Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 789–794

ON PRIME AND SEMIPRIME RINGS WITH PERMUTING 3-DERIVATIONS

Yong-Soo Jung and Kyoo-Hong Park

Abstract. Let R be a 3-torsion free semiprime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that the trace is centralizing on I. Then the trace of Δ is commuting on I. In particular, if R is a 3!-torsion free prime ring and Δ is nonzero under the same condition, then R is commutative.

1. Introduction and preliminaries

Throughout this paper, R will represent an associative ring, and Z will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We will also use the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Let S be a nonempty subset of R. Then a map $f : R \to R$ is said to be *commuting* (resp. *centralizing*) on S if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z$) for all $x \in S$. Recall that R is *semiprime* if $xRx = \{0\}$ implies $x = 0$ and R is *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$.

An additive map $d : R \to R$ is called a *derivation* if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

By a *bi-derivation* we mean a bi-additive map $D : R \times R \to R$ (i.e., D is additive in both arguments) which satisfies the relations

$$D(xy, z) = D(x, z)y + xD(y, z),$$
$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y \in R$. Let D be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in R$. The map $\tau : R \to R$ defined by $\tau(x) = D(x, x)$ for all $x, y \in R$ is called the trace of D. If R is a noncommutative 2-torsion free prime ring and $D : R \times R \to R$ is a symmetric bi-derivation, then it follows from [1, Theorem 3.5] that $D = 0$.

A map $\Delta : R \times R \times R \to R$ will be said to be *permuting* if the equation $\Delta(x_1, x_2, x_3) = \Delta(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ holds for all $x_1, x_2, x_3 \in R$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, x)$

Received March 11, 2007.

2000 *Mathematics Subject Classification.* 16W20, 16W25.

Key words and phrases. prime ring, semiprime ring, commuting map, centralizing map, derivation, bi-derivation, 3-derivation.

for all $x \in R$, where $\Delta : R \times R \times R \to R$ is a permuting map, is called the *trace* of Δ. It is obvious that, in case when $\Delta : R \times R \times R \to R$ is a permuting map which is also 3-additive (i.e., additive in each argument), the trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + 3\Delta(x,x,y) + 3\Delta(x,y,y)$$

for all $x, y \in R$.

Since we have

$$\Delta(0,y,z) = \Delta(0+0,y,z) = \Delta(0,y,z) + \Delta(0,y,z)$$

for all $y, z \in R$, we obtain $\Delta(0,y,z) = 0$ for all $y, z \in R$. Hence we get

$$0 = \Delta(0,y,z) = \Delta(x-x,y,z) = \Delta(x,y,z) + \Delta(-x,y,z)$$

and so we see that $\Delta(-x,y,z) = -\Delta(x,y,z)$ for all $x, y, z \in R$. This tells us that δ is an odd function.

Here we introduce the following map:

A 3-additive map $\Delta : R \times R \times R \to R$ will be called a 3-*derivation* if the relations

$$\Delta(x_1x_2,y,z) = \Delta(x_1,y,z)x_2 + x_1\Delta(x_2,y,z),$$
$$\Delta(x,y_1y_2,z) = \Delta(x,y_1,z)y_2 + y_1\Delta(x,y_2,z)$$

and

$$\Delta(x,y,z_1z_2) = \Delta(x,y,z_1)z_2 + z_1\Delta(x,y,z_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i \in R$, $i = 1, 2$. If Δ is permuting, then the above three relations are equivalent to each other.

For example, let R be commutative. A map $\Delta : R \times R \times R \to R$ defined by $(x,y,z) \mapsto d(x)d(y)d(z)$ for all $x, y, z \in R$ is a permuting 3-derivation, where d is a derivation on R.

On the other hand, let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \; a, b \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is a complex field. It is clear that R is a noncommutative ring under matrix addition and matrix multiplication. We define a map $\Delta : R \times R \times R \to R$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that Δ is a permuting 3-derivation.

A study concerning the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of E. C. Posner [3] which states that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Since then, a great deal of work in this context has been done by a number of authors (see, e.g., [1] and references therein). For instance, as a study concerning centralizing (commuting) maps, J. Vukman [4, 5] investigated symmetric bi-derivations on prime and semiprime rings.

In this paper, we apply the results due to E. C. Posner [3] and J. Vukman [4] to permuting 3-derivations, respectively.

2. The main results

We first need the following well-known lemma [2].

Lemma 2.1. *Let R be a prime ring. Let $d : R \to R$ be a derivation and $a \in R$. If $ad(x) = 0$ holds for all $x \in R$, then we have either $a = 0$ or $d = 0$.*

We begin our investigation of permuting 3-derivations with the next result.

