

PERFECT AND CONCRETE FILTERS OF WFI-ALGEBRAS

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ABSTRACT. The notion of perfect filters and concrete filters in WFI-algebras is introduced, and several properties are investigated. Relations between a filter, a perfect filter and a concrete filter are given, and characterizations of a perfect filter and a concrete filter are provided. An extension property for a concrete filter and a perfect filter is established.

1. Introduction

In 1990, W. M. Wu [5] introduced the notion of fuzzy implication algebras (FI-algebra, for short), and investigated several properties. In [4], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebras, and investigated the relations between such FI-algebras and MV-algebras. In [1], Y. B. Jun discussed several aspects of WFI-algebras. He introduced the notion of associative (resp. normal, medial) WFI-algebras, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebras, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [3] introduced the notions of simulative and/or mutant WFI-algebras and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. Y. B. Jun et al. [2] introduced the concept of ideals of WFI-algebras. He gave relations between a filter and an ideal, and provided characterizations of an ideal. He also established an extension property for an ideal. In this paper we introduced the concept of perfect filters and concrete filters of WFI-algebras. We give relations between a filter, a perfect filter and a concrete filter, and provide characterizations of a perfect filter and a concrete filter. We establish an extension property for a perfect filter and a concrete filter.

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2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *WFI-algebra* we mean a system $\mathfrak{X} = (X, \ominus, 1) \in K(\tau)$ in which the following axioms hold:

- (a1) $(\forall x \in X) (x \ominus x = 1)$,
- (a2) $(\forall x, y \in X) (x \ominus y = y \ominus x = 1 \Rightarrow x = y)$,
- (a3) $(\forall x, y, z \in X) (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,
- (a4) $(\forall x, y, z \in X) ((x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z)) = 1)$.

For the convenience of notation, we shall write $[x, y_1, y_2, \dots, y_n]$ for

$$(\dots((x \ominus y_1) \ominus y_2) \ominus \dots) \ominus y_n.$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \dots, y]$, where y occurs n -times.

Proposition 2.1 ([1]). *In a WFI-algebra \mathfrak{X} , the following are true:*

- (b1) $x \ominus [x, y]^2 = 1$,
- (b2) $1 \ominus x = 1 \Rightarrow x = 1$,
- (b3) $1 \ominus x = x$,
- (b4) $x \ominus y = 1 \Rightarrow (y \ominus z) \ominus (x \ominus z) = 1, (z \ominus x) \ominus (z \ominus y) = 1$,
- (b5) $(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1)$,
- (b6) $[x, y]^3 = x \ominus y$.

A nonempty subset S of a WFI-algebra \mathfrak{X} is called a *subalgebra* of \mathfrak{X} if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI-algebra \mathfrak{X} is called a *filter* of \mathfrak{X} if it satisfies:

- (c1) $1 \in F$,
- (c2) $(\forall x \in F) (\forall y \in X) (x \ominus y \in F \Rightarrow y \in F)$.

A filter F of a WFI-algebra \mathfrak{X} is said to be *closed* [1] if F is also a subalgebra of \mathfrak{X} .

Proposition 2.2 ([1]). *Let F be a filter of a WFI-algebra \mathfrak{X} . Then F is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.*

Proposition 2.3 ([1]). *In a finite WFI-algebra, every filter is closed.*

We now define a relation “ \preceq ” on \mathfrak{X} by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to \preceq .

3. Perfect filters and concrete filters

In what follows, let \mathfrak{X} denote a WFI-algebra unless otherwise specified.

Definition 3.1 ([2]). A nonempty subset G of a WFI-algebra \mathfrak{X} is called an *ideal* of \mathfrak{X} if it satisfies:

- (c1) $1 \in G$,
- (c3) $(\forall x, y \in X) (\forall z \in G) ((x \ominus z) \ominus y \in G \Rightarrow x \ominus y \in G)$.

Note that every ideal is a closed filter, but the converse is not true (see [2]). In [2], a characterization of an ideal is provided as follows.

Lemma 3.2 ([2]). *Let G be a filter of \mathfrak{X} . Then the following are equivalent:*

- (i) G is an ideal of \mathfrak{X} .
- (ii) $(\forall x, y \in X) ((x \ominus 1) \ominus y \in G \Rightarrow x \ominus y \in G)$.
- (iii) $(\forall x, y, z \in X) ((x \ominus y) \ominus z \in G \Rightarrow x \ominus (y \ominus z) \in G)$.

