Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 795–806

# $F$-TRACELESS COMPONENT OF THE CONFORMAL CURVATURE TENSOR ON KÄHLER MANIFOLD

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Abstract. We investigate $F$-traceless component of the conformal curvature tensor defined by (3.6) in Kähler manifolds of dimension $\geq 4$, and show that the $F$-traceless component is invariant under concircular change. In particular, we determine Kähler manifolds with parallel $F$-traceless component and improve some theorems, provided in the previous paper ([2]), which are concerned with the traceless component of the conformal curvature tensor and the spectrum of the Laplacian acting on $p$ $(0 \leq p \leq 2)$-forms on the manifold by using the $F$-traceless component.

## 1. Introduction

Let $E$ be a real $n$-dimensional vector space and let $E^p_q$ the tensor space of tensors of type $(p,q)$. A fixed basis of $E$ determines a unique basis of $E^p_q$. The components of any tensor $A$ with respect to this basis will be denoted by $A^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q}$. A tensor $A \in E^p_q$ is said to be *traceless* ([2, 5, 6, 7]) if

$$A^{\cdots a_{i-1} t b_{i+1} \cdots}_{\cdots b_{j-1} t b_{j+1} \cdots} = 0, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q.$$

The trace decomposition problem is the problem of existence and uniqueness of a decomposition of a tensor $A \in E^p_q$ in which on term is traceless and the remaining terms are linear combinations of the Kronecker $\delta$-tensor with traceless coefficients ([5]).

Moreover, for an arbitrary tensor $F$ of type (1,1) which is traceless, namely $F^t_t = 0$, if a traceless tensor $A \in E^p_q$ satisfies

$$F^s_t A^{\cdots a_{i-1} t b_{i+1} \cdots}_{\cdots b_{j-1} s b_{j+1} \cdots} = 0, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q,$$

then $A$ is said to be *$F$-traceless* ([6, 7]).

Received March 15, 2007.
2000 *Mathematics Subject Classification.* 53C.
*Key words and phrases.* Kähler manifold, conformal curvature tensor, traceless decomposition, $F$-traceless decomposition, constant holomorphic sectional curvature, spectrum.
* This work was supported by the 2006 Inje University Grant.
** This work was supported by the Korea Research Foundation Grant. (R14-2002-003-01002-0).

Recently, in his paper [7], Mikes generalized the form of the trace decomposition theorem, formulated by Krupka ([5]), and derived the so-called *F-decomposition formula* of a tensor $A \in E^p_q$ for a traceless tensor $F$ of type (1,1) satisfying the condition $F^a_t F^t_b = e\delta^a_b, e = \pm 1$. In particular, Lakoma and Jukl ([6]) gave an explicit $F$-decomposition formula of a tensor $A$ of type $(1,3)$ which satisfies

$$(1.1)\qquad \begin{aligned} &A^a_{dcb} + A^a_{dbc} = 0, \quad A^a_{dcb} + A^a_{cbd} + A^a_{bdc} = 0,\\ &A^t_{tcb} = 0, \quad F_c{}^t F_b{}^s A^a_{dts} = A^a_{dcb}, \end{aligned}$$

where $F$ is a traceless tensor of type (1,1) with $e = -1$. By using the fact that the Riemannian curvature tensor of Kähler structure $(F, g)$ satisfies the condition (1.1), they proved (see also [7]) that the $F$-traceless component of the Riemannian curvature tensor in a Kähler manifold is the $H$-projective curvature tensor(for definition, see [14, p.262]).

In this paper we investigate $F$-traceless component of the conformal curvature tensor (for definition, see (2.1)) in a Kähler manifold by means of Mikes' analysis ([7]) and show that the $F$-traceless component is invariant under concircular change. In particular, as applications of the $F$-traceless component, we determine Kähler manifolds with vanishing $F$-traceless component.

