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ABELIAN-BY-NILPOTENT GROUPS WITH CHAIN CONDITIONS FOR NORMAL SUBGROUPS OF INFINITE ORDER OR INDEX

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ABSTRACT. We study the structure of abelian-by-nilpotent groups satisfying the maximal condition on infinite normal subgroups or the minimal condition on normal subgroups of infinite index.

1. Introduction

A group G is said to satisfy the *weak maximal condition on normal subgroups* if there are no infinite ascending chains $G_1 < G_2 < \cdots$ of normal subgroups of G such that all the indices $|G_{i+1} : G_i|$ are infinite. The *weak minimal condition on normal subgroups* is defined by substituting descending for ascending chains. Kurdachenko [2] considered groups satisfying the weak maximal or weak minimal conditions on normal subgroups.

A group G is said to satisfy *max-*∞n (the maximal condition on infinite normal subgroups) if there are no infinite ascending chains of infinite normal subgroups of G. Similarly a group G is said to satisfy *min-*∞n (the minimal condition on normal subgroups of infinite index) if there are no infinite descending chains of normal subgroups with infinite index in G. Since the chain conditions max-∞n and min-∞n are weaker than the chain conditions max-n and min-n (the maximal and minimal conditions on normal subgroups, respectively), we define a group satisfies max-∞n^* if it satisfies max-∞n, but not max-n and a group satisfies min-∞n^* if it satisfies min-∞n, but not min-n.

De Giovanni et al. [3] characterized the structure of groups satisfying max-∞n^* or min-∞n^*. In addition, the structure of nonfinitely generated solvable groups satisfying max-∞n^* and solvable groups satisfying min-∞n^* was investigated in detail. In this paper, we consider abelian-by-nilpotent groups with these chain conditions.

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2. Basic results

We say that a G-operator group N is *G-quasifinite* if N is infinite but every proper G-invariant subgroup of N is finite.

Lemma 2.1 ([3], Theorem 2.1). *Let G be a group satisfying max-∞n^*. Then there is an infinite normal subgroup R which has the following properties:*

(1) *R is the unique smallest normal subgroup such that G/R has max-n;*
(2) *R is the unique G-quasifinite normal subgroup;*
(3) *R is either an elementary abelian p-group or a radicable abelian p-group of finite rank for some prime p;*
(4) *R is a countably infinite, locally finite G-module.*

And the subgroup R in Lemma 2.1 is denoted by

$$\rho(G).$$

Lemma 2.2 ([3], Proposition 4.4). *If G is a nonfinitely generated soluble group with max-∞n^*, then $C_G(\rho(G))$ is a torsion group.*

Karbe [1] proved that the weak maximal or weak minimal conditions on normal subgroups are inherited by any subgroups of finite index. We aim to extend Wilson's theorem on groups with min-n to G-operator groups. This will be used for investigating abelian-by-nilpotent groups with min-∞n^*. Recall the statement of Wilson's theorem: if a group G satisfies min-n and H is a subgroup of G with finite index, then H satisfies min-n.

The following is the generalization of Wilson's theorem.

Proposition 2.3. *Let M be a G-operator group and let H be a subgroup of G of finite index. If M has min-G, then it has min-H.*

Proof. First note that the case $M = G$ is Wilson's Theorem. The proof is substantially Wilson's. Suppose that M does not in fact have min-H. Since G/H_G is finite, we may assume that $H \lhd G$. By min-G it follows that M contains a subgroup K which is G-invariant and minimal with respect to not satisfying min-H.

Consider the set $\mathfrak{S}$ of all finite nonempty subsets X of G with the following property: if

$$K_1 > K_2 > \cdots \tag{1}$$

is an infinite descending chain of H-invariant subgroups of K, then

$$K = K_i^X \tag{2}$$

holds for all i. It is not evident that such subsets exist, so our first concern is to produce one.

Let T be a transversal to H in G; thus $G = HT$. For any chain the above type we have $K_i^T = K_i^{HT} = K_i^G \leqslant K$ since K_i is H-invariant and K is G-invariant. If $K_i^T \neq K$, then K_i^T has the property min-H by minimality of

K. But this implies that $K_j = K_{j+1}$ for some $j \geq i$. By this contradiction $K_i^T = K$ for all i and $T \in \mathfrak{S}$.