We give another characterization of an ideal.

Theorem 3.3. *Let G be a filter of \mathfrak{X} . Then the following are equivalent:*

- (i) G is an ideal of \mathfrak{X} .
- (ii) $(\forall x, y, z \in X) ((x \ominus 1) \ominus (y \ominus z) \in G \Rightarrow x \ominus (y \ominus z) \in G)$.
- (iii) $(\forall x, z \in X) (\forall y \in G) ((x \ominus 1) \ominus (y \ominus z) \in G \Rightarrow x \ominus z \in G)$.

Proof. (i) \Rightarrow (ii) is by Lemma 3.2(ii).

(ii) \Rightarrow (iii) Assume that (ii) is valid. Let $x, z \in X$ and $y \in G$ be such that $(x \ominus 1) \ominus (y \ominus z) \in G$. Then $y \ominus (x \ominus z) = x \ominus (y \ominus z) \in G$ by (ii), and so $x \ominus z \in G$ by (c2).

(iii) \Rightarrow (i) Suppose that (iii) is true and $(x \ominus 1) \ominus y \in G$ for all $x, y \in X$. Then $(x \ominus 1) \ominus (1 \ominus y) = (x \ominus 1) \ominus y \in G$. Since $1 \in G$, it follows from (iii) that $x \ominus y \in G$ so from Lemma 3.2 that G is an ideal of \mathfrak{X} . \square

Lemma 3.4. *A WFI-algebra \mathfrak{X} satisfies the following equality.*

$$(\forall x, y, z \in X) (((x \ominus y) \ominus (x \ominus z), 1]^2 = (z \ominus 1) \ominus (y \ominus 1)).$$

Proof. For every $x, y, z \in X$, we get

$$\begin{aligned} & [(x \ominus y) \ominus (x \ominus z), 1]^2 = [x \ominus y, 1]^2 \ominus [x \ominus z, 1]^2 \quad \text{by (b5)} \\ & = ((x \ominus z) \ominus 1) \ominus [x \ominus y, 1]^3 \quad \text{by (a3)} \\ & = ((x \ominus z) \ominus 1) \ominus ((x \ominus y) \ominus 1) \quad \text{by (b6)} \\ & = (x \ominus y) \ominus [x \ominus z, 1]^2 \quad \text{by (a3)} \\ & = (x \ominus y) \ominus ([x, 1]^2 \ominus [z, 1]^2) \quad \text{by (b5)} \\ & = (x \ominus y) \ominus ((z \ominus 1) \ominus [x, 1]^3) \quad \text{by (a3)} \\ & = (x \ominus y) \ominus ((z \ominus 1) \ominus (x \ominus 1)) \quad \text{by (b6)} \\ & = (z \ominus 1) \ominus (x \ominus ((x \ominus y) \ominus 1)) \quad \text{by (a3)} \\ & = (z \ominus 1) \ominus (x \ominus ((x \ominus 1) \ominus (y \ominus 1))) \quad \text{by (b5)} \\ & = (z \ominus 1) \ominus (y \ominus (x \ominus [x, 1]^2)) \quad \text{by (a3)} \\ & = (z \ominus 1) \ominus (y \ominus 1). \quad \text{by (a3) and (a1)} \end{aligned}$$

This completes the proof. \square

Lemma 3.5. *A filter F of \mathfrak{X} satisfies the following implication.*

$$(\forall x \in X) (x \in F \Rightarrow [x, 1]^2 \in F).$$

Proof. For every $x \in F$, we have

$$x \ominus [x, 1]^2 = (x \ominus 1) \ominus (x \ominus 1) = 1 \in F,$$

and so $[x, 1]^2 \in F$ by (c2). \square

Definition 3.6. A nonempty subset H of \mathfrak{X} is called a *concrete filter* of \mathfrak{X} if it satisfies:

- (c1) $1 \in H$,
- (c4) $(\forall x, z \in X) (\forall y \in H) ((x \ominus y) \ominus (x \ominus z) \in H \Rightarrow z \in H)$.