## 2. Preliminaries

Let $(M, F, g)$ be a Kähler manifold of real dimension $n$ with almost complex structure $F$ and Kähler metric $g$. Let $M$ be covered by a system of coordinate neighborhoods $\{U; x^i\}$, where here and in the sequel the indices $a, b, c, d, e, h, i, j, s, t$ run over the range $\{1, 2, \dots, n\}$ and we use the Einstein convention with respect to this system of indices. We denote by $g_{ba}$, $\nabla_a$, $R_{dcb}{}^a$, $R_{ba}$, $r$ and $F^a_b$ local components of $g$, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor, the scalar curvature and $F$ of $M$, respectively. Then the following relations hold on $M$;

$$(2.1)\qquad \begin{aligned} &R_{dct}{}^a F_b{}^t = R_{dcb}{}^t F_t{}^a, \quad R_{tcb}{}^a F_d{}^t = -R_{dtb}{}^a F_c{}^t,\\ &F_b{}^t R_t{}^a = R_b{}^t F_t{}^a, \quad F_c{}^t R_{tb} = -R_{ct} F_b{}^t, \quad F_c{}^t F_b{}^s R_{ts} = R_{cb},\\ &\nabla_t R_{dcb}{}^t = \nabla_d R_{cb} - \nabla_c R_{db}, \quad \nabla_b r = 2\nabla_t R_b{}^t. \end{aligned}$$

Thus the tensor $S_{cb} := F_c{}^t R_{tb}$ satisfies

$$(2.2)\qquad \begin{aligned} &S_{cb} = -\frac{1}{2}F^{ts}R_{cbts} = F^{ts}R_{tcbs} = -S_{bc}, \quad F_c{}^t S_{tb} = -S_{ct}F_b{}^t = -R_{cb}\\ &\nabla_a S_{cb} = F_c{}^t \nabla_a R_{tb}, \ \nabla_t S_c{}^t = \frac{1}{2}F_c{}^t \nabla_t r, \ F_d{}^t \nabla_t S_{cb} = F_d{}^t F_c{}^s \nabla_t R_{sb} \end{aligned}$$

(cf. [8, 14]).

On the other hand, in the previous paper [2], the present authors investigated a curvature-like tensor field $\overset{*}{\to} C$ defined on $M$ of which local components are given by

(2.3)

$$\begin{aligned}\overset{*}{\to} C_{dcb}{}^a = R_{dcb}{}^a &+ \frac{1}{n}(R_d{}^a g_{cb} - R_c{}^a g_{db} + \delta_d^a R_{cb} - \delta_c^a R_{db} \\ &- S_d{}^a F_{cb} + S_c{}^a F_{db} - F_d{}^a S_{cb} + F_c{}^a S_{db} + 2S_{dc}F_b{}^a + 2F_{dc}S_b{}^a) \\ &+ \frac{(n+4)r}{n^2(n+2)}(F_d{}^a F_{cb} - F_c{}^a F_{db} - 2F_{dc}F_b{}^a) \\ &- \frac{(n^2+5n+12)r}{n^2(n-1)(n+2)}(\delta_d^a g_{cb} - \delta_c^a g_{db}) - \frac{2(n-4)}{n(n-1)}(\delta_d^a R_{cb} - \delta_c^a R_{db}),\end{aligned}$$

where $S_{cb} = F_c{}^t R_{tb} = -S_{bc}$ and $S_b{}^a = S_{bt}g^{ta}$. In fact $\overset{*}{\to} C$ is the traceless component in the sense of Krupka ([5]) of the conformal curvature tensor $C$, provided in [3], whose local components are given by

$$\begin{aligned}C_{dcb}{}^a = R_{dcb}{}^a &+ \frac{1}{n}(R_d{}^a g_{cb} - R_c{}^a g_{db} + \delta_d^a R_{cb} - \delta_c^a R_{db} \\ &- S_d{}^a F_{cb} + S_c{}^a F_{db} - F_d{}^a S_{cb} + F_c{}^a S_{db} + 2S_{dc}F_b{}^a + 2F_{dc}S_b{}^a) \\ &+ \frac{(n+4)r}{n^2(n+2)}(F_d{}^a F_{cb} - F_c{}^a F_{db} - 2F_{dc}F_b{}^a) \\ &- \frac{(3n+4)r}{n^2(n+2)}(\delta_d^a g_{cb} - \delta_c^a g_{db}).\end{aligned} \tag{2.4}$$

In fact $C$ and $\overset{*}{\to} C$ are related as follows;

$$C_{dcb}{}^a = \overset{*}{\to} C_{dcb}{}^a + \frac{2(n-4)}{n(n-1)}\{\delta_d^a(R_{cb} - \frac{r}{n}g_{cb}) - \delta_c^a(R_{db} - \frac{r}{n}g_{db})\}. \tag{2.5}$$

