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A NOTE ON DECOMPOSITION OF COMPLETE EQUIPARTITE GRAPHS INTO GREGARIOUS 6-CYCLES

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ABSTRACT. In [8], it is shown that the complete multipartite graph $K_{n(2t)}$ having n partite sets of size $2t$, where $n \geq 6$ and $t \geq 1$, has a decomposition into *gregarious* 6-cycles if $n \equiv 0, 1, 3$ or 4 (mod 6). Here, a cycle is called *gregarious* if it has at most one vertex from any particular partite set. In this paper, when $n \equiv 0$ or 3 (mod 6), another method using difference set is presented. Furthermore, when $n \equiv 0$ (mod 6), the decomposition obtained in this paper is ∞-*circular*, in the sense that it is invariant under the mapping which keeps the partite set which is indexed by ∞ fixed and permutes the remaining partite sets cyclically.

1. Introduction

Decompositions of graphs into edge-disjoint cycles have been an active research area for many years. Especially, decompositions by cycles of a fixed length have been considered in a number of different ways. Recently, it was shown that a complete graph of odd order, or a complete graph of even order minus an 1-factor, has a decomposition into k-cycles if k divides the number of edges (see [1], [11] and [12] as well as their references). One of the key factors for all these works was the cycle decomposition of complete bipartite graphs obtained by Sotteau ([13]). Many authors began to consider cycle decompositions with special properties ([4], [5], [9], [10]). Then, Billington and Hoffman ([2]) introduced the notion of a *gregarious* cycle in a tripartite graph. However, the definition of gregarious cycles has been modified in later research papers ([2], [4], [7]) for general partite graphs. Recently, Billington and Hoffman ([3]) and Cho and et el. ([7]) independently produced gregarious 4-cycle decompositions for certain complete multipartite graphs. In [8], Cho and Gould showed that $K_{n(2t)}$ has a decomposition into 6-cycles for all $t \geq 1$ if $n \equiv 0, 1, 3$ or 4 (mod 6).

In this paper, as a note to the earlier paper ([8]), the author shows another method of proof when $n \equiv 0$ or 3 (mod 6), which uses the complete difference

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set of a set of numbers. Furthermore, when $n \equiv 0 \pmod 6$, the decomposition obtained is invariant under the mapping which keeps one partite set fixed and permutes the remaining partite sets cyclically.

First of all, we make the definition of gregarious cycles clear. We call a cycle in a multipartite graph *gregarious* if it has at most one vertex from any particular partite set.

For simplicity, we will call a graph γ_6*-decomposable* if it is decomposable into gregarious 6-cycles, and a decomposition into gregarious 6-cycles will be called a γ_6*-decomposition.*

Let $K(m_1, m_2, \ldots, m_n)$ denote the complete multipartite graph with partite sets of size m_i, $i = 1, \ldots, n$. If all sizes are the same and equal to m, we denote it by $K_{n(m)}$, and call the graph a *complete equipartite graph.* Thus, the graph $K_{n(1)}$ means the complete graph K_n with n-vertices.

Lemma 1.1 ([1], [12]). *Let n be an odd integer and m be any positive integer with $m \leq n$. Then, K_n has a decomposition into m-cycles if and only if m divides $\frac{n(n-1)}{2}$.*

Lemma 1.2 ([5]). *If $K(m_1, m_2, \ldots, m_n)$ is decomposable into cycles, then $m_1, m_2, \ldots, m_n$ have the same parity, and furthermore n must be odd if the parity is odd.*

The following lemma is proved in [5] for decompositions into arbitrary (non-necessarily gregarious) cycles, by the standard "expanding points method". However, exactly the same method can be applied for decompositions into gregarious cycles.

Lemma 1.3. *If $K(m_1, m_2, \ldots, m_n)$ is decomposable into gregarious k-cycles for an even integer k, then so is $K(m_1t, m_2t, \ldots, m_nt)$ for every integer $t \geq 1$.*

Proof. Let $\mathcal{C}$ be a decomposition of $K(m_1, m_2, \ldots, m_n)$ into gregarious k-cycles. Expand each vertex v of $K(m_1, m_2, \ldots, m_n)$ to a set of t vertices $v^{(1)}$, $v^{(2)}$, $\ldots$, $v^{(t)}$, and make edges $u^{(i)}v^{(j)}$ for $i, j = 1, 2, \ldots, t$ if uv is an edge of $K(m_1, m_2, \ldots, m_n)$. Then, the resulting graph is $K(m_1t, m_2t, \ldots, m_nt)$. For each k-cycle $\langle v_1, v_2, \ldots, v_k \rangle$ in $\mathcal{C}$, we choose t^2 gregarious k-cycles

$$\langle v_1^{(i)}, v_2^{(j)}, v_3^{(i)}, v_4^{(j)}, \ldots, v_{k-1}^{(i)}, v_k^{(j)} \rangle \qquad (1 \leq i \leq t,\ 1 \leq j \leq t)$$

of $K(m_1t, m_2t, \ldots, m_nt)$. Let $\mathcal{C}^*$ be the collection of all such gregarious k-cycles obtained from each cycles in $\mathcal{C}$, then $\mathcal{C}^*$ is a decomposition of

$$K(m_1t, m_2t, \ldots, m_nt)$$

into gregarious k-cycles. □

2. Decomposition of $K_{n(m)}$ into gregarious 6-cycles

Lemma 2.1. *K_n is γ_6-decomposable if and only if $n \equiv 0, 1, 4$ or $9 \pmod{12}$. If n is one of such an integer then $K_{n(m)}$ is γ_6-decomposable for all $m \geq 1$.*

Proof. If K_n has decomposition into cycles, the degree of each vertex must be even. Thus, n should be odd. Now, if n is an odd number, then 6 divides $\frac{n(n-1)}{2}$ if and only if $n \equiv 0, 1, 4$ or 9 (mod 12). The conclusions follows by Lemmas 1.1 and 1.3. □

If m is odd and $K_{n(m)}$ is γ_6-decomposable, then n is also odd by Lemma 1.2. Since 6 must divides the number $\frac{n(n-1)m^2}{2}$ of edges of $K_{n(m)}$, 12 divides $n(n-1)m^2$. Since m and n are odd, we have $n \equiv 1 \pmod 4$ and $nm \equiv 0 \pmod 3$. If $n \equiv 0 \pmod 3$ then $n \equiv 9 \pmod{12}$ and so $K_{n(m)}$ is γ_6-decomposable by Lemma 2.1. So it remains to settle the cases when $n \equiv 1$ or 5 (mod 12) and $m \equiv 0 \pmod 3$. The decomposition problem for these cases is not settled yet.

In [8], it is proven that, for $n \geq 6$, $K_{n(2)}$ is γ_6-decomposable if and only if 6 divide $2n(n-1)$, the number of edges in $K_{n(2)}$ and, in such a case, $K_{n(2t)}$ is also γ_6-decomposable for every positive integer t. The authors used a difference set method for $n \equiv 1$ or 4 (mod 6). For $n \equiv 0$ or 3 (mod 6), they presented $K_{n(2t)}$ as a *join* of two graphs which are already known to have γ_6-decomposable and showed that the join is γ_6-decomposable. We will not explain the join of graphs here. Anyway, such decompositions do not have nice symmetry as the case when $n \equiv 1$ or 4 (mod 6). In the following sections, we use a method using the difference set of the extended number system $\mathbb{Z}_{n-1}^{\infty}$ to prove the following.