We now select a minimal element of $\mathfrak{S}$, say X. If $x \in X$, then $Xx^{-1} \in \mathfrak{S}$ because K is G-invariant. Of course Xx^{-1} is also minimal in $\mathfrak{S}$ and it contains 1. Thus we may assume that $1 \in X$. If in fact X contains no other element, then (1) and (2) are inconsistent, so that K has min-H. Consequently the set

$$Y = X \backslash \{1\}$$

is nonempty. Therefore Y does not belong to $\mathfrak{S}$ by minimality of X.

It follows that there exists an infinite descending chain $K_1 > K_2 > \cdots$ of H-invariant subgroups of K such that $K_j^Y \neq K$ for some j. Define

$$L_i = K_i \cap K_i^Y.$$

Then L_i is a H-invariant subgroup of K. Also $L_i \geqslant L_{i+1}$. Suppose that $L_i = L_{i+1}$; since $X \in \mathfrak{S}$, we must have $K = K_{i+1}^X$ and

$$K_i = K_i \cap K_{i+1}^X = K_i \cap (K_{i+1} K_{i+1}^Y) \leqslant K_{i+1} L_i = K_{i+1},$$

contradicting $K_i > K_{i+1}$. Hence $L_i > L_{i+1}$ for all i. Therefore $L_i^X = K$ for all i, which shows that

$$K_j = K_j \cap L_j^X = K_j \cap (L_j L_j^Y) \leqslant L_j (K_j \cap K_j^Y) = L_j.$$

Hence $K_j = L_j$. Finally, by definition of L_j we obtain $K_j^Y = K_j^X = K$, a contradiction. □

3. Abelian-by-nilpotent groups with max-∞n^* or min-∞n^*

Our first result describes abelian-by-nilpotent groups with max-∞n^*.

Theorem 3.1. *An abelian-by-nilpotent group G satisfies max-∞n^* if and only if there is an infinite abelian normal subgroup R such that G/R is finitely generated, R is G-quasifinite, and $C_G(R)$ is torsion.*

Proof. Suppose that G satisfies max-∞n^*. Then, since finitely generated abelian-by-nilpotent groups satisfy max-n, G is not finitely generated. Hence the result follows from Lemmas 2.1 and 2.2.

Conversely, suppose that G has the structure indicated, but does not satisfy max-∞n. Let $G_1 < G_2 < \cdots$ be an infinite ascending chain of infinite normal subgroups of G.

Case: *$G_i \cap R$ is infinite for some i.* Since R is G-quasifinite, $G_i \cap R = R$ and so $R \leqslant G_i$. Hence G/R does not have max-n, a contradiction.

Case: *$G_i \cap R$ is finite for all i.* Since $G_i R/R \simeq G_i/G_i \cap R$ is infinite, $G_i R/R$ is not torsion. Hence $G_i R/R$ has an element xR of infinite order. Since $\langle x \rangle R \leqslant G_i R$, we can assume that $x \in G_i$. If $[R, x^j] = 1$, then $x^j \in C_G(R)$, a contradiction. Hence $[R, x^j]$ is finite with bounded order. Since $[R, x^j]^{\langle x \rangle} = [R, x^j]$, it follows that $[R, x^j, x^k] = 1$ for some $k > 0$ and all

$j > 0$. Hence $[R, x^k, x^k] = 1$. If l is the order of $[R, x^k]$, then $[R, x^{lk}] \leqslant [R, x^k]^l [R, x^k, x^k] = 1$. Hence $x^{lk} \in C_G(R)$, a contradiction.

Thus G satisfies max-∞n. Finally, R cannot be finitely generated since otherwise it would be finite. Hence G does not have max-n. □

Proposition 3.2. *A torsion abelian-by-nilpotent group G with max-∞n^* is Chernikov.*

Proof. Let $R = \rho(G)$. Then G/R is finitely generated solvable torsion, so it is finite. Write $G = XR$ where X is finite abelian-by-nilpotent. We will show that R is not an elementary abelian p-group. Let

$$U = R^p (X \cap R)$$

for some prime p. We pass to the group

$$\overline{G} = G/U = (XU/U) \ltimes (R/U).$$

Thus we can assume that $G = X \ltimes R$ with R an elementary abelian p-group and X a finite nilpotent group. Write $X = P \times Q$ where P is the p-component and Q is the p'-component. Write

$$C = C_R(P).$$

Then C is a $\mathbb{Z}_p Q$-module. By Maschke's Theorem, C is completely reducible, that is, a direct sum of simple submodules-the latter are X-invariant and so are normal in G. Thus C is a direct summand of finitely many simple submodules; hence C is finite.