It is known ([3]) that the conformal curvature tensor $C$ is invariant under conformal change, provided $n \geq 4$ and a Kähler manifold with $C = 0$ is of constant holomorphic sectional curvature, provided $n \geq 6$. Moreover, $C$ satisfies

$$\begin{aligned}&C_{dcb}{}^a + C_{cdb}{}^a = 0, \quad C_{dcb}{}^a + C_{cbd}{}^a + C_{bdc}{}^a = 0, \\ &C_{dct}{}^t = 0, \quad F_d{}^t F_c{}^s C_{tsb}{}^a = C_{dcb}{}^a.\end{aligned} \tag{2.6}$$

On the other hand, the traceless component $\overset{*}{\to} C$ of $C$ is invariant under concircular change, provided $n \geq 4$, and moreover, a Kähler manifold ($n \geq 4$) is of constant holomorphic sectional curvature if and only if the manifold is Einstein and $\overset{*}{\to} C = 0$ (for details, see [2]).

## 3. F-decomposition formula of the conformal curvature tensor

In this section, we provide the following theorem as a $F$-decomposition formula of a tensor $A$ of type $(1,3)$ for almost complex structure $F$ which satisfies

$$\begin{aligned}&A_{dcb}{}^a + A_{cdb}{}^a = 0, \quad A_{dcb}{}^a + A_{cbd}{}^a + A_{bdc}{}^a = 0, \\ &A_{dct}{}^t = 0, \quad F_d{}^t F_c{}^s A_{tsb}{}^a = A_{dcb}{}^a.\end{aligned} \tag{3.1}$$

It is clear that the conditions (1.1) and (3.1) about the tensor $A$ are slightly different, but the proof of the theorem is omitted since we can derive the formula (3.2) by quite similar method as shown in [6]

**Lemma 3.1.** *Let $A$ be a tensor of type $(1,3)$ with properties (3.1) and let $F$ be an almost complex structure. If $n \geq 6$, then there exists a unique $F$-decomposition of $A$ of the form*

$$A_{dcb}{}^a = {}^F A_{dcb}{}^a + \delta_d^a C_{cb} + \delta_c^a D_{db} + \delta_b^a E_{dc} + F_d^a G_{cb} + F_c^a H_{db} + F_b^a I_{dc}, \tag{3.2}$$

*where the tensors $C_{cb}, D_{cb}, E_{cb}, G_{cb}, H_{cb}, I_{cb}$ have the following forms*

$$C_{cb} = \frac{1}{n} A_{cb}, \quad D_{cb} = -\frac{1}{n} A_{cb} = -C_{cb}, \quad E_{cb} = 0, \quad G_{cb} = \frac{1}{n} F_b{}^t A_{ct} = -G_{bc},$$

$$H_{cb} = -\frac{1}{n} F_b{}^t A_{ct} = -G_{cb}, \quad I_{cb} = \frac{2}{n} F_b{}^t A_{ct} = 2G_{cb},$$

$${}^F A_{dcb}{}^a = A_{dcb}{}^a - \frac{1}{n}(\delta_d^a A_{cb} - \delta_c^a A_{db} + F_d^a F_b{}^t A_{ct} - F_c^a F_b{}^t A_{dt} + 2F_b^a F_c{}^t A_{dt})$$

*and $A_{cb} := A_{tcb}{}^t$.*

Here we notice that the tensor ${}^F A_{dcb}{}^a$ given in (3.2) is $F$-traceless and $A_{cb} = A_{bc}$ which is a direct consequence of (3.1).

As already shown in (2.6), the conformal curvature tensor $C$ in a Kähler manifold satisfies the conditions (3.1) and consequently Lemma 3.1 yields the following $F$-decomposition formula of $C$ ;

(3.3)

$$C_{dcb}{}^a = {}^F C_{dcb}{}^a + \frac{2(n-4)}{n^2}\{\delta_d^a (R_{cb} - \frac{r}{n} g_{cb}) - \delta_c^a (R_{db} - \frac{r}{n} g_{db})$$
$$+ F_d{}^a F_b{}^t (R_{ct} - \frac{r}{n} g_{ct}) - F_c{}^a F_b{}^t (R_{dt} - \frac{r}{n} g_{dt}) + 2F_b{}^a F_c{}^t (R_{dt} - \frac{r}{n} g_{dt})\}$$

because of

$$C_{cb} = \frac{1}{n} C_{tcb}{}^t = \frac{2(n-4)}{n^2}(R_{cb} - \frac{r}{n} g_{cb}), \quad G_{cb} = \frac{2(n-4)}{n^2} F_b{}^t (R_{ct} - \frac{r}{n} g_{ct}).$$

