ON THE COMMUTANT OF MULTIPLICATION OPERATORS
WITH ANALYTIC POLYNOMIAL SYMBOLS

B. Khani Robati

Abstract. Let $B$ be a certain Banach space consisting of analytic functions defined on a bounded domain $G$ in the complex plane. Let $\varphi$ be an analytic polynomial or a rational function and let $M_{\varphi}$ denote the operator of multiplication by $\varphi$. Under certain condition on $\varphi$ and $G$, we characterize the commutant of $M_{\varphi}$ that is the set of all bounded operators $T$ such that $TM_{\varphi} = M_{\varphi}T$. We show that $T = M_{\Psi}$ for some function $\Psi$ in $B$.

1. Introduction

Let $B$ be a Banach space consisting of analytic functions defined on a bounded domain $G$ in the complex plane such that $1 \in B, zB \subset B$, and for every $\lambda \in G$ the evaluation functional at $\lambda$, $e_{\lambda}: B \to \mathbb{C}$, given by $f \mapsto f(\lambda)$, is bounded. Also assume $\text{ran}(M_z - \lambda) = \text{ker} e_{\lambda}$ for every $\lambda \in G$ and if $f \in B$ and $|f(\lambda)| > c > 0$ for every $\lambda \in G$, then $\frac{1}{f}$ is a multiplier of $B$.

Throughout this article unless otherwise is explicitly stated, we assume that $G$ is a bounded domain in the complex plane and by a Banach space of analytic functions $B$ on $G$ we mean one satisfying the above conditions.

Some examples of such spaces are as follows:

1) The algebra $A(G)$ which is the algebra of all continuous functions on the closure of $G$ that are analytic on $G$.

2) The Bergman space of analytic functions defined on $G$, $L^p_a(G)$ for $1 \leq p < \infty$.

3) The spaces $D_\alpha$ of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $D$, for which

$$||f||_{\alpha}^2 = \sum(n + 1)^{\alpha}|\hat{f}(n)|^2 < \infty$$

for every $\alpha \geq 1$ or $\alpha \leq 0$.

Received August 10, 2006; Revised August 28, 2007.
2000 Mathematics Subject Classification. Primary 47B35; Secondary 47B38.
Key words and phrases. commutant, multiplication operators, Banach space of analytic functions, univalent function, bounded point evaluation.

©2007 The Korean Mathematical Society

683
4) The analytic Lipschitz spaces \( \text{Lip}(\alpha, \overline{G}) \) for \( 0 < \alpha < 1 \), i.e., the space of all analytic functions defined on \( G \) that satisfy a Lipschitz condition of order \( \alpha \).

5) The subspace \( \text{lip}(\alpha, \overline{G}) \) of \( \text{Lip}(\alpha, \overline{G}) \) consisting of functions \( f \) in \( \text{Lip}(\alpha, \overline{G}) \) for which

\[
\lim_{z \to w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0.
\]

6) The classical Hardy spaces \( H^p \) for \( 1 \leq p \leq \infty \).

In this article \( r(z) = p(z)/q(z) \) is a rational function such that \( p(z) \) and \( q(z) \) are polynomials without common factors. Also the poles of \( r(z) \) which are exactly the zeros of \( q(z) \) are off \( \overline{G} \). Let \( r \) be a rational function and \( \lambda \in \overline{G} \). If \( r(z) \) has a zero of order one at \( \lambda \in \overline{G} \) and \( r(z) \neq 0 \) for all \( z \neq \lambda \) in \( \overline{G} \), then we say that \( r \) has only a simple zero in \( \overline{G} \).

A complex valued function \( \phi \) defined on \( G \) is called a multiplier of \( B \) if \( \phi B \subset B \), i.e., \( \phi f \) is in \( B \) for every \( f \) in \( B \), and the set of all multipliers of \( B \) is denoted by \( \mathcal{M}(B) \). By the Closed Graph Theorem, it is easy to see that every multiplier \( \phi \) defines a bounded linear operator \( M_\phi : f \to \phi f \) on \( B \). The algebra of all bounded operators on \( B \) is denoted by \( L(B) \). Let \( T \in L(B) \) and \( TM_z = M_z T \), it is easy to see that \( T = M_\phi \) for some function \( \varphi \in \mathcal{M}(B) \). A good source on this topics is \([7]\). We denote by \( \{ M_\phi \}^\prime \) the set of operators \( T \in L(B) \) such that \( M_\phi T = TM_\phi \), i.e., the commutant of \( M_\phi \).

