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# BOUNDEDNESS AND CONTINUITY OF SOLUTIONS FOR STOCHASTIC DIFFERENTIAL INCLUSIONS ON INFINITE DIMENSIONAL SPACE

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Abstract. For the stochastic differential inclusion on infinite dimensional space of the form $dX_t \in \sigma(X_t)dW_t + b(X_t)dt$, where $\sigma, b$ are set-valued maps, $W$ is an infinite dimensional Hilbert space valued $Q$-Wiener process, we prove the boundedness and continuity of solutions under the assumption that $\sigma$ and $b$ are closed convex set-valued satisfying the Lipschitz property using approximation.

## 1. Introduction

Let $H$ and $U$ be two separable Hilbert spaces and denote by $L = L(U, H)$ the set of all linear bounded operators from $U$ into $H$. The set $L$ is a linear space and, equipped with the operator norm, becomes a Banach space. However if both spaces are infinite dimensional, then $L$ is not a separable space. Let $Q$ be a symmetric nonnegative operator in $L(U) = L(U, U)$ and $W(t), t \geq 0$, be a $U$-valued $Q$-Wiener process. Let $U_0 = Q^{1/2}U$ and $L_2^0 = L_2(U_0, H)$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathfrak{F}_t)_{t\geq 0}$ of sub $\sigma$-fields of $\mathfrak{F}$ each containing all $P$-null sets. We consider the following stochastic differential inclusion (1.1) on infinite dimensional Hilbert space $H$.

$$dX_t \in \sigma(X_t)dW_t + b(X_t)dt, \tag{1.1}$$

where $\sigma : H \to \mathcal{P}(L_2^0)$, $b : H \to \mathcal{P}(H)$ are set-valued maps. For finite dimensional case, the study of the existence and properties of solution for these stochastic differential inclusions have been developed by many authors ([1], [2], [3], [4]). Furthermore the results for the viable solutions have been made ([2], [5], [6]). Yun and Shigekawa ([8]) proved the existence of solution for the stochastic differential inclusion (1.1) on finite dimensional space under the condition that $\sigma$ and $b$ satisfy the Lipschitz condition.

In this paper, we prove the boundedness and continuity of solutions for (1.1).

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## 2. Preliminaries

We prepare the definition of solution for stochastic differential inclusion and some results for the stochastic differential equation on infinite dimensional Hilbert space. We consider two Hilbert spaces $H$ and $U$, and a symmetric nonnegative operator $Q \in L(U)$. We consider first the case when $\text{Tr } Q < +\infty$. Then there exists a complete orthonormal system $\{e_k\}$ in $U$, and a bounded sequence of nonnegative real numbers $\lambda_k$ such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$.

**Definition 2.1.** An $U$-valued stochastic process $W(t), t \geq 0$, is called a $Q$-Wiener process if

(i) $W(0) = 0$,
(ii) $W$ has continuous trajectories,
(iii) $W$ has independent increments,
(iv) $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$, $t \geq s \geq 0$.

If a process $W(t), t \in [0,T]$ satisfies (i) - (iii) and (iv) for $t, s \in [0,T]$, then we say that $W$ is a $Q$-Wiener process on $[0,T]$. Using the Kolmogorov extension theorem, for arbitrary trace class symmetric nonnegative operator $Q$ on a separable Hilbert space $U$ there exists a $Q$-Wiener process $W(t), t \geq 0$ ([4, Proposition 4.2]).

For an $L = L(U, H)$-valued elementary process $\Phi$ one defines the stochastic integral by the formula

$$\int_0^t \Phi(s) dW(s) = \sum_{m=0}^{k-1} \Phi_m (W_{t_{m+1} \wedge t} - W_{t_m \wedge t})$$

and denote it by $\Phi \cdot W(t), t \in [0,T]$.

It is useful, at this moment, to introduce the subspace $U_0 = Q^{1/2}(U)$ of $U$ which, endowed with the inner product

$$\langle u, v \rangle_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle u, e_k \rangle \langle v, e_k \rangle = \langle Q^{-1/2} u, Q^{-1/2} v \rangle,$$

is a Hilbert space.

