z, \varphi_i(z)) = 0$ for every $z \in V_i$ and $i = 1, 2, \ldots, n-1$. Since φ_i is continuous, there exists an open subset U_i of V_i such that $\lambda \in U_i$ and $\varphi_i(U_i) \subseteq \Omega_i$. Let $U_0 = \cap_{i=1}^{n-1} U_i$. Then $U = U_0 \cap G$ is a nonempty open subset of G. Suppose that $w \in U$. Then $w \in U_i$ for every i and so, $(w, \varphi_1(w)), (w, \varphi_2(w)), \ldots,$ $(w, \varphi_{n-1}(w))$ are zeros of F. Hence $\varphi_1(w), \varphi_2(w), \ldots, \varphi_{n-1}(w)$ are $n-1$ distinct roots of the equation $r(z) - r(w) = 0$, which are outside of $\overline{G}$. □

Proposition 2.2. *Let $\mathcal{B}$ be a Banach space of analytic functions on G. If $r(z)$ satisfies the conditions of Theorem 2.1, then*

$$\{M_r\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}.$$

Proof. By Theorem 2.1, there is an open set $U \subseteq G$ such that for every $w \in U$ the function $r(z) - r(w)$ has only a simple zero in $\overline{G}$. Assume that $T \in \{M_r\}'$. By Theorem 1.1, for every $w \in U$ and each $f \in \mathcal{B}$, we have $T(f)(w) = T(1)(w)f(w)$. Since two analytic function $T(f)$ and $T(1)f$ are equal on U and G is connected, we have $T(f) = T(1)f$ for all $f \in \mathcal{B}$, and the proof is complete. □

Example 2.3. Let $\mathcal{B}$ be a Banach space of analytic functions on D, where D is the unit disk and let $r(z) = z^n/(2-z)$ for some positive integer $n \geq 2$. It is easy to see that $|r(1)| = 1$ and $|r(z)| < 1$ for $z \in \overline{D} - \{1\}$. Using Theorem 2.1, with $\lambda = 1$ and Proposition 2.2, we have

$$\{M_r\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}.$$

In the reminder of this article we assume that $G \subset D$ and $\varphi(z)$ is a polynomial.

Theorem 2.4. *Let $\mathcal{B}$ be a Banach space of analytic functions on G. Let $n \geq 2$ be an integer, $a \neq 0$ and b be two complex numbers, and let $p(z) = z^n + az + b$. Then*

a) If $0 \in \overline{G}$ and $|a| > 1$, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

b) If $a = |a|e^{i\theta}$ belongs to $\overline{D}$ and $e^{\frac{i\theta}{n-1}} \in \overline{G}$, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

Proof. *a)* If in Theorem 2.1 we set $\lambda = 0$, then it is easy to see that $p(z)-p(\lambda) = z^n + az$ has $n-1$ distinct zeros outside of $\overline{D}$. Hence by Proposition 2.2, we have $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

b) Set $\lambda = e^{\frac{i\theta}{n-1}}$, we can see that $p(\lambda) = p(z)$ if and only if $\lambda^n + a\lambda = z^n + az$. But

$$| \lambda^n + a\lambda | = | \lambda || \lambda^{n-1} + a | = | e^{i\theta} + | a | e^{i\theta} | = 1 + | a | .$$

So if $|z| < 1$, then $| z^n + az | \leq |z^n| + |a||z| < 1 + |a| = | \lambda^n + a\lambda |$, which implies that z isn't a zero of $p(z)-p(\lambda)$. In the next step assume that $z_0 \in \partial D$ is a root of equation $p(z)-p(\lambda) = 0$. Therefore, $| {z_0}^{n-1} + a | = | {z_0}^n + az_0 | = | \lambda^n + a\lambda | = 1 + | a | = | {z_0}^{n-1} | + | a |$. Thus $\arg {z_0}^{n-1} = \arg a + 2k\pi$ for some integer k. Since $| z_0 | = 1$, we have ${z_0}^{n-1} = \lambda^{n-1}$. It follows that $(z_0 - \lambda)(a + \lambda^{n-1}) = 0$, which implies that $z_0 = \lambda$. To complete the proof of the theorem, it suffices to show that $n-1$ zeros of $p(z)-p(\lambda)$ outside of $\overline{D}$ are distinct. Since the absolute value of each zero of $p'(z)$ is $| \frac{a}{n} |^{\frac{1}{n-1}}$ which is less than one, we conclude that the zeros of $p(z) - p(\lambda)$ outside of $\overline{D}$ are distinct. Hence by Proposition 2.2, the proof is complete. □