On the other hand, using (2.2) and (2.4), we can easily verify that $C_{dcba} := C_{dcb}{}^t g_{ta}$ is skew-symmetric with respect to the indices $b$ and $a$. Hence $\overset{*}{\to} C_{dcba} := \overset{*}{\to} C_{dcb}{}^t g_{ta}$ and ${}^F C_{dcba} := {}^F C_{dcb}{}^t g_{ta}$ satisfy the following equalities ;

$$\overset{*}{\to} C_{dcba} + \overset{*}{\to} C_{dcab} = -\frac{2(n-4)}{n(n-1)}(g_{da} R_{cb} + g_{db} R_{ca} - g_{ca} R_{db} - g_{cb} R_{da}), \tag{3.4}$$

$$\begin{aligned} {}^F C_{dcba} + {}^F C_{dcab} = &-\frac{2(n-4)}{n^2}\{g_{da} R_{cb} + g_{db} R_{ca} - g_{ca} R_{db} - g_{cb} R_{da} \\ &+ F_{da} F_b{}^t (R_{ct} - \frac{r}{n} g_{ct}) + F_{db} F_a{}^t (R_{ct} - \frac{r}{n} g_{ct}) \\ &- F_{ca} F_b{}^t (R_{dt} - \frac{r}{n} g_{dt}) - F_{cb} F_a{}^t (R_{dt} - \frac{r}{n} g_{dt})\}. \end{aligned} \tag{3.5}$$

By means of (2.5) and (3.3)-(3.5) we can easily prove

**Theorem 3.2.** *On a Kähler manifold of dimension* $\geq 6$, $\overset{*}{\to} C = {}^F C$ *if and only if the manifold is Einstein.*

*Proof.* It is clear from (2.5) and (3.3) that $\overset{*}{\to} C = {}^F C$ on an Einstein Kähler manifold. Conversely, if $\overset{*}{\to} C = {}^F C$ on a Kähler manifold, then (3.4) and (3.5) yield

$$\frac{n-4}{n^2(n-1)}(nR_{cb} - rg_{cb}) = 0,$$

and consequently the manifold is Einstein, provided $n \geq 6$. □

By means of (2.2), (2.4) and (3.3) we can give the local components of the $F$-traceless component ${}^F C$ as follows ;

(3.6)

$$\begin{aligned}
&{}^F C_{dcb}{}^a \\
&= R_{dcb}{}^a - \frac{r}{n(n+2)}(\delta_d^a g_{cb} - g_{db}\delta_c^a + F_d{}^a F_{cb} - F_c{}^a F_{db} - 2F_{dc}F_b{}^a) \\
&+ \frac{1}{n}\{(R_d{}^a - \frac{r}{n}\delta_d^a)g_{cb} + \delta_d^a(R_{cb} - \frac{r}{n}g_{cb}) - (R_c{}^a - \frac{r}{n}\delta_c^a)g_{db} - \delta_c^a(R_{db} - \frac{r}{n}g_{db}) \\
&- F_d{}^t(R_t{}^a - \frac{r}{n}\delta_t^a)F_{cb} - F_d{}^a F_c{}^t(R_{bt} - \frac{r}{n}g_{bt}) + F_c{}^t(R_t{}^a - \frac{r}{n}\delta_t^a)F_{db} \\
&+ F_c{}^a F_d{}^t(R_{bt} - \frac{r}{n}g_{bt}) + 2F_d{}^t(R_{tc} - \frac{r}{n}g_{tc})F_b{}^a + 2F_{dc}F_b{}^t(R_t{}^a - \frac{r}{n}\delta_t^a)\} \\
&- \frac{2(n-4)}{n^2}\{\delta_d^a(R_{cb} - \frac{r}{n}g_{cb}) - \delta_c^a(R_{db} - \frac{r}{n}g_{db}) + F_d{}^a F_b{}^t(R_{ct} - \frac{r}{n}g_{ct}) \\
&- F_c{}^a F_b{}^t(R_{dt} - \frac{r}{n}g_{dt}) + 2F_b{}^a F_c{}^t(R_{dt} - \frac{r}{n}g_{dt})\}.
\end{aligned}$$

It is clear from (3.6) that

$$\tag{3.7} \begin{aligned} \|{}^F C\|^2 = \|R\|^2 &+ \frac{8(3n^4 - 29n^3 + 87n^2 - 104n + 48)}{n^4}\|R_1\|^2 \\ &- \frac{8(3n^5 - 22n^4 + 29n^3 + 70n^2 - 160n + 96)}{n^5(n+2)}r^2. \end{aligned}$$