In the construction of the stochastic integral for more general processes an important role will be played by the space of all Hilbert-Schmidt operators $L_2^0 = L_2(U_0, H)$ from $U_0$ into $H$. The space $L_2^0$ is also a separable Hilbert space, equipped with the norm

$$\begin{aligned} ||\Psi||_{L_2^0}^2 &= \sum_{h,k=1}^{\infty} |\langle \Psi_{g_h}, f_k \rangle|^2 = \sum_{h,k=1}^{\infty} \lambda_h |\langle \Psi_{e_h}, f_k \rangle|^2 \\ &= ||\Psi Q^{1/2}||^2 = \text{Tr } [\Psi Q \Psi^*], \end{aligned}$$

where $\{g_j\}$, with $g_j = \sqrt{\lambda_j} e_j, j = 1, 2, \ldots, \{e_j\}$ and $\{f_j\}$ are complete orthonormal bases in $U_0, U$ and $H$ respectively. Clearly, $L \subset L_2^0$, but not all operators from $L_2^0$ can be regarded as restrictions of operators from $L$. The space $L_2^0$ contains genuinely unbounded operators on $U$ ([4]).

Let $\Phi(t), t \in [0,T]$, be a measurable $L_2^0$-valued process; we define the norms

$$|||\Phi|||_t = \{E\int_0^t ||\Phi(s)||^2_{L_2^0} ds\}^{1/2}$$
$$= \{E\int_0^t \mathrm{Tr}\ (\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^* ds\}^{1/2}, \quad t \in [0,T].$$

**Definition 2.2** ([4, Proposition 4.5]). If a process $\Phi$ is elementary and $|||\Phi|||_t < \infty$ then the process $\Phi \cdot W$ is a continuous, square integrable $H$-valued martingale on $[0,T]$ and

$$E|\Phi \cdot W(t)|^2 = |||\Phi|||_t^2, \quad 0 \le t \le T.$$

Let us consider a stochastic differential inclusion on infinite dimensional space

$$dX_t \in \sigma(X_t)dW_t + b(X_t)dt, \tag{1.1}$$

with initial value $X_0 = x$, where $\sigma : H \to \mathcal{P}(L_2^0)$, $b : H \to \mathcal{P}(H)$ are set-valued maps and $x$ is an $H$-valued $\mathfrak{F}_0$-measurable random variable.

**Definition 2.3.** A stochastic process $X = \{X_t,\ t \in [0,T]\}$ is said to be a solution of (1.1) on $[0,T]$ with the initial condition $X_0 = x$ if there are predictable random processes $\xi : \Omega \times [0,T] \to L_2^0$, $\eta : \Omega \times [0,T] \to H$ such that $\xi(t) \in \sigma(X_t)$, $\eta(t) \in b(X_t)$ for every $t \in [0,T]$ almost surely and

$$X_t = x + \int_0^t \xi(s)\ dW_s + \int_0^t \eta(s)\ ds.$$

## 3. Main result

For a Banach space $X$ with the norm $||\cdot||$ and for non-empty sets $A, A'$ in $X$, we denote $||A|| = \sup\{||a|| \mid a \in A\}$, $d(a,A') = \inf\{d(a,a') \mid a' \in A'\}$, $d(A,A') = \sup\{d(a,A') \mid a \in A\}$ and $d_H(A,A') = \max\{d(A,A'), d(A',A)\}$, a Hausdorff metric. We can prove the existence of solution for the stochastic differential inclusion (1.1) under Lipschitz condition using approximation. From now we assume that the coefficients $\sigma$ and $b$ in (1.1) are closed convex set-valued functions which are Lipschitz continuous, i.e., there exists constants $L > 0$ and $K > 0$ such that

$$\begin{cases} d_H(\sigma(x),\sigma(y)) \le L|x-y|, \ d_H(b(x),b(y)) \le L|x-y| \\ ||\sigma(x)|| \le K(1+|x|), \ ||b(x)|| \le K(1+|x|). \end{cases}$$

**Theorem 3.1.** *There exists a solution $X_t, t \in [0,T]$, for the stochastic differential inclusion* (1.1).

*Proof.* For arbitrary $\xi_t^0$ and $\eta_t^0$, define $(X_t^n), (\xi_t^n)$, and $(\eta_t^n)$ as the following by induction.