Theorem 2.1. *Let $n \geq 6$ and $n \equiv 0$ or 3 (mod 6). There is a systematic procedure to produce a γ_6-decomposition of $K_{n(2t)}$ with a nice symmetry for all $t \geq 1$ by using the complete difference set of the extended number system $\mathbb{Z}_{n-1}^{\infty}$.*

Due to Theorem 1.3, we will consider the decomposition of $K_{n(2)}$ only in the following sections.

3. Cycles from feasible sequences of differences

In this section, we assume $n \equiv 0$ or 3 (mod 6) with $n \geq 6$. Let $\mathbb{Z}_{n-1}^{\infty} = \{\infty, 0, 1, 2, \ldots, n-2\}$. The arithmetic in $\mathbb{Z}_{n-1}^{\infty}$ is done modulo $n-1$ when ∞ is not involved. When ∞ is involved, we make the convention that $a \pm \infty = \infty \pm a = \infty$ for $a = 0, 1, \ldots, n-2$ and $\infty \pm \infty = 0$. In this paper, all arithmetic is done in $\mathbb{Z}_{n-1}^{\infty}$.

Let the partite sets of $K_{n(2)}$ be $A_\infty = \{\infty, \overline{\infty}\}$, $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, …, and $A_{n-2} = \{n-2, \overline{n-2}\}$. Thus, elements in $\mathbb{Z}_{n-1}^{\infty}$ are used as indices of the partite sets as well as the vertices.

Let $\mathcal{D}_{n-1}^* = \{\infty, \pm 1, \ldots, \pm\frac{n-1}{2}\}$ if n is odd, and $\mathcal{D}_{n-1}^* = \{\infty, \pm 1, \ldots, \pm\frac{n-2}{2}\}$ if n is even. Then, $\mathcal{D}_{n-1}^*$ is a complete set of differences of two distinct numbers in $\mathbb{Z}_{n-1}^{\infty}$. A sequence $\rho = (r_1, r_2, \ldots, r_6)$ of differences in $\mathcal{D}_{n-1}^*$ is called a *feasible sequence*, or an *f-sequence* for simplicity, if

(i) $\rho = (r_1, r_2, \ldots, r_6)$, where $r_i \in \mathcal{D}_{n-1}^* \setminus \{\infty\}$ for $i = 1, 2, \ldots, 6$, $\sum_{i=1}^{6} r_i = 0$, and $\sum_{i=p}^{q} r_i \neq 0$ for p, q with $1 < p$ or $q < 6$, or

(ii) $\rho = (r_1, r_2, r_3, r_4, \infty, \infty)$, where $r_i \in \mathcal{D}^*_{n-1} \setminus \{\infty\}$ for $i = 1, 2, 3, 4$, and $\sum_{i=p}^{q} r_i \neq 0$ for p, q with $1 < p$ or $q < 4$.

Note that any proper partial sum of consecutive entries of an f-sequence is nonzero.

Let $\rho = (r_1, r_2, \ldots, r_6)$ be an f-sequence. A sequence $\sigma_\rho = \langle 0, s_1, \ldots, s_5 \rangle$ of elements in $\mathbb{Z}^\infty_{n-1}$ is called the *sequence of initial sums*, or an *s-sequence* for short, of ρ if $s_j = \sum_{i=1}^{j} r_i$ for $j = 1, 2, 3, 4, 5$. Thus, if we put $s_0 = 0$ then $s_j = s_{j-1} + r_j$ if $1 \leq j \leq 5$, and $s_5 + r_6 = 0$ or $s_5 = \infty$ by the definition of f-sequences. For example, when $n = 12$, $\sigma_{\rho_0} = \langle 0, 3, 10, 4, 8, 5 \rangle$ for $\rho_0 = (3, -4, 5, 4, -3, -5)$ and $\sigma_{\rho_1} = \langle 0, 1, 3, 4, 2, \infty \rangle$ for $\rho_1 = (1, 2, 1, -2, \infty, \infty)$.

Intuitively, an s-sequence represents the ordering of partite sets which a 6-cycle traverses, and the feasibility of the corresponding f-sequence guarantees that the cycle is proper and gregarious. Now, the following lemma is trivial by definitions.

Lemma 3.1. *Let $\sigma = \langle 0, s_1, s_2, \ldots, s_5 \rangle$ be the sequence of initial sums of a sequence $\rho = (r_1, r_2, \ldots, r_6)$ of differences. Then, ρ is an f-sequence if and only if $0, s_1, s_2, \ldots, s_5$ are mutually distinct and $\sum_{i=1}^{6} r_i = 0$.*

Let ϕ^+ and ϕ^- be the mappings of $\mathbb{Z}^\infty_{n-1}$ into $\bigcup_{i \in \mathbb{Z}^\infty_{n-1}} A_i$ defined by $\phi^+(i) = i$ and $\phi^-(i) = \bar{i}$ for all i in $\mathbb{Z}^\infty_{n-1}$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \ldots, \phi_5)$ where each ϕ_i is ϕ^+ or ϕ^- for $i = 0, 1 \ldots, 5$. Given a flag ϕ^*, we also use the same notation ϕ^* to denote the mapping defined by $\phi^*\langle s_0, s_1, \ldots, s_5 \rangle = \langle \phi_0(s_0), \phi_1(s_1), \ldots, \phi_5(s_5) \rangle$ for every s-sequence $\langle s_0, s_1, \ldots, s_5 \rangle$. Note that

$$\phi^*\langle s_0, s_1, \ldots, s_5 \rangle$$

is a γ_6-cycle of $K_{n(2)}$.

Let τ be the permutation $(0, 1, \ldots, n-2)(\infty)$ on $\mathbb{Z}^\infty_{n-1}$, that is, $\tau(i) = i + 1$ for all i in $\mathbb{Z}^\infty_{n-1}$. In this sense, we may call τ a *translation* on $\mathbb{Z}^\infty_{n-1}$. We extend τ to a mapping τ_* on 6-cycles of $K_{n(2)}$ by defining

$$\begin{aligned} \tau_*\langle \phi_0(s_0), \phi_1(s_1), \ldots, \phi_5(s_5) \rangle &= \langle \phi_0(\tau(s_0)), \phi_1(\tau(s_1)), \ldots, \phi_5(\tau(s_5)) \rangle \\ &= \langle \phi_0(s_0+1), \phi_1(s_1+1), \ldots, \phi_5(s_5+1) \rangle. \end{aligned}$$

Note that τ_*^{n-1} is the identity mapping and, by convention, τ_*^0 is the identity mapping.