Theorem 2.5. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G. Let $n \geq 2$ be an integer, $a \neq 0$ and b be two complex numbers, and let $p(z) = z^n + az^{n-1} + b$. Also assume that $a = |a|e^{i\theta_0}$ and $e^{i\theta_0} \in \overline{G}$. If $(n-1)^{n-1}|a^n| \neq n^n(1 + |a|)$, then*

$$\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}.$$

Proof. Set $\lambda = e^{i\theta_0}$, we can see that $p(\lambda) = p(z)$ if and only if $\lambda^n + a\lambda^{n-1} = z^n + az^{n-1}$. But

$$| \lambda^n + a\lambda^{n-1} | = | e^{in\theta_0} + | a | e^{in\theta_0} | = 1 + | a | .$$

So if $| z | < 1$, then $| z^n + az^{n-1} | \leq | z^n | + | a || z^{n-1} | < 1 + | a | = | \lambda^n + a\lambda^{n-1} |$, which implies that z isn't a zero of $p(z) - p(\lambda)$. Now assume that $z_0 \in \partial D$ is a root of equation $p(z) - p(\lambda) = 0$. Therefore, $| z_0 + a | = | {z_0}^n + az_0^{n-1} | = | \lambda^n + a\lambda^{n-1} | = 1 + | a | = | z_0 | + | a |$. Thus $\operatorname{Arg} z_0 = \operatorname{Arg}(a)$. Since $| z_0 | = 1$, we have $z_0 = \lambda$. To complete the proof of the theorem we need only to show that $n-1$ zeros of $p(z) - p(\lambda)$ outside of $\overline{D}$ are distinct. Since the only nonzero root of the equation $p'(z) = 0$ is $\frac{-a(n-1)}{n}$, it suffices to show that $\frac{-a(n-1)}{n}$ is not a root of equation $p(z)-p(\lambda) = 0$. But $p(\frac{-a(n-1)}{n}) = p(\lambda)$ implies that $(-1)^{n-1}a^n\frac{(n-1)^{n-1}}{n^{n-1}} = n(e^{in\theta_0} + |a|e^{in\theta_0})$ and so $|a|^n\frac{(n-1)^{n-1}}{n^{n-1}} = n(1+|a|)$. But by assumption this relation is not true. Therefore, we conclude that $n-1$ zeros of $p(z) - p(\lambda)$ outside $\overline{D}$ are distinct and by Proposition 2.2, the proof is complete. □

Theorem 2.6. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G and let $p = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ such that $a_1 \neq 0$. If there is $z_0 \in \partial D \cap \partial G$ such that all nonzero terms $a_i{z_0}^i$ for $i \geq 1$ are positive or all are negative also $p(z) - p(z_0)$ has n distinct zeros, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Proof. By assumption $p(z_0) - a_0$ is real and $| p(z_0) - a_0 |=| a_1z_0 | + | a_2{z_0}^2 | + \cdots + | a_n{z_0}^n |$. Hence $p(z) - p(z_0) = 0$ implies that

$$| a_1z + a_2z^2 + \cdots + a_nz^n |=| a_1 | + | a_2 | + \cdots + | a_n | .$$

For $z \in D$, we have

$$| a_1z + a_2z^2 + \cdots + a_nz^n |<| a_1 | + | a_2 | + \cdots + | a_n | .$$

So $p(z) - p(z_0)$ has no zero in D. On the other hand if $z \in \partial D$ is a zero of $p(z) - p(z_0)$, then

$$\begin{aligned} | a_1 | + | a_2 | + \cdots + | a_n | &= | a_1z | + | a_2z^2 | + \cdots + | a_nz^n | \\ &= | a_1z + a_2z^2 + \cdots + a_nz^n | . \end{aligned}$$

Hence $\text{Arg}(a_1z + a_2z^2 + \cdots + a_nz^n) = \text{Arg}\ (a_1z)$. Since $p(z) = p(z_0)$, we have $p(z) - a_0 = p(z_0) - a_0$. Thus $p(z) - a_0$ is a real number and also $\text{Arg}(a_1z + a_2z^2+\cdots+a_nz^n) = \text{Arg}(a_1z_0)$. Hence $\text{Arg}(a_1z) = \text{Arg}(a_1z_0)$ and since $| z |= 1$, we have $z = z_0$. Therefore, $p(z) - p(z_0)$ has $n - 1$ distinct zeros outside $\overline{D}$ and by Proposition 2.2, the proof is complete. □

An easy application of the above theorem is when p is a polynomial of degree 3 such that its coefficients satisfy the conditions of Theorem 2.6. Let a and b be zeros of $p'(z)$. If $p(a)$ and $p(b)$ are not equal to $p(z_0)$, then $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

Using the same argument as used in the proof of Theorem 2.6, we have the following two propositions.