From (3.6), we have

**Theorem 3.3.** *A Kähler manifold of dimension* $\geq 4$ *is of constant holomorphic sectional curvature if and only if the manifold is Einstein and* ${}^F C = 0$ *everywhere.*

*Proof.* If a Kähler manifold of dimension $\geq 4$ is of constant holomorphic sectional curvature, then the local components of the Riemannian curvature tensor are given by

$$R_{dcb}{}^a = \frac{r}{n(n+2)}(\delta_d^a g_{cb} - g_{db}\delta_c^a + F_d{}^a F_{cb} - F_c{}^a F_{db} - 2F_{dc}F_b{}^a),$$

which and (3.6) imply that the manifold is Einstein and ${}^F C = 0$. The converse is trivial. □

From (3.5) and (3.6), we have immediately

**Theorem 3.4.** *On a Kähler manifold of dimension* $\geq 6$, *if* ${}^FC = 0$ *everywhere, then the manifold is of constant holomorphic sectional curvature.*

Finally we give the next theorem as a geometric meaning of the $F$-traceless component ${}^FC$ of the conformal curvature tensor $C$.

**Theorem 3.5.** *The $F$-traceless component ${}^FC$ of the conformal curvature tensor $C$ on a Kähler manifold of dimension $\geq 4$ is invariant under concircular change.*

*Proof.* We consider a conformal change of the Riemannian metrics $g_{ba}$ and $'g_{ba}$ as follows :

$$'g_{ba} = e^{2\rho} g_{ba}$$

for a smooth function $\rho$. It is well known (cf. [10]) that the curvature tensors $R$ and $'R$ corresponding to $g$ and $'g$ are related by

$$'R_{dcb}{}^a = R_{dcb}{}^a + \rho_{db}\delta_c^a - \rho_{cb}\delta_d^a + g_{db}\rho_c^a - g_{cb}\rho_d^a,$$

where $\rho_a$ denotes the local components of the gradient vector of $\rho$ and

$$\rho_{ba} = \nabla_b\rho_a - \rho_b\rho_a + \frac{1}{2}\rho_t\rho^t g_{ba}, \quad \rho^a = \rho_t g^{at} \quad \rho_b^a = \rho_{bt}g^{ta}.$$

Hence we have

$$'R_{ba} = R_{ba} - (n-2)\rho_{ba} - \rho_t^t g_{ba}, \quad 'r\ e^{2\rho} = r - 2(n-1)\rho_t^t,$$

where $'R_{ba}$ and $'s$ denote the Ricci tensor and the scalar curvature corresponding to $'g$, respectively, and $\rho_t^t = \rho_{ba}g^{ba}$.

On the other hand, it follows from (3.3) that the $F$-traceless component of the conformal curvature tensor $'C_{dcb}{}^a$ corresponding to $'g$ is given by

$$\begin{aligned}
&'C_{dcb}{}^a \\
&= {}'^F C_{dcb}{}^a + \frac{2(n-4)}{n^2}\{\delta_d^a('R_{cb} - \frac{'r}{n}'g_{cb}) - \delta_c^a('R_{db} - \frac{'r}{n}'g_{db}) \\
&\quad + F_d{}^aF_b{}^t('R_{ct} - \frac{'r}{n}'g_{ct}) - F_c{}^aF_b{}^t('R_{dt} - \frac{'r}{n}'g_{dt}) + 2F_b{}^aF_c{}^t('R_{dt} - \frac{'r}{n}'g_{dt})\},
\end{aligned}$$

from which, using the equality

$$'R_{cb} - \frac{'r}{n}'g_{cb} = R_{cb} - \frac{r}{n}g_{cb} - (n-2)(\rho_{cb} - \frac{1}{n}\rho_t^t g_{cb}),$$

and taking account of the fact ([3]) that $C_{dcb}{}^a$ is invariant under the conformal change, provided $n \geq 4$, we can easily obtain

$$\begin{aligned}
&{}^FC_{dcb}{}^a \\
&= {}'^F C_{dcb}{}^a - \frac{2(n-2)(n-4)}{n^2}\{\delta_d^a(\rho_{cb} - \frac{1}{n}\rho_t^t g_{cb}) - \delta_c^a(\rho_{db} - \frac{1}{n}\rho_t^t g_{db}) \\
&\quad + F_d{}^aF_b{}^s(\rho_{cs} - \frac{1}{n}\rho_t^t g_{cs}) - F_c{}^aF_b{}^s(\rho_{ds} - \frac{1}{n}\rho_t^t g_{ds}) + 2F_b{}^aF_c{}^s(\rho_{ds} - \frac{1}{n}\rho_t^t g_{ds})\}.
\end{aligned}$$