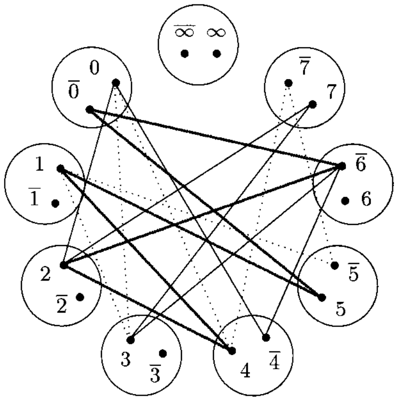
Given an s-sequence σ and a flag ϕ^*, we can generate the class $\{\tau_*^i(\phi^*(\sigma)) \mid 0 \leq i \leq n-2\}$ containing $n-1$ γ_6-cycles, which we call an *full class* generated from $\phi^*(\sigma)$. Sometimes, when n is odd, we need to generate the class $\{\tau_*^i(\phi^*(\sigma)) \mid 0 \leq i \leq \frac{n-1}{2} - 1\}$ or $\{\tau_*^i(\phi^*(\sigma)) \mid \frac{n-1}{2} \leq i \leq n-2\}$ containing $\frac{n-1}{2}$ γ_6-cycles, which we call an *half class* generated from $\phi^*(\sigma)$. The cycle $\phi^*(\sigma)$ is called the *starter cycle* of the relevant class. For example, if $\sigma_\rho = \langle s_0, s_1, \ldots, s_5 \rangle$ from $\rho = (r_1, r_2, \ldots, r_6)$ and $\phi^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+, \phi^-)$, then the γ_6-cycles in the full class generated by $\phi^*(\sigma)$ are as below:

$$\begin{array}{llllllll}
\langle & 0, & \overline{s_1}, & \overline{s_2}, & s_3, & s_4, & \overline{s_5} & \rangle, \\
\langle & 1, & \overline{s_1+1}, & \overline{s_2+1}, & s_3+1, & s_4+1, & \overline{s_5+1} & \rangle, \\
\langle & 2, & \overline{s_1+2}, & \overline{s_2+2}, & s_3+2, & s_4+2, & \overline{s_5+2} & \rangle, \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\langle & n-1, & \overline{s_1+n-1}, & \overline{s_2+n-1}, & s_3+n-1, & s_4+n-1, & \overline{s_5+n-1} & \rangle.
\end{array}$$

If neither ∞ nor $\overline{\infty}$ appears in a column of the above table, that column either has all i or has all $\overline{i}$ for $i = 0, 1, \ldots, n-1$. Note that, if $q-p = r_1$ then the edge $p\overline{q}$ appears exactly once as the first edge of a cycle above. Similarly, each of the edges $\overline{p}\,\overline{q}$ with $q-p = r_2$, $\overline{p}q$ with $q-p = r_3$, pq with $q-p = r_4$, $p\overline{q}$ with $q-p = r_5$ and $\overline{p}q$ with $q-p = r_6$, appears exactly once in the above.

The above procedure is the principal method we are going to use to obtain γ_6-decompositions of $K_{n(2)}$. The main problem then is how to choose appropriate f-sequences and flags, which will be discussed in the next section.

An edge joining a vertex in A_i and a vertex in A_j is called an *edge of distance* d if $i-j = \pm d$ for some d in $\mathcal{D}^*_{n-1}$ with $0 < d \leq \frac{n-1}{2}$. For example, the edges $0\overline{4}$, 73, $\overline{7}\,\overline{2}$ and $\overline{8}3$ are all edges of distance 4 in $K_{10(2)}$. An edge involving the vertex ∞ or $\overline{\infty}$, such as $\infty\overline{3}$ and $6\overline{\infty}$, are called an *edge of infinite distance*. When n is odd and $d = \frac{n-1}{2}$, an edge of distance d is called a *diagonal edge*.

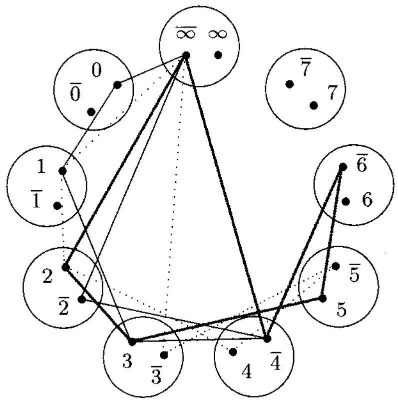


—— : $\Phi_1^*(\sigma_\eta) = \langle 0, 2, 7, 3, \overline{6}, \overline{4}\rangle$

······ : $\tau_*(\Phi_1^*(\sigma_\eta)) = \langle 1, 3, 0, 4, \overline{7}, \overline{5}\rangle$

—— : $\tau_*^2(\Phi_1^*(\sigma_\eta)) = \langle 2, 4, 1, 5, \overline{0}, \overline{6}\rangle$

Figure 1



—— : $\Psi_1^*(\sigma_{\delta_1}) = \langle 0, 1, 3, \overline{4}, \overline{2}, \overline{\infty}\rangle$

······ : $\tau_*(\Psi_1^*(\sigma_{\delta_1})) = \langle 1, 2, 4, \overline{5}, \overline{3}, \overline{\infty}\rangle$

—— : $\tau_*^2(\Psi_1^*(\sigma_{\delta_1})) = \langle 2, 3, 5, \overline{6}, \overline{4}, \overline{\infty}\rangle$

Figure 2

4. Proof of Theorem 2.1

For i with $1 \leq i \leq \frac{n-2}{2}-2$, using differences in $\{\pm i, \pm(i+1), \pm(i+2)\}$, we generate two special full classes as follow. Put $\eta = (i, -(i+1), i+2, i+1, -i, -(i+2))$, then we have the s-sequence $\sigma_\eta = \langle 0, i, n-2, i+1, 2i+2, i+2\rangle$. Since $2i+2 \leq 2(\frac{n-2}{2}-2)+2 = n-4$, all entries of σ_η are mutually distinct. Thus, η is indeed an f-sequence by Lemma 3.1. Now, we take two special flags $\Phi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^-)$ and $\Phi_2^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^-, \phi^+)$, and generate two full classes C and D from the starter cycles $\Phi_1^*(\sigma_\eta)$ and $\Phi_2^*(\sigma_\eta)$, respectively. These two classes are called the *standard classes* generated from η. They are listed below and depicted by the picture in Figure 1.

$$
(C)\quad \begin{array}{l}
\langle\ 0,\ i,\ n-2, i+1, \overline{2i+2}, \overline{i+2}\,\rangle,\\
\langle\ 1,\ i+1,\ 0,\ i+2, \overline{2i+3}, \overline{i+3}\,\rangle,\\
\langle\ 2,\ i+2,\ 1,\ i+3, \overline{2i+4}, \overline{i+4}\,\rangle,\\
\quad\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
\langle n-2, i-1, n-3,\ i,\ \overline{2i+1}, \overline{i+1}\,\rangle.
\end{array}
\qquad
(D)\quad \begin{array}{l}
\langle\ \overline{0},\ i,\ \overline{n-2}, \overline{i+1}, \overline{2i+2}, i+2\,\rangle,\\
\langle\ \overline{1},\ i+1,\ \overline{0},\ \overline{i+2}, \overline{2i+3}, i+3\,\rangle,\\
\langle\ \overline{2},\ i+2,\ \overline{1},\ \overline{i+3}, \overline{2i+3}, i+3\,\rangle,\\
\quad\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
\langle \overline{n-2}, i-1, \overline{n-3},\ \overline{i},\ \overline{2i+1}, i+1\,\rangle.
\end{array}
$$

The following Lemma is easily checked from the above table.