Proposition 2.7. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G. Let $p = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ such that $a_1 \neq 0$ and $\text{Arg}\, a_i = \theta_0$ for $a_i \neq 0$ with $i \geq 1$. If $p(z) - p(1)$ has n distinct zeros and $1 \in \partial G$, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Proposition 2.8. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G and let $p = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ such that $a_1 \neq 0$. Also assume that for each $a_i \neq 0$ with $i \geq 1$, $\text{Arg}\, a_i = \theta_0$ for i odd and $\text{Arg}\, a_i = \theta_0 + \pi$ or $\text{Arg}\, a_i = \theta_0 - \pi$ for i even . If $p(z) - p(-1)$ has n distinct zeros and $-1 \in \partial G$, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Theorem 2.9. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G, and let $p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 2$. If the maximum value of $\mid p(z) \mid$ on $\overline{D}$ is obtained at a unique point $z_0 \in \partial G$ and $\mid a_1 \mid +2 \mid a_2 \mid +3 \mid a_3 \mid + \cdots + (n-1) \mid a_{n-1} \mid < n \mid a_n \mid$ or none of the zeros of $p^{'}(z)$ is a root of equation $p(z) - p(z_0) = 0$, then $\{M_p\}^{'} = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Proof. By assumption $|p(z)| < |p(z_0)|$ for all $z \in \overline{D} - \{z_0\}$. Hence $p(z) - p(z_0)$ has no zero in $\overline{G} - \{z_0\}$. If none of the zeros of $p^{'}(z)$ is a root of equation $p(z) - p(z_0) = 0$, then $n-1$ zeros of $p(z) - p(z_0)$ outside $\overline{D}$, and therefore outside $\overline{G}$ are distinct. Thus by Proposition 2.2, the proof is complete. Otherwise by assumption $|a_1| + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| \ < n|a_n|$. Hence $|p^{'}(z) - na_n z^{n-1} \mid < \mid na_n z^{n-1} \mid$ for all $z \in \partial D$ and therefore by the Rouche's Theorem $p^{'}(z)$ has $n-1$ zeros inside D, which implies that $n-1$ zeros of $p(z) - p(z_0)$ outside $\overline{D}$ are distinct. Now by Proposition 2.2, the proof is complete. □

Corollary 2.10. *Let $\mathcal{B}$ be a Banach spaces of analytic functions on G, let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 2$ with nonnegative real coefficients and let $1 \in \partial G$. If there is a positive integer $m \leq n$ such that a_m and a_{m-1} are not equal to zero and*

$$\mid a_1 \mid +2 \mid a_2 \mid +3 \mid a_3 \mid + \cdots + (n-1) \mid a_{n-1} \mid < n \mid a_n \mid,$$

then $\{M_p\}^{'} = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

Proof. It is easy to see that $|p(1)| > |p(z)|$ for all $z \in \overline{D} - \{1\}$. Indeed, if $z = e^{i\theta}$ for some θ, $-\pi < \theta \leq \pi$ and $\mid p(z) \mid = \mid p(1) \mid$, we have

$$\mid a_m e^{im\theta} + a_{m-1} e^{i(m-1)\theta} \mid = a_m + a_{m-1}.$$

Therefore, $m\theta = (m-1)\theta + 2k\pi$ for some integer k. Hence $z = 1$, and by Theorem 2.9, the proof is complete. □

In the next example we present three applications of some of the above theorems.

Example 2.11. Let $\mathcal{B}$ be a Banach spaces of analytic functions on G.

a) Let $p(z) = -z^3 + 6z^2 - 9z + 5$, and let $z_0 = -1$ belongs to $\overline{G}$. Since $p^{'}(z) = -3z^2 + 12z - 9$ has zeros 1 and 3, which aren't the roots of equation $p(z) - p(-1) = 0$ by Theorem 2.6, or Proposition 2.8, we have $\{M_p\}^{'} = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

b) Let $p(z) = 3z^7 + 2z^5 + 4z^2 + 5$ and let $z_0 = 1$ belongs to $\overline{G}$. Then by Theorem 2.9, we have $\{M_p\}^{'} = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

c) Let $p(z) = iz^3 + 3iz^2 + 3iz - 3$ and $1 \in \overline{G}$. Since $p^{'}(z) = 3iz^2 + 6iz + 3i$ has a zero of order 2 at $z = -1$ and -1 isn't a root of equation $p(z) - p(1) = 0$, we conclude that the roots of this equation are distinct. Hence by Proposition 2.7, we have $\{M_p\}^{'} = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

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Department of Mathematics
College of Sciences
Shiraz University
Shiraz 71454, IRAN
E-mail address: khani@susc.ac.ir