The commutant of Toeplitz operator on certain Hilbert spaces of functions was studied in several papers. See, for example, \([1-4, 8]\). Also the commutant of multiplication operators on Banach spaces of functions were investigated for certain multiplication operators. See for instance \([5-7, 9-11]\). In section 2 of this article we investigate the commutant of the operator \( M_\phi \), when \( \varphi \) is an analytic polynomial or a rational function. By the Implicit Function Theorem under certain condition on \( \varphi \) and \( G \), we characterize the commutant of \( M_\phi \).

In fact, when \( p \) is a polynomial of degree one it is an univalent function and it is well known that \( \{ M_\phi \}^\prime = \{ M_\psi : \Psi \in \mathcal{M}(B) \} \). Hence we consider certain polynomials with degree \( n \geq 2 \). We conclude the introduction with a theorem that will be used in the proof of Proposition 2.2.

**Theorem 1.1.** Let \( B \) be a Banach space of analytic functions and let \( \phi \in \mathcal{M}(B) \cap A(G) \). If for some \( \lambda \in G \), \( \phi - \phi(\lambda) \) has only a simple zero in \( \overline{G} \), then \( T(f)(\lambda) = T(1)(\lambda) f(\lambda) \) for each \( f \in B \) and every \( T \in \{ M_\phi \}^\prime \).

**Proof.** See \([6, \text{Theorem 2.1}]\). \( \square \)

2. The main results

In \([4]\) Ž. Čučković and Dashan Fan have shown that if \( G = \{ z \in \mathbb{C} : r < |z| < 1 \} \), \( B = L^2_\alpha(G) \) and \( p(z) = a_2 z^2 + \cdots + a_n z^n \), where \( a_i \geq 0 \) and \( p(z) - p(1) \) has \( n \) distinct zeros, then \( \{ M_p \}^\prime = \{ M_\Psi : \Psi \in \mathcal{M}(B) \} \). In this
section we extend the result obtained in [4] to various domains $G$, to Banach spaces of analytic functions and to certain polynomial or rational symbols. The proof of the next theorem is similar to a part of the proof of Theorem 4 in [4].

**Theorem 2.1.** Suppose that $G$ is an open set in $\mathbb{C}$. Let $r(z) = p(z)/q(z)$ be a rational function with poles off $\overline{G}$, let $n = \max\{\deg(p), \deg(q)\} \geq 2$ and let $\lambda \in \overline{G}$. If $r(z) - r(\lambda)$ has $n - 1$ distinct zeros outside of $\overline{G}$, then there is an open set $U \subseteq G$ such that for every $w \in U$ the function $r(z) - r(w)$ has only a simple zero in $G$.

**Proof.** Let $A$ be the set of zeros of $q$ and let $\Omega = \mathbb{C} - A$. Assume that $z_1, z_2, \ldots, z_{n-1}$ are distinct zeros of $r(z) - r(\lambda)$ outside $\overline{G}$. We now choose open subsets $\Omega_1, \Omega_2, \ldots, \Omega_n$ of $\Omega$ such that $z_i \in \Omega_i$ and $\Omega_i \cap \overline{\Omega_j} = \emptyset$ for every $i = 1, 2, \ldots, n - 1$, and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Assume that $F : \Omega \times \Omega \to \mathbb{C}$ defined by $F(z, w) = r(z) - r(w)$. Then $F(\lambda, w)$ has $n - 1$ zeros outside $\overline{G}$. Since by the hypothesis the zeros of $F(\lambda, w)$ are simple, we have

$$\frac{\partial F}{\partial w}(\lambda, z_i) = (r(\lambda) - r(w))^\prime(z_i) \neq 0 \quad \text{for } i = 1, 2, \ldots, n - 1.$$

Thus by the Implicit Function Theorem, for each $i$, there exists an open neighborhood $V_i$ and a continuous map $\phi_i : V_i \to \mathbb{C}$ such that $\lambda \in V_i$, $\phi_i(\lambda) = z_i$ and $F(z, \phi_i(z)) = 0$ for every $z \in V_i$ and $i = 1, 2, \ldots, n - 1$. Since $\phi_i$ is continuous, there exists an open subset $U_i$ of $V_i$ such that $\lambda \in U_i$ and $\phi_i(U_i) \subseteq \Omega_i$. Let $U_0 = \cap_{i=1}^{n-1} U_i$. Then $U = U_0 \cap G$ is a nonempty open subset of $G$. Suppose that $w \in U$. Then $w \in U_i$ for every $i$ and so, $(w, \phi_1(w)), (w, \phi_2(w)), \ldots, (w, \phi_{n-1}(w))$ are zeros of $F$. Hence $\phi_1(w), \phi_2(w), \ldots, \phi_{n-1}(w)$ are $n - 1$ distinct roots of the equation $r(z) - r(w) = 0$, which are outside of $\overline{G}$.