Lemma 4.1. *In the cycles of the classes C and D above, each of the edges of the form $p\,q$, $p\,\overline{q}$, $\overline{p}\,q$ and $\overline{p}\,\overline{q}$ of distance d appears exactly once for $d = i$, $i+1$ and $i+2$. The classes C and D are invariant under τ_*.*

With the differences in $\{\infty, \pm 1, \pm 2\}$, we produce two f-sequences $\delta_1 = (1, 2, 1, -2, \infty, \infty)$ and $\delta_2 = (1, 2, -1, 2, \infty, \infty)$, and get the corresponding s-sequences $\sigma_{\delta_1} = \langle 0, 1, 3, 4, 2, \infty\rangle$ and $\sigma_{\delta_2} = \langle 0, 1, 3, 2, 4, \infty\rangle$. Take two special flags $\Psi_1^* = (\phi^+, \phi^+, \phi^+, \phi^-, \phi^-, \phi^-)$ and $\Psi_2^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^+, \phi^+)$, and generate two full classes E_1 and E_2 from starter cycles $\Psi_1^*(\sigma_{\delta_1})$ and $\Psi_2^*(\sigma_{\delta_2})$, respectively, as below. They are depicted by the picture in Figure 2.

$$
(E_1)\quad \begin{array}{l}
\langle\ 0,\ 1,\ 3, \overline{4}, \overline{2}, \overline{\infty}\,\rangle,\\
\langle\ 1,\ 2,\ 4, \overline{5}, \overline{3}, \overline{\infty}\,\rangle,\\
\langle\ 2,\ 3,\ 5, \overline{6}, \overline{4}, \overline{\infty}\,\rangle,\\
\quad\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
\langle n-3, n-2, 1, \overline{2}, \overline{0}, \overline{\infty}\,\rangle,\\
\langle n-2,\ 0,\ 2, \overline{3}, \overline{1}, \overline{\infty}\,\rangle.
\end{array}
\qquad
(E_2)\quad \begin{array}{l}
\langle\ \overline{0},\ 1,\ \overline{3}, \overline{2}, 4, \infty\,\rangle,\\
\langle\ \overline{1},\ 2,\ \overline{4}, \overline{3}, 5, \infty\,\rangle,\\
\langle\ \overline{2},\ 3,\ \overline{5}, \overline{4}, 6, \infty\,\rangle,\\
\quad\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
\langle \overline{n-3}, n-2, \overline{1}, \overline{0}, 2, \infty\,\rangle,\\
\langle \overline{n-2},\ 0,\ \overline{2}, \overline{1}, 3, \infty\,\rangle.
\end{array}
$$

The following lemma is easily checked from the above table.

Lemma 4.2. *In the cycles of the classes E_1 and E_2 above, each of the edges of the form $p\,q$, $p\,\overline{q}$, $\overline{p}\,q$ and $\overline{p}\,\overline{q}$ of distance d appears exactly once for $d = 1$, 2 and ∞. The classes E_1 and E_2' are invariant under τ_*.*

We divide the proof of Theorem 2.1 into two cases depending on n.

Case (1). Suppose $n \equiv 0 \pmod 6$ and put $n = 6k$ $(k \geq 1)$. Since $K_{n(2)}$ has $4 \cdot \binom{6k}{2} = 12k(6k-1)$ edges, we need to produce $2k(6k-1) = 2k(n-1)$ disjoint 6-cycles. In fact, we will produce $2k$ full classes. In this case,

$$\mathbb{Z}_{n-1}^{\infty} = \{\infty, 0, 1, \ldots, 6k-2\} \quad \text{and} \quad \mathcal{D}_{n-1}^{*} = \{\infty, \pm 1, \pm 2, \ldots, \pm(3k-1)\}.$$

Starting with the differences in $\{\infty, \pm 1, \pm 2\}$, we obtain two full classes E_1 and E_2 of Lemma 4.2. If $k > 1$ then the set $\mathcal{D}^*_{n-1} \setminus \{\infty, \pm 1, \pm 2\} = \{\pm 3, \pm 4, \ldots, \pm(3k-1)\}$ is not empty. Partition this set into $k-1$ subsets $\{\pm 3i, \pm(3i+1), \pm(3i+2)\}$ for $i = 1, 2, \ldots, k-1$. For each i, we put $\eta_i = (3i, -(3i+1), 3i+2, 3i+1, -3i, -(3i+2))$ and generate two standard classes C_i and D_i from η_i, as in Lemma 4.1. If $k = 1$, we do not have these classes. Put

$$\mathcal{C}^* = \left(\bigcup_{i=1}^{2} E_i\right) \cup \left(\bigcup_{i=1}^{k-1} C_i\right) \cup \left(\bigcup_{i=1}^{k-1} D_i\right).$$

Then, by Lemmas 4.1 and 4.2, γ_6-cycles in $\mathcal{C}^*$ involve each edge of $K_{n(2)}$ exactly once. Thus, we have the following theorem.

Theorem 4.1. *For $n \equiv 0 \pmod 6$, the class $\mathcal{C}^*$ above is a γ_6-decomposition of $K_{n(2)}$ and is ∞-circular in the sense that it is invariant under τ_*.*

Example 4.1. Let $n = 12$. Then $\mathcal{D}^*_{n-1} = \{\infty, \pm 1, \pm 2, \ldots, \pm 5\}$. By the procedure above, we have $\delta_1 = (1, 2, 1, -2, \infty, \infty)$, $\delta_2 = (1, 2, -1, 2, \infty, \infty)$ and $\eta_1 = (3, -4, 5, 4, -3, -5)$. The corresponding s-sequences are $\sigma_{\delta_1} = \langle 0, 1, 3, 4, 2, \infty \rangle$, $\sigma_{\delta_1} = \langle 0, 1, 3, 2, 4, \infty \rangle$ and $\sigma_{\eta_1} = \langle 0, 3, 10, 4, 8, 5 \rangle$. The γ_6-decomposition is given below:

$$\begin{array}{llll}
\langle 0, 1, 3, \bar{4}, \bar{2}, \overline{\infty} \rangle, & \langle \bar{0}, 1, \bar{3}, \bar{2}, 4, \infty \rangle, & \langle 0, 3, 10, 4, \bar{8}, \bar{5} \rangle, & \langle \bar{0}, 3, \overline{10}, \bar{4}, \bar{8}, 5 \rangle, \\
\langle 1, 2, 4, \bar{5}, \bar{3}, \overline{\infty} \rangle, & \langle \bar{1}, 2, \bar{4}, \bar{3}, 5, \infty \rangle, & \langle 1, 4, 0, 5, \bar{9}, \bar{6} \rangle, & \langle \bar{1}, 4, \bar{0}, \bar{5}, \bar{9}, 6 \rangle, \\
\langle 2, 3, 5, \bar{6}, \bar{4}, \overline{\infty} \rangle, & \langle \bar{2}, 3, \bar{5}, \bar{4}, 6, \infty \rangle, & \langle 2, 5, 1, 6, \overline{10}, \bar{7} \rangle, & \langle \bar{2}, 5, \bar{1}, \bar{6}, \overline{10}, 7 \rangle, \\
\langle 3, 4, 6, \bar{7}, \bar{5}, \overline{\infty} \rangle, & \langle \bar{3}, 4, \bar{6}, \bar{5}, 7, \infty \rangle, & \langle 3, 6, 2, 7, \bar{0}, \bar{8} \rangle, & \langle \bar{3}, 6, \bar{2}, \bar{7}, \bar{0}, 8 \rangle, \\
\langle 4, 5, 7, \bar{8}, \bar{6}, \overline{\infty} \rangle, & \langle \bar{4}, 5, \bar{7}, \bar{6}, 8, \infty \rangle, & \langle 4, 7, 3, 8, \bar{1}, \bar{9} \rangle, & \langle \bar{4}, 7, \bar{3}, \bar{8}, \bar{1}, 9 \rangle, \\
\langle 5, 6, 8, \bar{9}, \bar{7}, \overline{\infty} \rangle, & \langle \bar{5}, 6, \bar{8}, \bar{7}, 9, \infty \rangle, & \langle 5, 8, 4, 9, \bar{2}, \overline{10} \rangle, & \langle \bar{5}, 8, \bar{4}, \bar{9}, \bar{2}, 10 \rangle, \\
\langle 6, 7, 9, \overline{10}, \bar{8}, \overline{\infty} \rangle, & \langle \bar{6}, 7, \bar{9}, \bar{8}, 10, \infty \rangle, & \langle 6, 9, 5, 10, \bar{3}, \bar{0} \rangle, & \langle \bar{6}, 9, \bar{5}, \overline{10}, \bar{3}, 0 \rangle, \\
\langle 7, 8, 10, \bar{0}, \bar{9}, \overline{\infty} \rangle, & \langle \bar{7}, 8, \overline{10}, \bar{9}, 0, \infty \rangle, & \langle 7, 10, 6, 0, \bar{4}, \bar{1} \rangle, & \langle \bar{7}, 10, \bar{6}, \bar{0}, \bar{4}, 1 \rangle, \\
\langle 8, 9, 0, \bar{1}, \overline{10}, \overline{\infty} \rangle, & \langle \bar{8}, 9, \bar{0}, \overline{10}, 1, \infty \rangle, & \langle 8, 0, 7, 1, \bar{5}, \bar{2} \rangle, & \langle \bar{8}, 0, \bar{7}, \bar{1}, \bar{5}, 2 \rangle, \\
\langle 9, 10, 1, \bar{2}, \bar{0}, \overline{\infty} \rangle, & \langle \bar{9}, 10, \bar{1}, \bar{0}, 2, \infty \rangle, & \langle 9, 1, 8, 2, \bar{6}, \bar{3} \rangle, & \langle \bar{9}, 1, \bar{8}, \bar{2}, \bar{6}, 3 \rangle, \\
\langle 10, 0, 2, \bar{3}, \bar{1}, \overline{\infty} \rangle. & \langle \overline{10}, 0, \bar{2}, \bar{1}, 3, \infty \rangle. & \langle 10, 2, 9, 3, \bar{7}, \bar{4} \rangle. & \langle \overline{10}, 2, \bar{9}, \bar{3}, \bar{7}, 4 \rangle.
\end{array}$$

Case (2). Suppose $n = 9$. We treat this case as a special case. A 6-cycle decomposition exists for K_9 by Lemma 1, and the following is an example.

$$\langle 0, 1, 6, 7, 3, 4 \rangle, \quad \langle 1, 2, 7, \infty, 4, 5 \rangle, \quad \langle 2, 3, \infty, 0, 5, 6 \rangle,$$
$$\langle 0, 2, 5, 7, 1, 3 \rangle, \quad \langle 3, 5, \infty, 1, 4, 6 \rangle, \quad \langle 6, \infty, 2, 4, 7, 0 \rangle,$$

where $\mathbb{Z}_8^\infty$ is used for the vertex set of K_9. From this, we obtain the following γ_6-decomposition of $K_{9(2)}$ by the method in Lemma 1.3.

$$\begin{array}{llll}
\langle 0, 1, 6, 7, 3, 4 \rangle, & \langle 0, \bar{1}, 6, \bar{7}, 3, \bar{4} \rangle, & \langle \bar{0}, 1, \bar{6}, 7, \bar{3}, 4 \rangle, & \langle \bar{0}, \bar{1}, \bar{6}, \bar{7}, \bar{3}, \bar{4} \rangle, \\
\langle 1, 2, 7, \infty, 4, 5 \rangle, & \langle 1, \bar{2}, 7, \overline{\infty}, 4, \bar{5} \rangle, & \langle \bar{1}, 2, \bar{7}, \infty, \bar{4}, 5 \rangle, & \langle \bar{1}, \bar{2}, \bar{7}, \overline{\infty}, \bar{4}, \bar{5} \rangle, \\
\langle 2, 3, \infty, 0, 5, 6 \rangle, & \langle 2, \bar{3}, \infty, \bar{0}, 5, \bar{6} \rangle, & \langle \bar{2}, 3, \overline{\infty}, 0, \bar{5}, 6 \rangle, & \langle \bar{2}, \bar{3}, \overline{\infty}, \bar{0}, \bar{5}, \bar{6} \rangle, \\
\langle 0, 2, 5, 7, 1, 3 \rangle, & \langle 0, \bar{2}, 5, \bar{7}, 1, \bar{3} \rangle, & \langle \bar{0}, 2, \bar{5}, 7, \bar{1}, 3 \rangle, & \langle \bar{0}, \bar{2}, \bar{5}, \bar{7}, \bar{1}, \bar{3} \rangle, \\
\langle 3, 5, \infty, 1, 4, 6 \rangle, & \langle 3, \bar{5}, \infty, \bar{1}, 4, \bar{6} \rangle, & \langle \bar{3}, 5, \overline{\infty}, 1, \bar{4}, 6 \rangle, & \langle \bar{3}, \bar{5}, \overline{\infty}, \bar{1}, \bar{4}, \bar{6} \rangle, \\
\langle 6, \infty, 2, 4, 7, 0 \rangle. & \langle 6, \overline{\infty}, 2, \bar{4}, 7, \bar{0} \rangle. & \langle \bar{6}, \infty, \bar{2}, 4, \bar{7}, 0 \rangle. & \langle \bar{6}, \overline{\infty}, \bar{2}, \bar{4}, \bar{7}, \bar{0} \rangle.
\end{array}$$