Example 3.7. Let $X = \{1, a, b, c\}$ be a set with the following Cayley table:

\ominus	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a *WFI*. It is easy to verify that the set $H := \{1, a\}$ is a concrete filter of \mathfrak{X} .

Proposition 3.8. Every concrete filter H of \mathfrak{X} satisfies the following implication.

$$(\forall x \in X) ([x, 1]^2 \in H \Rightarrow x \in H).$$

Proof. Let $x \in X$ be such that $[x, 1]^2 \in H$. Then $(x \ominus 1) \ominus (x \ominus x) = [x, 1]^2 \in H$. Since $1 \in H$, it follows from (c4) that $x \in H$. This completes the proof. \square

Theorem 3.9. Every concrete filter is a filter.

Proof. Let H be a concrete filter of \mathfrak{X} and let $x, y \in X$ be such that $x \ominus y \in H$ and $x \in H$. Then $(1 \ominus x) \ominus (1 \ominus y) = x \ominus y \in H$ and so $y \in H$ by (c4). Hence H is a filter of \mathfrak{X} . \square

The converse of Theorem 3.9 is not true in general as seen in the following example.

Example 3.10. Let $X = \{1, a, b, c\}$ be a set with the following Cayley table:

\ominus	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	b	c	1	a
c	b	b	1	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a *WFI*-algebra. It is easy to verify that the set $G := \{1, b\}$ is a filter of \mathfrak{X} . But it is not a concrete filter of \mathfrak{X} because $[c, 1]^2 \in G$ and $c \notin G$.

We give conditions for a filter to be a concrete filter.

Theorem 3.11. *A filter F of \mathfrak{X} is a concrete filter of \mathfrak{X} if and only if it satisfies the following implication.*

$$(3.1) \quad (\forall x, y, z \in X) ((x \odot y) \odot (x \odot z) \in F \Rightarrow y \odot z \in F).$$

Proof. Suppose that F is a concrete filter of \mathfrak{X} and let $x, y, z \in X$ be such that $(x \odot y) \odot (x \odot z) \in F$. Using (a3) and (a4), we have

$$1 = (x \odot y) \odot ((y \odot z) \odot (x \odot z)) = (y \odot z) \odot ((x \odot y) \odot (x \odot z)),$$

which implies that

$$((y \odot z) \odot ((x \odot y) \odot (x \odot z))) \odot ((y \odot z) \odot (y \odot z)) = 1 \odot 1 = 1 \in F.$$

It follows from (c4) that $y \odot z \in F$. Conversely, let F be a filter of \mathfrak{X} that satisfies (3.1). Let $x, z \in X$ and $y \in F$ be such that $(x \odot y) \odot (x \odot z) \in F$. Then $y \odot z \in F$ by (3.1), and so $z \in F$ by (c2). Hence F is a concrete filter of \mathfrak{X} . \square

Theorem 3.12. *A filter F of \mathfrak{X} is a concrete filter of \mathfrak{X} if and only if it satisfies the following implication.*

$$(3.2) \quad (\forall x \in X) ([x, 1]^2 \in F \Rightarrow x \in F).$$

Proof. The necessity is by Proposition 3.8. Let F be a filter of \mathfrak{X} that satisfies (3.2). Let $x, y, z \in X$ be such that $(x \odot y) \odot (x \odot z) \in F$. Using (a3), (b5), (b6) and Lemmas 3.4 and 3.5, we get

$$[y \odot z, 1]^2 = (z \odot 1) \odot (y \odot 1) = [(x \odot y) \odot (x \odot z), 1]^2 \in F.$$

It follows from (3.2) that $y \odot z \in F$ so from Theorem 3.11 that F is a concrete filter of \mathfrak{X} . \square

Let \mathfrak{X} be a WFI-algebra given in Example 3.10. Then the set $G = \{1, b\}$ is an ideal of \mathfrak{X} which is not a concrete filter of \mathfrak{X} . This shows that an ideal is not a concrete filter in general. For any subset G of \mathfrak{X} , consider an implication

$$(3.3) \quad x \odot 1 \in G \Rightarrow x \in G.$$

Note that there exists an ideal G of \mathfrak{X} in which (3.3) is not valid. For example, the ideal $G = \{1, b\}$ in Example 3.10 does not satisfy (3.3) because $a \odot 1 = 1 \in G$ and $a \notin G$.