**Proposition 2.2.** Let $B$ be a Banach space of analytic functions on $G$. If $r(z)$ satisfies the conditions of Theorem 2.1, then

$$\{M_r\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}.$$

**Proof.** By Theorem 2.1, there is an open set $U \subseteq G$ such that for every $w \in U$ the function $r(z) - r(w)$ has only a simple zero in $\overline{G}$. Assume that $T \in \{M_r\}'$. By Theorem 1.1, for every $w \in U$ and each $f \in B$, we have $T(f)(w) = T(1)(w)f(w)$. Since two analytic function $T(f)$ and $T(1)f$ are equal on $U$ and $G$ is connected, we have $T(f) = T(1)f$ for all $f \in B$, and the proof is complete.

**Example 2.3.** Let $B$ be a Banach space of analytic functions on $D$, where $D$ is the unit disk and let $r(z) = z^n/(2 - z)$ for some positive integer $n \geq 2$. It is easy to see that $|r(1)| = 1$ and $|r(z)| < 1$ for $z \in \overline{D} - \{1\}$. Using Theorem 2.1, with $\lambda = 1$ and Proposition 2.2, we have

$$\{M_r\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}.$$
Theorem 2.4. Let $B$ be a Banach space of analytic functions on $G$. Let $n \geq 2$ be an integer, $a \neq 0$ and $b$ be two complex numbers, and let $p(z) = z^n + az + b$. Then

- If $0 \in \overline{G}$ and $|a| > 1$, then $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.
- If $a = |a|e^{i\theta}$ belongs to $\overline{D}$ and $e^{i\theta} \in \overline{G}$, then $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

Proof. a) If in Theorem 2.1 we set $\lambda = 0$, then it is easy to see that $p(z) - p(\lambda) = z^n + az$ has $n - 1$ distinct zeros outside of $\overline{D}$. Hence by Proposition 2.2, we have $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

b) Set $\lambda = e^{\frac{n\pi}{n-1}}$, we can see that $p(\lambda) = p(z)$ if and only if $\lambda^n + a\lambda = z^n + az$. But

$$\lambda^n + a\lambda = \lambda \times \lambda^{n-1} + a | = e^{i\theta} + a | e^{i\theta} = 1 + | a |.$$  

So if $|z| < 1$, then $|z^n + az| \leq |z^n| + |a||z| < 1 + |a| = |\lambda^n + a\lambda|$, which implies that $z$ isn’t a zero of $p(z) - p(\lambda)$. In the next step assume that $z_0 \in \partial D$ is a root of equation $p(z) - p(\lambda) = 0$. Therefore, $|z_0^{n-1} + a| = |z_0^n + az_0| = |\lambda^n + a\lambda| = 1 + |a| = |z_0^{n-1}| + |a|$. Thus arg $z_0^{n-1} = \arg a + 2k\pi$ for some integer $k$. Since $|z_0| = 1$, we have $z_0^{n-1} = \lambda^{n-1}$. It follows that $(z_0 - \lambda)(\lambda^{n-1} - 1) = 0$, which implies that $z_0 = \lambda$. To complete the proof of the theorem, it suffices to show that $n - 1$ zeros of $p(z) - p(\lambda)$ outside of $\overline{D}$ are distinct. Since the absolute value of each zero of $p'(z)$ is $|a|/n$, which is less than one, we conclude that the zeros of $p(z) - p(\lambda)$ outside of $\overline{D}$ are distinct. Hence by Proposition 2.2, the proof is complete.

Theorem 2.5. Let $B$ be a Banach spaces of analytic functions on $G$. Let $n \geq 2$ be an integer, $a \neq 0$ and $b$ be two complex numbers, and let $p(z) = z^n + az^{n-1} + b$. Also assume that $a = |a|e^{i\theta_0}$ and $e^{i\theta_0} \in \overline{G}$. If $(n - 1)^{n-1}|a^n| \neq n^n(1 + |a|)$, then

$$\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(B)\}.$$  

Proof. Set $\lambda = e^{i\theta_0}$, we can see that $p(\lambda) = p(z)$ if and only if $\lambda^n + a\lambda^{n-1} = z^n + az^{n-1}$. But