Theorem 4.2. *$K_9(2)$ is γ_6-decomposable.*

Case (3). Suppose $n \equiv 3 \pmod 6$ with $n \geq 15$ and put $n = 6k+3$ $(k \geq 2)$. Since $K_{n(2)}$ has $4 \cdot \binom{6k+3}{2} = 2(6k+3)(6k+2) = 6(2k+1)(n-1)$ edges, we need to produce $(2k+1)(n-1)$ disjoint 6-cycles. In fact, we will produce $2k-1$ full classes and 4 half classes. We have

$$\mathbb{Z}_{n-1}^{\infty} = \{\infty, 0, 1, \ldots, 6k+1\} \quad \text{and} \quad \mathcal{D}_{n-1}^{*} = \{\infty, \pm 1, \pm 2, \ldots, \pm(3k+1)\}.$$

Note that $3k+1 = -(3k+1)$ in $\mathbb{Z}_{n-1}^{\infty}$. With the differences in $\{\pm(3k-3), \pm(3k-2), \pm(3k-1), \pm 3k, \pm(3k+1)\}$, we produce two f-sequences

$$\rho_1 = (3k-3,\ -(3k-2),\ 3k-1,\ 3k+1,\ -(3k-1),\ -3k) \quad \text{and}$$
$$\rho_2 = (3k-3,\ 3k-2,\ -3k,\ -(3k-3),\ 3k,\ -(3k-2)).$$

Then, we have

$$\sigma_{\rho_1} = \langle 0,\ 3k-3,\ 6k+1,\ 3k-2,\ 6k-1,\ 3k \rangle \quad \text{and}$$
$$\sigma_{\rho_2} = \langle 0,\ 3k-3,\ 6k-5,\ 3k-5,\ 6k,\ 3k-2 \rangle.$$

We take four flags

$$\phi_1^* = (\phi^-, \phi^-, \phi^+, \phi^-, \phi^-, \phi^+), \qquad \phi_3^* = (\phi^-, \phi^+, \phi^-, \phi^+, \phi^+, \phi^-),$$
$$\phi_2^* = (\phi^-, \phi^-, \phi^+, \phi^+, \phi^-, \phi^-), \qquad \phi_4^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^+, \phi^+).$$

Since σ_{ρ_1} has diagonal edges, we need to generate half classes from it, instead of full classes, to avoid double appearances of edges of the form $p\,q$ or $\overline{p}\,\overline{q}$. We generate the following 4 half classes from σ_{ρ_1} and the above four flags.

$$F_1 = \{\tau_*^i(\phi_1^*(\sigma_{\rho_1})) \mid 0 \le i \le \tfrac{n-1}{2} - 1\}, \quad F_3 = \{\tau_*^i(\phi_3^*(\sigma_{\rho_1})) \mid 0 \le i \le \tfrac{n-1}{2} - 1\},$$
$$F_2 = \{\tau_*^i(\phi_2^*(\sigma_{\rho_1})) \mid \tfrac{n-1}{2} \le i \le n-2\}, \quad F_4 = \{\tau_*^i(\phi_4^*(\sigma_{\rho_1})) \mid \tfrac{n-1}{2} \le i \le n-2\}.$$

The γ_6-cycles in these classes are as below. Note that $6k+2 = 0$ and $\frac{n-1}{2} - 1 = 3k$.

$$(F_1)\quad \begin{array}{l} \langle\ \overline{0},\ \overline{3k-3},\ 6k+1,\ \overline{3k-2},\ \overline{6k-1},\ 3k\ \rangle, \\ \langle\ \overline{1},\ \overline{3k-2},\ 0,\ \overline{3k-1},\ \overline{6k},\ 3k+1\ \rangle, \\ \langle\ \overline{3},\ \overline{3k-1},\ 1,\ \overline{3k},\ \overline{6k+1},\ 3k+2\ \rangle, \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \langle\ \overline{3k},\ \overline{6k-3},\ 3k-1,\ \overline{6k-2},\ \overline{3k-3},\ 6k\ \rangle. \end{array} \qquad (F_3)\quad \begin{array}{l} \langle\ \overline{0},\ 3k-3,\ \overline{6k+1},\ 3k-2,\ 6k-1,\ \overline{3k}\ \rangle, \\ \langle\ \overline{1},\ 3k-2,\ \overline{0},\ 3k-1,\ 6k,\ \overline{3k+1}\ \rangle, \\ \langle\ \overline{3},\ 3k-1,\ \overline{1},\ 3k,\ 6k+1,\ \overline{3k+2}\ \rangle, \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \langle\ \overline{3k},\ 6k-3,\ \overline{3k-1},\ 6k-2,\ 3k-3,\ \overline{6k}\ \rangle. \end{array}$$

$$(F_2)\quad \begin{array}{l} \langle\ \overline{3k+1},\ \overline{6k-2},\ 3k,\ 6k-1,\ \overline{3k-2},\ \overline{6k+1}\ \rangle, \\ \langle\ \overline{3k+2},\ \overline{6k-1},\ 3k+1,\ 6k,\ \overline{3k-1},\ \overline{0}\ \rangle, \\ \langle\ \overline{3k+3},\ \overline{6k},\ 3k+2,\ 6k+1,\ \overline{3k},\ \overline{1}\ \rangle, \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \langle\ \overline{6k+1},\ \overline{3k-4},\ 6k,\ 3k-3,\ \overline{6k-2},\ \overline{3k-1}\ \rangle. \end{array} \qquad (F_4)\quad \begin{array}{l} \langle\ \overline{3k+1},\ 6k-2,\ \overline{3k},\ \overline{6k-1},\ 3k-2,\ 6k+1\ \rangle, \\ \langle\ \overline{3k+2},\ 6k-1,\ \overline{3k+1},\ \overline{6k},\ 3k-1,\ 0\ \rangle, \\ \langle\ \overline{3k+3},\ 6k,\ \overline{3k+2},\ \overline{6k+1},\ 3k,\ 1\ \rangle, \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \langle\ \overline{6k+1},\ 3k-4,\ \overline{6k},\ \overline{3k-3},\ 6k-2,\ 3k-1\ \rangle. \end{array}$$

Then, in the γ_6-cycles of F_1, F_2, F_3 and F_4, each of the following edges appears exactly once. Note that an edge $p\,\overline{q}$ with $q - p = r$ and an edge $\overline{q}\,p$ with $p - q = -r$ are the same edge.

(i) Diagonal edges, i.e., edges of distance $3k+1$. Edges $p\,q$, $\overline{p}\,\overline{q}$, $p\,\overline{q}$, $\overline{p}\,q$ with $q - p = 3k+1$ appear as the fourth edges in F_3, F_1, F_2, F_4, respectively.

(ii) Edges of distance $3k-1$. Edges $p\,q$ with $q - p = 3k-1$ appear as the third edges in F_2 or as the fifth edges in F_4 in the form $q\,p$. Edges $\overline{p}\,\overline{q}$ appear as the third edges in F_4 or as the fifth edges in F_2 in the form $\overline{q}\,\overline{p}$. Edges $p\,\overline{q}$ appear as the third edges in F_1 or as the fifth edges in

F_3 in the form $\overline{q}\,p$. Edges $\overline{p}\,q$ appear as the third edges in F_3 or as the fifth edges in F_1 in the form $q\,\overline{p}$.