$$X_t^n = x + \int_0^t \xi_s^n dW_s + \int_0^t \eta_s^n ds,$$
$$\xi_t^{n+1} = P_{\sigma(X_t^n)}\xi_t^n,\ \eta_t^{n+1} = P_{b(X_t^n)}\eta_t^n,$$

where $P_A x$ is the nearest point of $A$ from $x$ for closed convex set $A$. We claim that $(X_t^n)$ converges and the limit becomes a solution. Since

$$\begin{aligned} ||\xi_t^{n+1} - \xi_t^n||_{L_2^0} &\le d_H(\sigma(X_t^n), \sigma(X_t^{n-1})) \\ &\le L\left|\int_0^t (\xi_s^n - \xi_s^{n-1})dW_s + \int_0^t (\eta_s^n - \eta_s^{n-1})ds\right|, \end{aligned}$$

we have

$$\begin{aligned} &E\left[\sup_{0\le s\le t} ||||\xi_s^{n+1} - \xi_s^n||_{L_2^0}||^p\right]^{1/p} \\ &\le LE\left[\sup_{0\le s\le t}\left|\int_0^s (\xi_v^n - \xi_v^{n-1})dW_v\right|^p\right]^{1/p} + LE\left[\sup_{0\le s\le t}\left|\int_0^s (\eta_v^n - \eta_v^{n-1})dv\right|^p\right]^{1/p} \\ &\le LC_1E\left[\left\{\int_0^t ||\xi_s^n - \xi_s^{n-1}||_{L_2^0}^2 ds\right\}^{p/2}\right]^{1/p} + LE\left[\left(\int_0^t |\eta_s^n - \eta_s^{n-1}|ds\right)^p\right]^{1/p} \\ &\qquad \text{(by Burkholder's inequality)} \\ &\le LC_1\left|\left|\int_0^t ||\xi_s^n - \xi_s^{n-1}||_{L_2^0}^2 ds\right|\right|_{p/2}^{1/2} + L\left|\left|\int_0^t |\eta_s^n - \eta_s^{n-1}|ds\right|\right|_p \\ &\le LC_1\left\{\int_0^t ||||\xi_s^n - \xi_s^{n-1}||_{L_2^0}^2||_{p/2}ds\right\}^{1/2} + L\int_0^t ||\eta_s^n - \eta_s^{n-1}||_p ds \\ &= LC_1\left\{\int_0^t ||||\xi_s^n - \xi_s^{n-1}||_{L_2^0}||_p^2 ds\right\}^{1/2} + L\int_0^t ||\eta_s^n - \eta_s^{n-1}||_p ds. \end{aligned}$$

By the same way,

$$\begin{aligned} &E\left[\sup_{0\le s\le t} |\eta_s^{n+1} - \eta_s^n|^p\right]^{1/p} \\ &\le LC_1\left\{\int_0^t ||||\xi_s^n - \xi_s^{n-1}||_{L_2^0}||_p^2 ds\right\}^{1/2} + L\int_0^t ||\eta_s^n - \eta_s^{n-1}||_p ds. \end{aligned}$$

Taking $M > 0$ be such that

$$\frac{2LC_1}{2M+1} + \frac{2L}{M+1} \le 1,\ \ 2LC_1\sqrt{t} \le e^{Mt},\ \text{and}\ \ 2Lt \le e^{Mt},$$

we have

$$\left\| \sup_{0\le s\le t} \|\xi_s^{n+1} - \xi_s^n\|_{L_2^0} \right\|_p \le \frac{e^{Mt}}{2^n}\left\{ \sup_{0\le s\le t} \|\|\|\xi_s^1 - \xi_s^0\|_{L_2^0}\|_p + \sup_{0\le s\le t} \|\eta_s^1 - \eta_s^0\|_p \right\}, \tag{3.1}$$

$$\left\| \sup_{0\le s\le t} |\eta_s^{n+1} - \eta_s^n| \right\|_p \le \frac{e^{Mt}}{2^n}\left\{ \sup_{0\le s\le t} \|\|\|\xi_s^1 - \xi_s^0\|_{L_2^0}\|_p + \sup_{0\le s\le t} \|\eta_s^1 - \eta_s^0\|_p \right\}. \tag{3.2}$$

In fact, in case of $n = 1$,

$$\begin{aligned} &\left\| \sup_{0\le s\le t} \|\xi_s^2 - \xi_s^1\|_{L_2^0} \right\|_p \\ &\le LC_1\sqrt{t \sup_{0\le s\le t} \|\|\|\xi_s^1 - \xi_s^0\|_{L_2^0}\|_p^2} + Lt \sup_{0\le s\le t} \|\eta_s^1 - \eta_s^0\|_p \\ &\le \frac{e^{Mt}}{2}\left\{ \sup_{0\le s\le t} \|\|\|\xi_s^1 - \xi_s^0\|_{L_2^0}\|_p + \sup_{0\le s\le t} \|\eta_s^1 - \eta_s^0\|_p \right\}. \end{aligned}$$