Theorem 3.13. *Every ideal satisfying the condition (3.3) is a concrete filter.*

Proof. Let G be an ideal of \mathfrak{X} that satisfies (3.3) and assume that $[x, 1]^2 \in G$ for all $x \in X$. Then $x \odot 1 \in G$ by Lemma 3.2, and so $x \in G$ by (3.3). It follows from Theorem 3.12 that G is a concrete filter of \mathfrak{X} . \square

Definition 3.14. A nonempty subset J of \mathfrak{X} is called a *perfect filter* of \mathfrak{X} if it satisfies:

- (c1) $1 \in J$,
- (c5) $(\forall x, y \in X) (\forall z \in J) ((x \odot 1) \odot (z \odot y) \in J \Rightarrow y \odot x \in J)$.

Example 3.15. Let $\mathfrak{X} = (X; \ominus, 1)$ be a WFI-algebra given in Example 3.7. It is easy to verify that a set $J := \{1, a\}$ is a perfect filter of \mathfrak{X} .

Theorem 3.16. *Every perfect filter is a closed filter.*

Proof. Let J be a perfect filter of \mathfrak{X} and let $x, y \in X$ be such that $x \ominus y \in J$ and $x \in J$. Since $(1 \ominus 1) \ominus (x \ominus y) = x \ominus y \in J$, it follows from (c5) that $y \ominus 1 \in J$. Taking $x = z = 1$ in (c5) and using (a1) and (b3) implies that

$$(3.4) \quad y \in J \Rightarrow y \ominus 1 \in J.$$

Thus we know that $[y, 1]^2 \in J$, which implies $y \in J$ by taking $x = y$ and $z = y = 1$ in (c5). Hence J is a filter of \mathfrak{X} . Now let $x, y \in J$. Then $y \ominus 1 \in J$ by (3.4). Since $x \ominus (y \ominus x) = y \ominus (x \ominus x) = y \ominus 1 \in J$, it follows from (c2) that $y \ominus x \in J$. Thus J is a subalgebra of \mathfrak{X} . This completes the proof. \square

The converse of Theorem 3.16 is not true in general as seen in the following example.

Example 3.17. Let $\mathfrak{X} = (X; \ominus, 1)$ be a WFI-algebra given in Example 3.10. We know that $J := \{1\}$ is a closed filter of \mathfrak{X} . But it is not a perfect filter of \mathfrak{X} since $(a \ominus 1) \ominus (1 \ominus 1) = 1 \in J$, but $1 \ominus a = b \notin J$.

We now give characterizations of a perfect filter.

Theorem 3.18. *Let J be a filter of \mathfrak{X} . Then the following are equivalent.*

- (i) J is a perfect filter of \mathfrak{X} .
- (ii) $(\forall x, y, z \in X) ((x \ominus 1) \ominus (y \ominus z) \in J \Rightarrow (y \ominus z) \ominus x \in J)$.
- (iii) $(\forall x, y \in X) ((x \ominus 1) \ominus y \in J \Rightarrow y \ominus x \in J)$.

Proof. (i) \Rightarrow (ii) Assume that J is a perfect filter of \mathfrak{X} . Let $x, y, z \in X$ be such that $(x \ominus 1) \ominus (y \ominus z) \in J$. Then

$$\begin{aligned} & (x \ominus 1) \ominus (((x \ominus 1) \ominus (y \ominus z)) \ominus (y \ominus z)) \\ &= ((x \ominus 1) \ominus (y \ominus z)) \ominus ((x \ominus 1) \ominus (y \ominus z)) \\ &= 1 \in J, \end{aligned}$$

and so $(y \ominus z) \ominus x \in J$ by (c5). Thus (ii) is valid.

(ii) \Rightarrow (iii) Taking $y = 1$ and $z = y$ in (ii) and using (b3), we obtain (iii).