(iii) Edges $\overline{p}\,q$ and $\overline{p}\,\overline{q}$ with $q-p=3k$. Edges $\overline{p}\,\overline{q}$ appear in the form $\overline{q}\,\overline{p}$ as the last edges in F_2 or F_3. Edges $\overline{p}\,q$ appear in the form $q\,\overline{p}$ as the last edges in F_1 or F_4.

(iv) Edges $\overline{p}\,q$ and $p\,\overline{q}$ with $q-p=3k-2$. Edges $\overline{p}\,q$ appear in the form $q\,\overline{p}$ as the second edges in F_1 or F_2. Edges $p\,\overline{q}$ appear in the form $\overline{q}\,p$ as the second edges in F_3 or F_4.

(v) Edges $\overline{p}\,q$ and $\overline{p}\,\overline{q}$ with $q-p=3k-3$. Edges $\overline{p}\,q$ appear as the first edges in F_3 or F_4. Edges $\overline{p}\,\overline{q}$ appear as the first edges in F_1 or F_2.

Next, we take another flag $\phi_5^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+, \phi^+)$, and generate the full class

$$F_5 = \{\tau_*^i(\phi_5^*(\sigma_{\rho_2})) \mid 0 \le i \le n-2\}$$

from σ_{ρ_2} and ϕ_5^*. The γ_6-cycles in this class are as below:

$$
\begin{array}{llllllll}
(F_5) & \langle\ 0, & \overline{3k-3}, & \overline{6k-5}, & 3k-5, & 6k, & 3k-2\,\rangle, \\
& \langle\ 1, & \overline{3k-2}, & \overline{6k-4}, & 3k-4, & 6k+1, & 3k-1\,\rangle, \\
& \langle\ 2, & \overline{3k-1}, & \overline{6k-3}, & 3k-3, & 0, & 3k\,\rangle, \\
& \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots \\
& \langle\ 3k, & \overline{6k-3}, & \overline{3k-7}, & 6k-5, & 3k-2, & 6k-2\,\rangle, \\
& \langle\ 3k+1, & \overline{6k-2}, & \overline{3k-6}, & 6k-4, & 3k-1, & 6k-1\,\rangle, \\
& \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots & \ \ \vdots \\
& \langle\ 6k & \overline{3k-5}, & \overline{6k-7}, & 3k-7, & 6k-2, & 3k-4\,\rangle, \\
& \langle\ 6k+1, & \overline{3k-4}, & \overline{6k-6}, & 3k-6, & 6k-1, & 3k-3\,\rangle.
\end{array}
$$

In the γ_6-cycles of F_5, each of the following edges appears exactly once.

(i) Edges of the form $p\,q$ or $p\,\overline{q}$ with $q-p=3k$. Edges $p\,q$ appear as the fifth edges. Edges $p\,\overline{q}$ appear in the form $\overline{q}\,p$ as the third edges.

(ii) Edges of the form $p\,q$ or $\overline{p}\,\overline{q}$ with $q-p=3k-2$. Edges $p\,q$ appear in the form $q\,p$ as the last edges. Edges $\overline{p}\,\overline{q}$ appear as the second edges.

(iii) Edges of the form $p\,q$ or $p\,\overline{q}$ with $q-p=3k-3$. Edges $p\,q$ appear in the form $q\,p$ as the fourth edges. Edges $p, \overline{q}$ appear as the first edges.

Therefore, in the γ_6-cycles of $\bigcup_{i=1}^{5} F_i$, every edge of distance $3k-3, 3k-2, 3k-1, 3k$ or $3k+1$ appears exactly once. Note that F_1, F_2, F_3, F_4 are not invariant under τ^*, while F_5 is.

Now, starting with the differences in $\{\infty, \pm1, \pm2\}$, we obtain two full classes E_1 and E_2, as in Lemma 4.2. Then, in the γ_6-cycles of these classes, every edge of distance $1, 2$ or ∞ appears exactly once. Note that, since $k \ge 2$, we have $2 < 3k-3$ and so cycles in E_i and cycles in F_j are disjoint for $i = 1, 2$ and $j = 1, 2, 3, 4, 5$.

If $k \ge 3$, the set

$$\mathcal{D}_{n-1}^* \setminus \{\infty, \pm1, \pm2, \pm(3k-3), \dots, \pm(3k+1)\} = \{\pm3, \pm4, \dots, \pm(3k-4)\}$$

is not empty. Partition this set into $k-2$ subsets $\{\pm3i, \pm(3i+1), \pm(3i+2)\}$ for $i = 1, 2, \dots, k-2$. For each i, put $\eta_i = (3i, -(3i+1), 3i+2, 3i+1, -3i, -(3i+2))$

and generate two standard classes C_i and D_i from η_i, as in Lemma 4.1. Then, in the γ_6-cycles in C_i and D_i, every edge of distance $3i, 3i+1$ and $3i+2$ appears exactly once. If $k = 2$, we do not have these classes.

Finally, put $\mathcal{C}^* = \left(\bigcup_{i=1}^{2} E_i\right) \cup \left(\bigcup_{i=1}^{5} F_i\right) \cup \left(\bigcup_{i=1}^{k-2} C_i\right) \cup \left(\bigcup_{i=1}^{k-2} D_i\right)$. Then, γ_6-cycles in $\mathcal{C}^*$ involve each edge of every distance in $K_{n(2)}$ exactly once. Thus, we have the following theorem.

Theorem 4.3. *For* $n \equiv 3 \pmod 6$ *with* $n \geq 15$, *the class* $\mathcal{C}^*$ *above is a* γ_6-*decomposition of* $K_{n(2)}$.

Example 4.2. Let $n = 15$. Using hexadecimal digits, let $\mathbb{Z}_{14}^{\infty} = \{\infty, 0, 1, \ldots, 9, a, b, c, d\}$. We have $\mathcal{D}_{14} = \{\pm\infty, \pm 1, \ldots, \pm 7\}$. Starting with f-sequences $\rho_1 = (3, -4, 5, 7, -5, -6)$ and $\rho_2 = (3, 4, -6, -3, 6, -4)$, we obtain s-sequences $\sigma_{\rho_1} = \langle 0, 3, d, 4, b, 6\rangle$ and $\sigma_{\rho_2} = \langle 0, 3, 7, 1, c, 4\rangle$. We generate classes F_1, F_2, F_3, F_4, F_5 as in Case (3). From f-sequences $\delta_1 = (1, 2, 1, -2, \infty, \infty)$ and $\delta_2 = (1, 2, -1, 2, \infty, \infty)$, we obtain s-sequences $\sigma_{\sigma_1} = \langle 0, 1, 3, 4, 2, \infty\rangle$ and $\sigma_{\delta_2} = \langle 0, 1, 3, 2, 3, \infty\rangle$. We generate two full classes E_1 and E_2 as in Lemma 4.2. These classes constitute a γ_6-decomposition of $K_{15(2)}$ and are listed below.