$$\lambda^n + a\lambda^{n-1} = e^{i\theta_0} + |a| e^{i\theta_0} = 1 + |a|.$$  

So if $z < 1$, then $|z^n + az^{n-1}| \leq |z^n| + |a||z^{n-1}| < 1 + |a| = |\lambda^n + a\lambda^{n-1}|$, which implies that $z$ isn’t a zero of $p(z) - p(\lambda)$. Now assume that $z_0 \in \partial D$ is a root of equation $p(z) - p(\lambda) = 0$. Therefore, $|z_0^n + az_0^{n-1}| = |\lambda^n + a\lambda^{n-1}| = 1 + |a| = |z_0^n| + |a|$. Thus Arg $z_0 = \text{Arg}(a)$. Since $|z_0| = 1$, we have $z_0 = \lambda$. To complete the proof of the theorem we need only to show that $n - 1$ zeros of $p(z) - p(\lambda)$ outside of $\overline{D}$ are distinct. Since the only nonzero root of the equation $p'(z) = 0$ is $-a(n-1)/n$, it suffices to show that $-a(n-1)/n$ is not a root of equation $p(z) - p(\lambda) = 0$. But $p(-a(n-1)/n) = p(\lambda)$ implies
that \((-1)^{n-1}a_n^{(n-1)^{n-1}} = n(e^{in\theta_0} + |a|e^{in\theta_0})\) and so \(a_n^{(n-1)^{n-1}} = n(1 + |a|)\). But by assumption this relation is not true. Therefore, we conclude that \(n - 1\) zeros of \(p(z) - p(\lambda)\) outside \(\overline{D}\) are distinct and by Proposition 2.2, the proof is complete.

**Theorem 2.6.** Let \(B\) be a Banach spaces of analytic functions on \(G\) and let \(p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\) be a polynomial of degree \(n \geq 2\) such that \(a_1 \neq 0\). If there is \(z_0 \in \partial D \cap \partial G\) such that all nonzero terms \(a_i z_0^i\) for \(i \geq 1\) are positive or all are negative also \(p(z) - p(z_0)\) has \(n\) distinct zeros, then \(\{M_p\}' = \{M_{\Psi} : \Psi \in \mathcal{M}(B)\}\).

**Proof.** By assumption \(p(z_0) - a_0\) is real and \(|p(z_0) - a_0| = |a_1 z_0| + |a_2 z_0^2| + \cdots + |a_n z_0^n|\). Hence \(p(z) - p(z_0) = 0\) implies that

\[
|a_1 z + a_2 z^2 + \cdots + a_n z^n| = |a_1| + |a_2| + \cdots + |a_n|.
\]

For \(z \in D\), we have

\[
|a_1 z + a_2 z^2 + \cdots + a_n z^n| < |a_1| + |a_2| + \cdots + |a_n|.
\]

So \(p(z) - p(z_0)\) has no zero in \(D\). On the other hand if \(z \in \partial D\) is a zero of \(p(z) - p(z_0)\), then

\[
|a_1| + |a_2| + \cdots + |a_n| = |a_1 z| + |a_2 z^2| + \cdots + |a_n z^n| = |a_1 z + a_2 z^2 + \cdots + a_n z^n|.
\]

Hence \(\text{Arg}(a_1 z + a_2 z^2 + \cdots + a_n z^n) = \text{Arg}(a_1 z)\). Since \(p(z) = p(z_0)\), we have \(p(z) - a_0 = p(z_0) - a_0\). Thus \(p(z) - a_0\) is a real number and also \(\text{Arg}(a_1 z + a_2 z^2 + \cdots + a_n z^n) = \text{Arg}(a_1 z_0)\). Hence \(\text{Arg}(a_1 z) = \text{Arg}(a_1 z_0)\) and since \(|z| = 1\), we have \(z = z_0\). Therefore, \(p(z) - p(z_0)\) has \(n - 1\) distinct zeros outside \(\overline{D}\) and by Proposition 2.2, the proof is complete.

An easy application of the above theorem is when \(p\) is a polynomial of degree 3 such that its coefficients satisfy the conditions of Theorem 2.6. Let \(a\) and \(b\) be zeros of \(p'(z)\). If \(p(a)\) and \(p(b)\) are not equal to \(p(z_0)\), then \(\{M_p\}' = \{M_{\Psi} : \Psi \in \mathcal{M}(B)\}\).

Using the same argument as used in the proof of Theorem 2.6, we have the following two propositions.

**Proposition 2.7.** Let \(B\) be a Banach spaces of analytic functions on \(G\). Let \(p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\) be a polynomial of degree \(n \geq 2\) such that \(a_1 \neq 0\) and \(\text{Arg}(a_i) = \theta_0\) for \(a_i \neq 0\) with \(i \geq 1\). If \(p(z) - p(1)\) has \(n\) distinct zeros and \(1 \in \partial G\), then \(\{M_p\}' = \{M_{\Psi} : \Psi \in \mathcal{M}(B)\}\).