Lemma 2.2. *Let R be a noncommutative $3!$-torsion free prime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that δ is commuting on I, where δ is the trace of Δ. Then we have $\Delta = 0$.*

Proof. Suppose that

$$[\delta(x), x] = 0 \quad \text{for all} \quad x \in I. \tag{2.1}$$

The substitution $x = x + y$ to linearize (2.1) leads to

$$\begin{aligned} 0 = {} & [\delta(x), y] + [\delta(y), x] + 3[\Delta(x, x, y), x] + 3[\Delta(x, y, y), x] \\ & +3[\Delta(x, x, y), y] + 3[\Delta(x, y, y), y] \quad \text{for all} \quad x, y \in I. \end{aligned} \tag{2.2}$$

Putting $-x$ instead of x in (2.2) and comparing (2.2) with the result, we arrive at

$$[\Delta(x, y, y), x] + [\Delta(x, x, y), y] = 0 \quad \text{for all} \quad x, y \in I \tag{2.3}$$

since δ is odd. We set $x = x + y$ in (2.3) and then use (2.1) and (2.3) to obtain

$$[\delta(y), x] + 3[\Delta(x, y, y), y] = 0 \quad \text{for all} \quad x, y \in I. \tag{2.4}$$

Let us write in (2.4) yx instead of x. Then we get

$$\begin{aligned} 0 &= [\delta(y), yx] + 3[\Delta(yx, y, y), y] \\ &= y[\delta(y), x] + 3\delta(y)[x, y] + 3y[\Delta(x, y, y), y] \\ &= y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\} + 3\delta(y)[x, y] \end{aligned}$$

which implies that

$$\delta(y)[x, y] = 0 \quad \text{for all} \quad x, y \in I \tag{2.5}$$

on account of (2.4). Since I is a nonzero noncommutative prime ring, it follows from (2.5) and Lemma 2.1 that, for all $y \in I$ with $y \notin Z$, we have $\delta(y) = 0$ since for every fixed $y \in I$, a map $x \mapsto [x, y]$ is a derivation on I.

Now, let $x \in I$ with $x \in Z$ and $y \in I$ with $y \notin Z$. Then $x + y \notin Z$ and $-y \notin Z$. Thus we have

$$0 = \delta(x + y) = \delta(x) + 3\Delta(x, x, y) + 3\Delta(x, y, y)$$

and

$$0 = \delta(x - y) = \delta(x) - 3\Delta(x, x, y) + 3\Delta(x, y, y)$$

which shows that

$$\delta(x) + 3\Delta(x, y, y) = 0. \tag{2.6}$$

Replacing $y \in I\,(y \notin Z)$ by $2y$ in (2.6) and using (2.6), we obtain that $\Delta(x, y, y) = 0$ and so the relation (2.6) gives $\delta(x) = 0$ for all $x \in I$ with $x \in Z$. Therefore we conclude that $\delta(x) = 0$ for all $x \in I$.

On the other hand, since the relation $\delta(x + y) = \delta(x) + \delta(y) + 3\Delta(x, x, y) + 3\Delta(x, y, y)$ is fulfilled for all $x, y \in I$, it follows that

$$\Delta(x, x, y) + \Delta(x, y, y) = 0 \quad \text{for all } \; x, y \in I \tag{2.7}$$

and substituting $y + z$ for y in (2.7) and employing (2.7), we obtain that $2\Delta(x, y, z) = 0 = \Delta(x, y, z)$ for all $x, y, z \in I$.

Let us substitute $rx\,(r \in R)$ for x in the above relation $\Delta(x, y, z) = 0$ for all $x, y, z \in I$. Then we have $\Delta(r, y, z)x = 0$, that is, $\Delta(r, y, z)I = \{0\}$. Since R is prime, we get $\Delta(r, y, z) = 0$ for all $y, z \in I$ and $r \in R$. Also, substituting $ys\,(s \in R)$ for y in this relation, we have $y\Delta(r, s, z) = 0$ and so $I\Delta(r, s, z) = \{0\}$. Again, by primeness of R, we obtain that $\Delta(r, s, z) = 0$ for all $z \in I$ and $r, s \in R$. Furthermore, replacing z by $tz\,(t \in R)$ in the relation $\Delta(r, s, z) = 0$, we get $\Delta(r, s, t)z = 0$, i.e., $\Delta(r, y, t)I = \{0\}$. The primness of R implies that $\Delta(r, s, t) = 0$ for all $r, s, t \in R$ which completes the proof of the theorem. □

We continue with the following result for permuting 3-derivations on semiprime rings.