$\langle \bar{0}, \bar{3}, d, \bar{4}, \bar{b}, 6 \rangle$,	$\langle \bar{0}, 3, \bar{d}, 4, b, \bar{6} \rangle$,	$\langle 0, \bar{3}, \bar{7}, 1, c, 4 \rangle$,	$\langle 0, 1, 3, \bar{4}, \bar{2}, \bar{\infty} \rangle$,	$\langle \bar{0}, 1, \bar{3}, \bar{2}, 4, \infty \rangle$,
$\langle \bar{1}, \bar{4}, 0, \bar{5}, \bar{c}, 7 \rangle$,	$\langle \bar{1}, 4, \bar{0}, 5, c, \bar{7} \rangle$,	$\langle 1, \bar{4}, \bar{8}, 2, d, 5 \rangle$,	$\langle 1, 2, 4, \bar{5}, \bar{3}, \bar{\infty} \rangle$,	$\langle \bar{1}, 2, \bar{4}, \bar{3}, 5, \infty \rangle$,
$\langle \bar{2}, \bar{5}, 1, \bar{6}, \bar{d}, 8 \rangle$,	$\langle \bar{2}, 5, \bar{1}, 6, d, \bar{8} \rangle$,	$\langle 2, \bar{5}, \bar{9}, 3, 0, 6 \rangle$,	$\langle 2, 3, 5, \bar{6}, \bar{4}, \bar{\infty} \rangle$,	$\langle \bar{2}, 3, \bar{5}, \bar{4}, 6, \infty \rangle$,
$\langle \bar{3}, \bar{6}, 2, \bar{7}, \bar{0}, 9 \rangle$,	$\langle \bar{3}, 6, \bar{2}, 7, 0, \bar{9} \rangle$,	$\langle 3, \bar{6}, \bar{a}, 4, 1, 7 \rangle$,	$\langle 3, 4, 6, \bar{7}, \bar{5}, \bar{\infty} \rangle$,	$\langle \bar{3}, 4, \bar{6}, \bar{5}, 7, \infty \rangle$,
$\langle \bar{4}, \bar{7}, 3, \bar{8}, \bar{1}, a \rangle$,	$\langle \bar{4}, 7, \bar{3}, 8, 1, \bar{a} \rangle$,	$\langle 4, \bar{7}, \bar{b}, 5, 2, 8 \rangle$,	$\langle 4, 5, 7, \bar{8}, \bar{6}, \bar{\infty} \rangle$,	$\langle \bar{4}, 5, \bar{7}, \bar{6}, 8, \infty \rangle$,
$\langle \bar{5}, \bar{8}, 4, \bar{9}, \bar{2}, b \rangle$,	$\langle \bar{5}, 8, \bar{4}, 9, 2, \bar{b} \rangle$,	$\langle 5, \bar{8}, \bar{c}, 6, 3, 9 \rangle$,	$\langle 5, 6, 8, \bar{9}, \bar{7}, \bar{\infty} \rangle$,	$\langle \bar{5}, 6, \bar{8}, \bar{7}, 9, \infty \rangle$,
$\langle \bar{6}, \bar{9}, 5, \bar{a}, \bar{3}, c \rangle$,	$\langle \bar{6}, 9, \bar{5}, a, 3, \bar{c} \rangle$,	$\langle 6, \bar{9}, \bar{d}, 7, 4, a \rangle$,	$\langle 6, 7, 9, \bar{a}, \bar{8}, \bar{\infty} \rangle$,	$\langle \bar{6}, 7, \bar{9}, \bar{8}, a, \infty \rangle$,
$\langle \bar{7}, \bar{a}, 6, b, \bar{4}, \bar{d} \rangle$,	$\langle \bar{7}, a, \bar{6}, \bar{b}, 4, d \rangle$,	$\langle 7, \bar{a}, \bar{0}, 8, 5, b \rangle$,	$\langle 7, 8, a, \bar{b}, \bar{9}, \bar{\infty} \rangle$,	$\langle \bar{7}, 8, \bar{a}, \bar{9}, b, \infty \rangle$,
$\langle \bar{8}, \bar{b}, 7, c, \bar{5}, \bar{0} \rangle$,	$\langle \bar{8}, b, \bar{7}, \bar{c}, 5, 0 \rangle$,	$\langle 8, \bar{b}, \bar{1}, 9, 6, c \rangle$,	$\langle 8, 9, b, \bar{c}, \bar{a}, \bar{\infty} \rangle$,	$\langle \bar{8}, 9, \bar{b}, \bar{a}, c, \infty \rangle$,
$\langle \bar{9}, \bar{c}, 8, d, \bar{6}, \bar{1} \rangle$,	$\langle \bar{9}, c, \bar{8}, \bar{d}, 6, 1 \rangle$,	$\langle 9, \bar{c}, \bar{2}, a, 7, d \rangle$,	$\langle 9, a, c, \bar{d}, \bar{b}, \bar{\infty} \rangle$,	$\langle \bar{9}, a, \bar{c}, \bar{b}, d, \infty \rangle$,
$\langle \bar{a}, \bar{d}, 9, 0, \bar{7}, \bar{2} \rangle$.	$\langle \bar{a}, d, \bar{9}, \bar{0}, 7, 2 \rangle$,	$\langle a, \bar{d}, \bar{3}, b, 8, 0 \rangle$.	$\langle a, b, d, \bar{0}, \bar{c}, \bar{\infty} \rangle$.	$\langle \bar{a}, b, \bar{d}, \bar{c}, 0, \infty \rangle$,
$\langle \bar{b}, \bar{0}, a, 1, \bar{8}, \bar{3} \rangle$,	$\langle \bar{b}, 0, \bar{a}, \bar{1}, 8, 3 \rangle$,	$\langle b, \bar{0}, \bar{4}, c, 9, 1 \rangle$,	$\langle b, c, 0, \bar{1}, \bar{d}, \bar{\infty} \rangle$,	$\langle \bar{b}, c, \bar{0}, \bar{d}, 1, \infty \rangle$,
$\langle \bar{c}, \bar{1}, b, 2, \bar{9}, \bar{4} \rangle$,	$\langle \bar{c}, 1, \bar{b}, \bar{2}, 9, 4 \rangle$,	$\langle c, \bar{1}, \bar{5}, d, a, 2 \rangle$,	$\langle c, d, 1, \bar{2}, \bar{0}, \bar{\infty} \rangle$,	$\langle \bar{c}, d, \bar{1}, \bar{0}, 2, \infty \rangle$,
$\langle \bar{d}, \bar{2}, c, 3, \bar{a}, \bar{5} \rangle$.	$\langle \bar{d}, 2, \bar{c}, \bar{3}, a, 5 \rangle$.	$\langle d, \bar{2}, \bar{6}, 0, b, 3 \rangle$.	$\langle d, 0, 2, \bar{3}, \bar{1}, \bar{\infty} \rangle$.	$\langle \bar{d}, 0, \bar{2}, \bar{1}, 3, \infty \rangle$.

Proof of Theorem 2.1. By Theorems 4.1 - 4.3 and Theorem 1.3. □

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