(iii) \Rightarrow (i) Suppose that (iii) is valid. Let $x, y \in X$ and $z \in J$ be such that $(x \ominus 1) \ominus (z \ominus y) \in J$. Using (b1), (a3) and (a4), we get

$$z \preceq [z, y]^2 \preceq ((x \ominus 1) \ominus (z \ominus y)) \ominus ((x \ominus 1) \ominus y),$$

and so $((x \ominus 1) \ominus (z \ominus y)) \ominus ((x \ominus 1) \ominus y) \in J$. Since J is a filter of \mathfrak{X} , it follows from (c2) that $(x \ominus 1) \ominus y \in J$ so from (iii) that $y \ominus x \in J$. Therefore J is a perfect filter of \mathfrak{X} . \square

Proposition 3.19. *Every perfect filter J of \mathfrak{X} satisfies the following implication.*

$$(3.5) \quad (\forall x, y, z \in X) ((x \ominus y) \ominus z \in J \Rightarrow (y \ominus z) \ominus x \in J).$$

Proof. Let $x, y, z \in X$ be such that $(x \ominus y) \ominus z \in J$. Since

$$\begin{aligned} 1 &= (x \ominus 1) \ominus (x \ominus 1) && \text{by (a1)} \\ &= (x \ominus 1) \ominus (y \ominus (x \ominus y)) && \text{by (a3) and (a1)} \\ &\preceq (x \ominus 1) \ominus ((x \ominus y) \ominus z) \ominus (y \ominus z) && \text{by (a4) and (b4)} \\ &= ((x \ominus y) \ominus z) \ominus ((x \ominus 1) \ominus (y \ominus z)) && \text{by (a3)} \end{aligned}$$

and since J is a filter of \mathfrak{X} , it follows that $(x \ominus 1) \ominus (y \ominus z) \in J$ so from Theorem 3.18(ii) that $(y \ominus z) \ominus x \in J$. \square

The following example shows that there exists a concrete filter (resp. an ideal) which is not a perfect filter.

Example 3.20. (i) Consider a WFI-algebra $\mathfrak{X} = (X = \{1, a, b\}; \ominus, 1)$ with the following Cayley table:

\ominus	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1

Then $J := \{1\}$ is a concrete filter of \mathfrak{X} , but it is not a perfect filter of \mathfrak{X} since

$$(b \ominus 1) \ominus (1 \ominus a) = 1 \in J \text{ and } a \ominus b = a \notin J.$$

(ii) Consider a WFI-algebra $\mathfrak{X} = (X = \{1, a, b\}; \ominus, 1)$ with the following Cayley table:

\ominus	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $J := \{1\}$ is an ideal of \mathfrak{X} , but it is not a perfect filter of \mathfrak{X} since

$$(a \ominus 1) \ominus (1 \ominus 1) = 1 \in J \text{ and } 1 \ominus a = a \notin J.$$

Theorem 3.21. *A nonempty subset J of X is a perfect filter of \mathfrak{X} if and only if it is both a concrete filter and an ideal of \mathfrak{X} .*

Proof. Let J be a perfect filter of \mathfrak{X} . Then J is a filter of \mathfrak{X} by Theorem 3.16. Let $x \in X$ be such that $[x, 1]^2 \in J$. Then $(x \ominus 1) \ominus (1 \ominus 1) = [x, 1]^2 \in J$, which implies from (a1), (b3) and Theorem 3.18(ii) that $x \in J$. Hence, by Theorem 3.12, J is a concrete filter of \mathfrak{X} . Now let $x, y \in X$ be such that $(x \ominus 1) \ominus y \in J$. Then

$$\begin{aligned} 1 &= ((x \ominus 1) \ominus y) \ominus ((x \ominus 1) \ominus y) && \text{by (a1)} \\ &\preceq ((x \ominus 1) \ominus y) \ominus ((y \ominus 1) \ominus [x, 1]^2) && \text{by (a4) and (b4)} \\ &= ((x \ominus 1) \ominus y) \ominus ([y, 1]^3 \ominus [x, 1]^2) && \text{by (b6)} \\ &= ((x \ominus 1) \ominus y) \ominus (([y, 1]^2 \ominus (x \ominus 1)) \ominus 1) && \text{by (b5)} \\ &= ((x \ominus 1) \ominus y) \ominus [(y \ominus 1) \ominus x, 1]^2. && \text{by (b5)} \end{aligned}$$

Since J is a filter, it follows that $((x \ominus 1) \ominus y) \ominus [(y \ominus 1) \ominus x, 1]^2 \in J$ so that $[(y \ominus 1) \ominus x, 1]^2 \in J$. Since J is a concrete filter, we get $(y \ominus 1) \ominus x \in J$ by

Theorem 3.12, and so $x \odot y \in J$ by Theorem 3.18(iii). Using Lemma 3.2, J is an ideal of \mathfrak{X} .