Next observe that PR is nilpotent since P is a finite p-group and R is an elementary abelian p-group. The argument of the last paragraph shows that each $Z_{i+1}(PR)/Z_i(PR)$ is finite. Consequently PR is finite and so is R, a contradiction.

Therefore R is a radicable abelian p-group of finite rank for some prime p by Lemma 2.1. Hence G is Chernikov. □

We now consider abelian-by-nilpotent groups with min-∞n^*; in order to do this, we begin with polycyclic groups with min-∞n^*.

Lemma 3.3. *A polycyclic group G satisfies min-∞n^* if and only if it is a finite extension of a G-rationally irreducible free abelian subgroup of finite rank.*

Proof. Suppose that G satisfies min-∞n^*. Let A be a non-trivial free abelian normal subgroup of G. Then G/A must be finite since otherwise A has min-G. Now let B be a non-trivial G-invariant subgroup of A. Then G/B is finite, hence so is A/B, by the preceding argument. Consequently A is G-rationally irreducible.

Conversely, suppose that A is a G-rationally irreducible free abelian subgroup of finite rank such that G/A is finite. Suppose that G dose not satisfy min-∞n and let $G_1 > G_2 > \cdots$ be an infinite descending chain of normal subgroups of G with infinite index.

Case: *$G_i \cap A$ is finite for some i.* Since A is torsion-free, $G_i \cap A = 1$. Hence $G_iA/A \simeq G_i/G_i \cap A \simeq G_i$ is finite, a contradiction.

Case: *$G_i \cap A$ is infinite for all i.* Since $A/A \cap G_i \simeq AG_i/G_i$ is finite, and so is G/G_i, a contradiction. Thus G satisfies min-∞n. Finally, G does not satisfy min-n: for A does not have min-G. □

Now we can determine the structure of abelian-by-nilpotent groups with min-∞n^*.

Theorem 3.4. *An abelian-by-nilpotent group G satisfies min-∞n^* if and only if it has an infinite abelian normal subgroup A such that:*

either

(1) *A is a G-rationally irreducible free abelian subgroup of finite rank such that G/A is finite*

or else

(2) *G/A is infinite cyclic-by-finite, A has min-G, and $A/[A, x]$ is finite where x is any element of infinite order in G.*

Proof. Suppose that G has min-∞n^*. Let A be an abelian normal subgroup of G with G/A nilpotent. If A is finite, then G/A is a nilpotent group with min-∞n^*. Hence it is infinite cyclic-by-finite and so is G ([5], Lemma 3.1). Hence G has the structure given in (1). Thus we now assume that A is infinite.

Case: *G/A is finite.* We will show that G is polycyclic in this case. Most of the work is to show that A is not torsion. Suppose for now that we have shown this. Let $x \in A$ have infinite order and put $B = \langle x \rangle^G$. Since G/A is finite, B is a finitely generated infinite abelian normal subgroup of G. Hence G/B is finite since otherwise B has min-G. Therefore G is polycyclic. Thus it will suffice to argue that A is not torsion. Note that A does not have min-G. Assuming that A is torsion, we know that it is the direct sum of finitely many non-trivial primary components and only one primary component A_p can be infinite since A does not satisfy min-G. Let

$$A[p] = \{a \in A \mid a^p = 1\}.$$

If $A[p]$ is finite, then A has finite rank. Hence it is a direct sum of finitely many cyclic and quasicyclic groups. But then A has min, as must G, a contradiction. Therefore $A[p]$ is infinite elementary abelian.

If $G/A[p]$ is infinite, then $A[p]$ has min-G; and hence has min-A by Proposition 2.3. This implies that $A[p]$ is finite, a contradiction. Hence $G/A[p]$ is finite and $A[p]$ does not have min-G. If H is an infinite G-invariant subgroup of $A[p]$ of infinite index, then as before H has min-A by Proposition 2.3, a contradiction. Consequently every infinite G-invariant subgroup of $A[p]$ has finite index. It follows that $A[p]$ has max-∞G (the maximal condition for infinite G-invariant subgroups), so G has max-∞n. Hence G is Chernikov by Proposition 3.2 and so $A[p]$ is finite. By this contradiction A is not torsion.