We can prove similarly for $\eta$. Assume that the above inequalities hold for $n-1$. Then

$$\begin{aligned} &\left\| \sup_{0\le s\le t} \|\xi_s^{n+1} - \xi_s^n\|_{L_2^0} \right\|_p \\ &\le LC_1\left\{ \int_0^t \left(\frac{e^{Ms}}{2^{n-1}}\right)^2 \phi(t)^2 ds \right\}^{1/2} + L\int_0^t \frac{e^{Ms}}{s^{n-1}}\, ds \\ &= LC_1\phi(t)\frac{1}{2^{n-1}}\left\{ \frac{1}{2M+1}(e^{2Mt} - 1) \right\}^{1/2} + \frac{L}{2^{n-1}}\frac{1}{M+1}(e^{Mt} - 1)\phi(t) \\ &\le \frac{e^{Mt}}{2^n}\phi(t), \end{aligned}$$

where $\phi(t) = \sup_{0\le s\le t} \|\|\|\xi_s^1 - \xi_s^0\|_{L_2^0}\|_p + \sup_{0\le s\le t} \|\eta_s^1 - \eta_s^0\|_p$. For $\eta$, we can prove similarly. Thus the above inequalities (3.1) and (3.2) hold for every $n = 1, 2, \ldots$.

Since

$$\sum_{n=0}^{\infty} \left\| \sup_{0\le s\le t} \|\xi_s^{n+1} - \xi_s^n\|_{L_2^0} \right\|_p < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \left\| \sup_{0\le s\le t} |\eta_s^{n+1} - \eta_s^n| \right\|_p < \infty,$$

$(\xi_t^n)$ and $(\eta_t^n)$ converge in $L^p$. Denoting the limits by $\xi_t$ and $\eta_t$, respectively,

$$\lim_{n\to\infty} \left\| \sup_{0\le s\le t} \|\xi_s^n - \xi_s\|_{L_2^0} \right\|_p = 0 \quad \text{and} \quad \lim_{n\to\infty} \left\| \sup_{0\le s\le t} |\eta_s^n - \eta_s| \right\|_p = 0.$$

If we put

$$X_t = x + \int_0^t \xi_s \, dW_s + \int_0^t \eta_s \, ds,$$

then we have

$$\left|\left| \sup_{0\le s\le t} |X_s^n - X| \right|\right|_p \le C_1 \left\{ \int_0^t ||||\xi_s^n - \xi_s||_{L_2^0}||_p^2 \, ds \right\}^{1/2} + \int_0^t ||\eta_s^n - \eta_s||_p \, ds.$$

Letting $n \to \infty$, the right hand side tends to 0. Thus $(X_s^n)$ converges to $X_s$ in $L^p$. Furthermore, we have

$$\begin{aligned} d(\xi_s, \sigma(X_s)) &\le ||\xi_s - \xi_s^n||_{L_2^0} + d(\xi_s^n, \sigma(X_s)) \\ &\le ||\xi_s - \xi_s^n||_{L_2^0} + L|X_s^{n-1} - X_s|, \end{aligned}$$

and thus

$$\left|\left| \sup_{0\le s\le t} d(\xi_s, \sigma(X_s)) \right|\right|_p \le \left|\left| \sup_{0\le s\le t} ||\xi_s - \xi_s^n||_{L_2^0} \right|\right|_p + L \left|\left| \sup_{0\le s\le t} |X_s^{n-1} - X_s| \right|\right|_p .$$

Since the right hand side converges to 0, $\xi_s \in \sigma(X_s)$, a.e. Similarly, we can prove that $\eta_s \in b(X_s)$, a.e. Hence $X_t$ is a solution. □

Furthermore, we have the following theorem for boundedness of solutions. The proof is similar to that in case of finite dimensional space.

**Theorem 3.2.** *Let $X_t$ be any solution of* (1.1). *Then $X_t$ is bounded, i.e., for $p \ge 2$,*

$$E[\sup_{0\le s\le t} |X_s|^p] < \infty.$$

*Proof.* Let $X_t$ be a solution. Then there exist $\xi_s \in \sigma(X_s)$ and $\eta_s \in b(X_s)$ such that