Conversely, suppose that J is both a concrete filter and an ideal of \mathfrak{X} . Then J is a closed filter of \mathfrak{X} . Let $x, y \in X$ be such that $(x \odot 1) \odot y \in J$. Then $x \odot y \in J$ by Lemma 3.2. Note that

$$\begin{aligned} (x \odot y) \odot ((y \odot x) \odot 1) &= (x \odot y) \odot ((y \odot 1) \odot (x \odot 1)) && \text{by (b5)} \\ &= (y \odot 1) \odot (x \odot ((x \odot y) \odot 1)) && \text{by (a3)} \\ &= (y \odot 1) \odot (x \odot ((x \odot 1) \odot (y \odot 1))) && \text{by (b5)} \\ &= x \odot [x, 1]^2 = 1 \in J. && \text{by (a3) and (a1)} \end{aligned}$$

It follows from (c2) that $(y \odot x) \odot 1 \in J$ so that $[y \odot x, 1]^2 \in J$ because J is a closed filter. Using Theorem 3.12, we have $y \odot x \in J$ and thus J is a perfect filter of \mathfrak{X} by Theorem 3.18. \square

Theorem 3.22. *Every ideal satisfying the condition (3.3) is a perfect filter.*

Proof. Let J be an ideal of \mathfrak{X} that satisfies the condition (3.3). Let $x, y \in X$ and $z \in J$ be such that $(x \odot 1) \odot (z \odot y) \in J$. Then $z \odot (x \odot y) = x \odot (z \odot y) \in J$ by (a3) and Lemma 3.2(ii), and so $x \odot y \in J$ by (c2). Since J is a filter, it follows from (a3), (b5), (b6) and Lemma 3.5 that

$$\begin{aligned} (y \odot x) \odot 1 &= (y \odot 1) \odot (x \odot 1) = (y \odot 1) \odot [x, 1]^3 \\ &= [x, 1]^2 \odot [y, 1]^2 = ((x \odot 1) \odot (y \odot 1)) \odot 1 = [x \odot y, 1]^2 \in J. \end{aligned}$$

Using (3.3), we get $y \odot x \in J$. Hence J is a perfect filter of \mathfrak{X} . \square

The following example shows that there exists a concrete filter with the condition (3.3) which is not a perfect filter.

Example 3.23. Let $X = \{1, a, b, c\}$ be a set with the following Cayley table:

\odot	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	c	c	1	b
c	b	b	c	1

Then $\mathfrak{X} = (X; \odot, 1)$ is a *WFI*. It is easy to verify that the set $G := \{1, a\}$ is a concrete filter of \mathfrak{X} and satisfies the condition (3.3), but it is not a perfect filter of \mathfrak{X} because $(b \odot 1) \odot c = c \odot c = 1 \in G$, but $c \odot b = c \notin G$.

Theorem 3.24 ([2], Extension property for an ideal). *Let G and H be filters of \mathfrak{X} such that $G \subseteq H$. If G is an ideal of \mathfrak{X} , then so is H .*

Theorem 3.25 (Extension property for a concrete filter). *Let G and H be filters of \mathfrak{X} such that $G \subseteq H$. If G is a concrete filter of \mathfrak{X} , then so is H .*

Proof. Let $x \in X$ be such that $[x, 1]^2 \in H$. Since

$$1 = (x \odot 1) \odot ([x, 1]^2 \odot 1) \preceq [[x, 1]^2, 1]^2 \odot [x, 1]^2 = [[x, 1]^2 \odot x, 1]^2,$$

it follows that $[[x, 1]^2 \ominus x, 1]^2 \in G$. Since G is a concrete filter of \mathfrak{X} , we have $[x, 1]^2 \ominus x \in G \subseteq H$ by Theorem 3.12, and so $x \in H$ by (c2). Therefore H is a concrete filter of \mathfrak{X} by Theorem 3.12. \square

Combining Theorems 3.24 and 3.25, we have the extension property for a perfect filter.

Theorem 3.26 (Extension property for a perfect filter). *Let G and H be filters of \mathfrak{X} such that $G \subseteq H$. If G is a perfect filter of \mathfrak{X} , then so is H .*

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