Case: *G/A is infinite.* We will show that G has the structure given in (2) in this case. Since G/A is infinite, A has min-G and so G/A does not have min-n.

Thus G/A is cyclic-by-finite ([5], Lemma 3.1), and so it is finite-by-cyclic. Now we write $G = XA$ with X a finitely generated subgroup.

Let $z \in A$ have infinite order. Then, since A has min-G, it follows that $\langle z\rangle^G$ has min-G. Also $\langle z\rangle^G$ is a finitely generated G/A-module. Since G/A is finitely generated nilpotent, it is polycyclic. Hence $\langle z\rangle^G$ has max-G ([7], 15.3.3). It follows that $\langle z\rangle^G$ is finite, a contradiction. Therefore A is torsion.

Now let M be a maximal normal torsion subgroup of G containing A such that $G/M = \langle xM\rangle$ is infinite cyclic and M/A is finite.

Now we write $G/A = M/A \times \langle xA\rangle$ where $|x| = \infty$. We note that

$$\langle x, [M, x]\rangle \lhd \langle x, M\rangle = G.$$

If $M/[M, x]$ is infinite, then $\langle x, [M, x]\rangle$ has infinite index in G. Thus $\langle x, [M, x]\rangle$ has min-G. But then $\langle x^k, [M, x^k]\rangle$ is a G-invariant subgroup of $\langle x, [M, x]\rangle$ for each $k > 0$, which cannot be true. Hence $M/[M, x]$ is finite.

Write $G = XA$ where X is a finitely generated abelian-by-nilpotent group. Then X has max-n. Hence $X \cap A$ has max-X, and also min-X since $G = XA$. Therefore $X \cap A$ is finite. Now factor out by the finite normal subgroup $X \cap A$. Then

$$G = X \ltimes A \quad \text{and} \quad M = (M \cap X) \ltimes A.$$

Let $x \in G$ with $|x| = \infty$. Then $x = ya$ with $y \in X$, $a \in A$ and clearly $|y| = \infty$. Also $[A, x] = [A, y]$. So we can assume that $x \in X$. Then $[M, x] = [A, x]$ since $[M, x] \leqslant [A, x]\,[M \cap X, x]$ and $[M \cap X, x] \leqslant X \cap A = 1$. This argument shows that

$$[M, x] \leqslant [A, x](X \cap A).$$

Since $X \cap A$ is finite, so is $[M, x]/[A, x]$. Therefore $A/[A, x]$ is finite.

Conversely, if (1) holds, then G is polycyclic and the result follows from Lemma 3.3. Thus we assume that (2) holds. Suppose that $G_1 > G_2 > \cdots$ is an infinite descending chain of normal subgroups of G with infinite index.

Case: *G_iA/A is infinite for some i.* G_iA/A contains an element xA of infinite order where $x \in G_i$ and G/G_iA is finite. Since $[A, x] \leqslant A \cap G_i$, it follows that $A/A \cap G_i \simeq AG_i/G_i$ is finite and so is G/G_i, a contradiction.

Case: *G_iA/A is finite for all i.* There is an i such that $G_iA = G_{i+1}A$ and $G_i \cap A = G_{i+1} \cap A$, which implies that

$$G_{i+1} = G_{i+1} \cap G_iA = G_i(G_{i+1} \cap A) = G_i,$$

a contradiction.

Therefore G has min-∞n. Finally if G has min-n, then it is locally finite ([6], Theorem 5.25). Hence G/A is finitely generated locally finite and so is finite, a contradiction. □

Example 3.5 ([4], Example 3.4). Let $M = \langle a_1\rangle \times \langle a_2\rangle \times \cdots$ be an infinite elementary abelian p-group and let $X = \langle x\rangle$ be an infinite cyclic group acting on M via

$$a_1^x = a_1 \text{ and } a_{i+1}^x = a_{i+1}a_i$$

for all $i = 1, 2, \ldots$. Then $G = X \ltimes M$ is an abelian-by-nilpotent group with min-∞n^*.

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