Hence, if the conformal change is concircular, that is, if

$$\rho_{ba} = \frac{1}{n}\rho_t^t g_{ba},$$

then ${}^F C_{dcb}{}^a =' {}^F C_{dcb}{}^a$, which means that ${}^F C_{dcb}{}^a$ is invariant under the concircular change. □

## 4. Kähler manifolds with parallel $F$-traceless component

It is well known(cf. [14], p.72) that the differential form $S = \frac{1}{2}S_{cb}dx^c \wedge dx^b$ is closed. Thus we have

$$F_t{}^e \nabla_e S_{sa} = \nabla_a R_{st} - \nabla_s R_{at},$$

from which, transvecting with $F_c{}^t F_b{}^s$,

$$\nabla_c R_{ba} = F_c{}^t F_b{}^s (\nabla_a R_{st} - \nabla_s R_{at}), \quad F_c{}^t F_b{}^s \nabla_t R_{sa} = \nabla_a R_{cb} - \nabla_b R_{ca}. \tag{4.1}$$

Differentiating (3.6) covariantly and using (2.1) and (2.2), we can easily obtain

(4.2)

$$\begin{aligned}
\nabla_t {}^F C_{dcb}{}^t = {} & \frac{n^3 - 16n - 32}{2n^3(n+2)}\{(\nabla_d r)g_{cb} - (\nabla_c r)g_{db}\} \\
& - \frac{2(n-4)}{n^2}\{F_d{}^t F_b{}^s(\nabla_t R_{cs} - \frac{1}{n}g_{cs}\nabla_t r) \\
& - F_c{}^t F_b{}^s(\nabla_t R_{ds} - \frac{1}{n}g_{ds}\nabla_t r) + 2F_b{}^t F_c{}^s(\nabla_t R_{ds} - \frac{1}{n}g_{ds}\nabla_t r)\} \\
& - \frac{(n^2-8)}{2n^2(n+2)}(\nabla_t r)(F_{cb}F_d{}^t - F_{db}F_c{}^t - 2F_b{}^t F_{dc}) \\
& + \frac{n^3 - 5n^2 + 16n - 8}{n^2(n-1)}(\nabla_d R_{cb} - \nabla_c R_{db}).
\end{aligned}$$

Thus we have

**Theorem 4.1.** *On a Kähler manifold the $F$-traceless component of the conformal curvature tensor is parallel if and only if the manifold is locally symmetric.*

*Proof.* We assume $\nabla_t {}^F C = 0$. Then it follows from (4.2) with $\nabla_t {}^F C = 0$ that

$$\begin{aligned}
0 = {} & \frac{n^3 - 16n - 32}{2n^3(n+2)}\{(\nabla_d r)g_{cb} - (\nabla_c r)g_{db}\} \\
& - \frac{2(n-4)}{n^2}\{F_d{}^t F_b{}^s(\nabla_t R_{cs} - \frac{1}{n}g_{cs}\nabla_t r) \\
& - F_c{}^t F_b{}^s(\nabla_t R_{ds} - \frac{1}{n}g_{ds}\nabla_t r) + 2F_b{}^t F_c{}^s(\nabla_t R_{ds} - \frac{1}{n}g_{ds}\nabla_t r)\} \\
& - \frac{(n^2-8)}{2n^2(n+2)}(\nabla_t r)(F_{cb}F_d{}^t - F_{db}F_c{}^t - 2F_b{}^t F_{dc}) \\
& + \frac{n^3 - 5n^2 + 16n - 8}{n^2(n-1)}(\nabla_d R_{cb} - \nabla_c R_{db}).
\end{aligned} \tag{4.3}$$

Transvecting (4.3) with $g^{cb}$ and using (2.1), we can easily have

$$\frac{2n^5-7n^4-2n^3+27n^2+16n-24}{2n^3(n+2)(n-1)}\nabla_d r=0,$$

which yields that the scalar curvature $r$ is constant, and consequently (4.3) reduces to

$$\begin{aligned}&-\frac{2(n-4)}{n^2}(F_d{}^tF_b{}^s\nabla_tR_{cs}-F_c{}^tF_b{}^s\nabla_tR_{ds}+2F_b{}^tF_c{}^s\nabla_tR_{ds})\\&+\frac{n^3-5n^2+16n-8}{n^2(n-1)}(\nabla_dR_{cb}-\nabla_cR_{db})=0.\end{aligned}\tag{4.4}$$