**Proposition 2.8.** Let \(B\) be a Banach spaces of analytic functions on \(G\) and let \(p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\) be a polynomial of degree \(n \geq 2\) such that \(a_1 \neq 0\). Also assume that for each \(a_i \neq 0\) with \(i \geq 1\), \(\text{Arg}(a_i) = \theta_0 + \pi\) or \(\text{Arg}(a_i) = \theta_0 - \pi\) for \(i\) even. If \(p(z) - p(-1)\) has \(n\) distinct zeros and \(-1 \in \partial G\), then \(\{M_p\}' = \{M_{\Psi} : \Psi \in \mathcal{M}(B)\}\).
Theorem 2.9. Let $B$ be a Banach spaces of analytic functions on $G$, and let $p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 2$. If the maximum value of $|p(z)|$ on $\overline{D}$ is obtained at a unique point $z_0 \in \partial G$ and $|a_1| + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| < n |a_n|$ or none of the zeros of $p'(z)$ is a root of equation $p(z) - p(z_0) = 0$, then $\{M_p\} = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

Proof. By assumption $|p(z)| < |p(z_0)|$ for all $z \in \overline{D} - \{z_0\}$. Hence $p(z) - p(z_0)$ has no zero in $\overline{G} - \{z_0\}$. If none of the zeros of $p'(z)$ is a root of equation $p(z) - p(z_0) = 0$, then $n-1$ zeros of $p(z) - p(z_0)$ outside $\overline{D}$, and therefore outside $\overline{G}$ are distinct. Thus by Proposition 2.2, the proof is complete. Otherwise by assumption $|a_1| + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| < n |a_n|$. Hence $|p'(z) - p'(z_0)| = |< a_n z^{n-1} | < | a_n z^{n-1} |$ for all $z \in \partial D$ and therefore by the Rouche’s Theorem $p'(z)$ has $n-1$ zeros inside $D$, which implies that $n-1$ zeros of $p(z) - p(z_0)$ outside $\overline{D}$ are distinct. Now by Proposition 2.2, the proof is complete. □

Corollary 2.10. Let $B$ be a Banach spaces of analytic functions on $G$, let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 2$ with nonnegative real coefficients and let $1 \in \partial G$. If there is a positive integer $m \leq n$ such that $a_m$ and $a_{m-1}$ are not equal to zero and

$$ |a_1| + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| < n |a_n|,$$

then $\{M_p\} = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

Proof. It is easy to see that $|p(1)| > |p(z)|$ for all $z \in \overline{D} - \{1\}$. Indeed, if $z = e^{i\theta}$ for some $\theta$, $-\pi < \theta \leq \pi$ and $|p(z)| = |p(1)|$, we have

$$ |a_m e^{im\theta} + a_{m-1} e^{i(m-1)\theta}| = |a_m + a_{m-1}|.$$

Therefore, $m\theta = (m-1)\theta + 2k\pi$ for some integer $k$. Hence $z = 1$, and by Theorem 2.9, the proof is complete. □

In the next example we present three applications of some of the above theorems.

Example 2.11. Let $B$ be a Banach spaces of analytic functions on $G$.

a) Let $p(z) = -z^3 + 6z^2 - 9z + 5$, and let $z_0 = -1$ belongs to $\overline{G}$. Since $p'(z) = -3z^2 + 12z - 9$ has zeros 1 and 3, which aren’t the roots of equation $p(z) - p(-1) = 0$ by Theorem 2.6, or Proposition 2.8, we have $\{M_p\} = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

b) Let $p(z) = 3z^7 + 2z^5 + 4z^2 + 5$ and let $z_0 = 1$ belongs to $\overline{G}$. Then by Theorem 2.9, we have $\{M_p\} = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$.

c) Let $p(z) = iz^3 + 3iz^2 + 3iz - 3$ and $1 \in \overline{G}$. Since $p'(z) = 3iz^2 + 6iz + 3i$ has a zero of order 2 at $z = -i$ and $-1$ isn’t a root of equation $p(z) - p(1) = 0$, we conclude that the roots of this equation are distinct. Hence by Proposition 2.7, we have $\{M_p\} = \{M_\Psi : \Psi \in \mathcal{M}(B)\}$. 

References


Department of Mathematics
College of Sciences
Shiraz University
Shiraz 71454, IRAN
E-mail address: khani@susc.ac.ir