Theorem 2.3. *Let R be a noncommutative 3-torsion free semiprime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that δ is centralizing on I, where δ is the trace of Δ. Then δ is commuting on I.*

Proof. Assume that

$$[\delta(x), x] \in Z \quad \text{for all } \; x \in I. \tag{2.8}$$

By linearizing (2.8) and again using (2.8), we obtain

$$\begin{aligned} Z \ni\; & [\delta(x), y] + [\delta(y), x] + 3[\Delta(x, x, y), x] + 3[\Delta(x, y, y), x] \\ & + 3[\Delta(x, x, y), y] + 3[\Delta(x, y, y), y] \quad \text{for all } x, y \in I. \end{aligned} \tag{2.9}$$

We substitute $-x$ for x in (2.9) and compare (2.9) with the result to get

$$[\Delta(x, y, y), x] + [\Delta(x, x, y), y] \in Z \quad \text{for all } x, y \in I \tag{2.10}$$

since R is 3-torsion free.

Replacing x by $x + y$ in (2.10) and using (2.10), we have

$$[\delta(y), x] + 3[\Delta(x, y, y), y] \in Z \quad \text{for all } x, y \in I. \tag{2.11}$$

Taking $x = y^2$ in (2.11) and invoking (2.8) show that

$$Z \ni [\delta(y), y^2] + 3[\Delta(y^2, y, y), y] = 8[\delta(y), y]y \quad \text{for all } y \in I \tag{2.12}$$

and commuting with $\delta(y)$ in (2.12) gives

$$8[\delta(y), y]^2 = 0 \quad \text{for all } y \in I. \tag{2.13}$$

On the other hand, substituting x by yx in (2.11), we obtain

$$\begin{aligned} Z \ni\ & [\delta(y), yx] + 3[\Delta(yx, y, y), y] \\ =\ & y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\} \\ & +3\delta(y)[x, y] + 4[\delta(y), y]x \quad \text{for all } x, y \in I. \end{aligned}$$

Hence we have, for all $x, y \in I$,

$$[y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\},\ y] + [3\delta(y)[x, y] + 4[\delta(y), y]x,\ y] = 0$$

and so we get

$$3\delta(y)[[x, y], y] + 7[\delta(y), y][x, y] = 0 \quad \text{for all } x, y \in I \tag{2.14}$$

according to (2.11).

Substituting $\delta(y)x$ for x in (2.14), it follows that

$$\begin{aligned} 0 =\ & \delta(y)\{3\delta(y)[[x, y], y] + 7[\delta(y), y][x, y]\} \\ & +6\delta(y)[\delta(y), y][x, y] + 7[\delta(y), y]^2 x \quad \text{for all } x, y \in I \end{aligned}$$

which, by (2.14), implies that

$$6\delta(y)[\delta(y), y][x, y] + 7[\delta(y), y]^2 x = 0 \quad \text{for all } x, y \in I. \tag{2.15}$$

Letting $x = [\delta(y), y]$ in (2.15), we arrive at $7[\delta(y), y]^3 = 0$ and so we get

$$7[\delta(y), y]^2 R 7[\delta(y), y]^2 = 0.$$

Since R is semiprime, we deduce that

$$7[\delta(y), y]^2 = 0 \quad \text{for all } y \in I. \tag{2.16}$$

Hence, the relations (2.13) and (2.16) yield $[\delta(y), y]^2 = 0$ for all $y \in I$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $[\delta(y), y] = 0$ for all $y \in I$. This completes the proof of the theorem. □

The following result is an analogue of Posner's theorem [3, Theorem 2].

Theorem 2.4. *Let R be a $3!$-torsion free prime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a nonzero permuting 3-derivation $\Delta : R \times R \times R \to R$ such that δ is centralizing on I, where δ be the trace of Δ. Then R is commutative.*

Proof. Suppose that R is noncommutative. Then it follows from Theorem 2.3 that δ is commuting on I. Hence Lemma 2.2 gives $\Delta = 0$ which guarantees the conclusion of the theorem. □

References

[1] M. Brešar, *Commuting maps: a survey*, Taiwanese J. Math. **8** (2004), no. 3, 361–397.
[2] J. Mayne, *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984), no. 1, 122–126.
[3] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
[4] J. Vukman, *Symmetric bi-derivations on prime and semi-prime rings*, Aequationes Math. **38** (1989), no. 2-3, 245–254.
[5] ———, *Two results concerning symmetric bi-derivations on prime rings*, Aequationes Math. **40** (1990), no. 2-3, 181–189.

Yong-Soo Jung
Department of Mathematics
Sun Moon University
Chungnam 336-708, Korea
E-mail address: ysjung@sunmoon.ac.kr

Kyoo-Hong Park
Department of Mathematics Education
Seowon University
Chungbuk 361-742, Korea
E-mail address: parkkh@seowon.ac.kr