Substituting the second equation of (4.1) into (4.4), we have

$$\frac{n^3-7n^2+26n-16}{n^2(n-1)}(\nabla_dR_{cb}-\nabla_cR_{db})=0,\text{i.e.},\nabla_dR_{cb}-\nabla_cR_{db}=0,$$

which together with the first equation of (4.1) implies that the Ricci tensor is parallel, and consequently the manifold is locally symmetric. The converse is trivial. □

## 5. Spectrum of the Laplacian and $F$-traceless component of the conformal curvature tensor

Let $M$ be a compact Kähler manifold of real dimension $n$ and denote by $\Delta$ the Laplacian acting on $p$-forms on $M$, $0\le p\le n$. Then we have the spectrum for each $p$ :

$$\mathrm{Spec}^p(M,g)=\{0\le\lambda_{0,p}\le\lambda_{1,p}\le\lambda_{2,p}\le\cdots\uparrow+\infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many as times as its multiplicity indicates. Furthermore, the Minakshisundaram-Pleijel-Gaffney's formula for $\mathrm{Spec}^p(M,g)$ is given by

$$\sum_{\alpha=0}^{\infty}\exp(-\lambda_{\alpha,p}t)\sim(4\pi t)^{\frac{-n}{2}}\sum_{\alpha=0}^{\infty}a_{\alpha,p}t^{\alpha}\quad\text{as } t\to0^+,$$

where the constants $A_{\alpha,p}$ are spectral invariants. In particular, for $p=0$, we have

$$a_{0,0}=\int_M dM=\mathrm{Vol}(M,g),\tag{5.1}$$

$$a_{1,0}=\frac{1}{6}\int_M r\,dM,\tag{5.2}$$

$$a_{2,0}=\frac{1}{360}\int_M\{2\|R\|^2-2\|R_1\|^2+5r^2\}dM,\tag{5.3}$$

where $dM$ denotes the natural volume element of $(M,g)$ (cf. [1]). For $p=1$, we have

$$a_{0,1} = n\mathrm{Vol}(M,g), \tag{5.4}$$

$$a_{1,1} = \frac{n-6}{6}\int_M r\,dM, \tag{5.5}$$

$$a_{2,1} = \frac{1}{360}\int_M \{2(n-15)\|R\|^2 - 2(n-90)\|R_1\|^2 + 5(n-12)r^2\}dM \tag{5.6}$$

(cf. [10]). For $p=2$, we have

$$a_{0,2} = \frac{n(n-1)}{2}\mathrm{Vol}(M,g), \tag{5.7}$$

$$a_{1,2} = \frac{n^2-13n+24}{12}\int_M r\,dM, \tag{5.8}$$

$$\begin{aligned} a_{2,2} = \frac{1}{720}\int_M &\{2(n^2-31n+240)\|R\|^2 - 2(n^2-181n+1080)\|R_1\|^2 \\ &+ 5(n^2-25n+120)r^2\}dM \end{aligned} \tag{5.9}$$

(cf. [8, 11, 12]).

By using those spectral invariants and some properties concerning traceless component of the conformal curvature tensor, in the previous paper ([2]), we have investigated certain geometric properties of compact Kähler manifolds and provided the following lemma particulary.

**Lemma 5.1** ([2]). *Let $M$ and $M'$ be compact Kähler manifolds. Assume that* $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ *and* $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$. *Then* $\dim M = \dim M' = n$, *and*

(*a*) *for $4 \le n \le 14$ or $n \ge 24$, $M$ is of constant holomorphic sectional curvature if and only if $M'$ is, and $r' =$ constant $= r$.*

(*b*) *for $n \ge 4$, $M$ is Einstein if and only if $M'$ is. Moreover, in this case $r' = r$, provided $n = 4$ or $n \ge 10$.*

From now on we improve the above lemma (see Theorem 5.3) by using Theorem 3.3 and Theorem 3.4 concerning the $F$-traceless component of the conformal curvature tensor. For this purpose we first recall the following lemma due to Tanno ([11]).

**Lemma 5.2** ([11]). *Let $(M,g)$ and $(M',g')$ be compact orientable Riemannian manifolds with* $\mathrm{Vol}(M,g) = \mathrm{Vol}(M',g')$ *and $\int_M r\,dM = \int_{M'} r'\,dM'$. If $r' =$ constant, then $\int_M r^2\,dM \ge \int_{M'} r'^2\,dM'$ with equality if and only if $r =$ constant $= r'$.*

**Theorem 5.3.** *Let $M$ and $M'$ be compact Kähler manifolds. Assume that* $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ *and* $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$. *Then* $\dim M = \dim M' = n$, *and* (*a*) *for* $n \geq 4$, $M$ *is of constant holomorphic sectional curvature if and only if* $M'$ *is, and* $r' = \text{constant} = r$.

(*b*) *for* $n \geq 4$, $M$ *is Einstein if and only if* $M'$ *is. Moreover, in this case* $r' = r$, *provided* $n = 4$ *or* $n \geq 10$.

*Proof.* Our assumption $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ yields $a_{0,1} = a'_{0,1}$ and consequently it follows from (5.4) that $\mathrm{Vol}(M) = \mathrm{Vol}(M')$. Since $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$, $a_{1,2} = a'_{1,2}$ yields $\int_M r\, dM = \int_{M'} r'\, dM'$. Summing up, we have

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \quad \int_M r\, dM = \int_{M'} r'\, dM'. \tag{5.10}$$

Moreover, the assumptions $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ and $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$ give $a_{2,1} = a'_{2,1}$ and $a_{2,2} = a'_{2,2}$, from which together with (5.6) and (5.9), we have

$$\begin{aligned} &\int_M \{(5n^2 - 51n - 360)\|R\|^2 + (13n^2 - 147n + 360)r^2\}dM \\ &= \int_{M'} \{(5n^2 - 51n - 360)\|R'\|^2 + (13n^2 - 147n + 360){r'}^2\}dM, \end{aligned} \tag{5.11}$$

$$\begin{aligned} &\int_M \{2(5n + 24)\|R_1\|^2 + (n - 24)r^2\}dM \\ &= \int_{M'} 2(5n + 24)\|R'_1\|^2 + (n - 24){r'}^2\}dM'. \end{aligned} \tag{5.12}$$

(*a*) It follows from (3.7) and (5.11) that

$$\begin{aligned} &\int_M \{(5n^2 - 51n - 360)\|{}^F C\|^2 \\ &\quad - \frac{8(n-15)(5n+24)(3n^4 - 29n^3 + 87n^2 - 104n + 48)}{n^4}\|R_1\|^2 + c_{1,2} r^2\}dM \\ &= \int_{M'} \{(5n^2 - 51n - 360)\|{}^F C'\|^2 \\ &\quad - \frac{8(n-15)(5n+24)(3n^4 - 29n^3 + 87n^2 - 104n + 48)}{n^4}\|R'_1\|^2 + c_{1,2}{r'}^2\}dM,' \end{aligned}$$

where

$$c_{1,2} = \frac{13n^8 - n^7 - 2038n^6 + 2216n^5 + 54328n^4 - 118480n^3 - 132480n^2 + 421632n - 276480}{n^5(n+2)}.$$

Taking account of (5.12), the above equation reduces to

$$\begin{aligned} &(5n + 24)(n - 15)\{\int_M \|{}^F C\|^2 dM - \int_{M'} \|{}^F C'\|^2 dM'\} \\ &+ d_{1,2}(\int_M r^2 dM - \int_{M'} {r'}^2 dM') = 0, \end{aligned} \tag{5.13}$$

where

$$d_{1,2} = \frac{25n^8 - 561n^7 + 5986n^6 - 35148n^5 + 84528n^4 + 7664n^3 - 377856n^2 + 559872n - 276480}{n^5(n+2)},$$

which is positive for $n \geq 10$. On the other hand, since $\|Q\|^2 = \|R_1\|^2 - \frac{r^2}{n}$, (5.12) reduces to

$$\begin{aligned} \text{(5.14)} \quad & \int_M \{2(5n+24)\|Q\|^2 dM - \int_{M'} 2(5n+24)\|Q'\|^2 dM' \\ & + \frac{(n-6)(n-8)}{n}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0, \end{aligned}$$

from which together with Lemma 5.2, (5.13) and (5.14), it follows that, for $n \geq 16$, $M$ is of constant holomorphic sectional curvature if and only if $M'$ is and, moreover, $r' = \text{constant} = r$. Combining these results and (a) in Lemma 5.1, we complete the proof of our assertion.

(*b*) is easily obtained from Lemma 5.2 and (5.14). □

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