$$X_t = x + \int_0^t \xi_s dW_s + \int_0^t \eta_s ds.$$

Since

$$\begin{aligned} &E[\sup_{0\le s\le t} |X_s|^p] \\ &\le 3^{p-1}|x|^p + 3^{p-1}C_1 E\left[ \{\int_0^t |\xi_s|^2 ds\}^{p/2} \right] + 3^{p-1} E\left[ \{\int_0^t |\eta_s|^2 ds\}^p \right] \\ &\le 3^{p-1}|x|^p + 3^{p-1}C_1 T^{\frac{p-2}{2}} \int_0^t E[|\xi_s|^p] ds + 3^{p-1} T^{p-1} \int_0^t E[|\eta_s|^p] ds \\ &\le 3^{p-1}|x|^p + 3^{p-1}C_1 T^{\frac{p-2}{2}} \int_0^t K^p(1 + E[|X_s|^p]) 2^{p-1} ds \\ &\quad + 3^{p-1} T^{p-1} \int_0^t K^p(1 + E[|X_s|^p]) 2^{p-1} ds, \end{aligned}$$

if we put $\psi(t) = E[\sup_{0\le s\le t} |X_s|^p]$,

$$\psi(t) \le 3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}C_1 + 6^{p-1}K^pT^{\frac{p-2}{2}}C_1\int_0^t \psi(s)ds$$

$$+ 6^{p-1}K^pT^p + 6^{p-1}K^pT^{p-1}\int_0^t \psi(s)ds$$

$$= 3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}(C_1+1)$$

$$+ 6^{p-1}K^p(T^{\frac{p-2}{2}}C_1 + T^{p-1})\int_0^t \psi(s)ds.$$

By Gronwall's inequality,

$$\psi(t) \le (3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}(C_1+1))\cdot \exp(6^{p-1}K^p(T^{\frac{p-2}{2}}C_1 + T^{p-1})t).$$

Hence $X_t$ is bounded. □

Let

$$S(x) = \{X_t | X_t \text{ is a solution of (1.1) with initial point } X_0 = x\}.$$

**Theorem 3.3.** *$S(x)$ is closed.*

*Proof.* Let $(X_t^n)$ be a sequence in $S(x)$ converging to $X_t$, i.e.,

$$\lim_{n\to\infty} E[\sup_{0\le t\le T} |X_t^n - X_t|^p] = 0.$$

Since $(X_t^n)$ are solutions of (1.1), there sequences $(\xi_t^n)$ and $(\eta_t^n)$ such that

$$X_t^n = x + \int_0^t \xi_t^n dW_s + \int_0^t \eta_t^n ds.$$

For closed convex set $A \subset \mathbb{R}^d$, define $P_A(x) \in \mathbb{R}^d$ by $\|x - P_A(x)\| = d(x, A)$. Then $P_A(x)$ exists uniquely. Put $\hat{\xi}_t^n = P_{\sigma(X_t)}(\xi_t^n)$ and $\hat{\eta}_t^n = P_{b(X_t)}(\eta_t^n)$. Then by hypothesis,

$$|\hat{\xi}_t^n - \xi_t^n| \le d_H(\sigma(X_t), \sigma(X_t^n)) \le L|X_t - X_t^n|,$$
$$|\hat{\eta}_t^n - \eta_t^n| \le d_H(b(X_t), b(X_t^n)) \le L|X_t - X_t^n|.$$

Since

$$E[\int_0^T |\hat{\xi}_t^n|^p dt] \le E[\int_0^T |\sigma(X_t)|^p dt] \le E[2^pK^p\int_0^T (1+|X_t|^p)dt],$$

$(\hat{\xi}_t^n)$ and $(\hat{\eta}_t^n)$ are $L^p$-bounded. Taking suitable subsequence of $(\hat{\xi}_t^n)$ and convex combinations of subsequence, we can estimate the limit $\hat{\xi}_t$ by the following way ([7]).

$$E[\int_0^T |\hat{\xi}_t - \sum_{j=1}^{N_n} \lambda_j \hat{\xi}_t^j|^p dt] \le \frac{1}{2^n}.$$

Similarly, for $(\hat{\eta}_t)$,

$$E[\int_0^T |\hat{\eta}_t - \sum_{j=1}^{N_n} \lambda_j \hat{\eta}_t^j|^p dt] \leq \frac{1}{2^n}.$$

Since

$$\begin{aligned}
&|X_t - x - \int_0^t \hat{\xi}_s dW_s - \int_0^t \hat{\eta}_s ds| \\
&= \Big| X_t - \sum_{j=1}^{N_n} \lambda_j X_t^j + \sum_{j=1}^{N_n} \lambda_j X_t^j - x - \sum_j \lambda_j \int_0^t \xi_s^j dW_s - \sum_j \lambda_j \int_0^t \eta_s^j ds \\
&\quad + \sum_j \lambda_j \int_0^t (\xi_s^j - \hat{\xi}_s^j) dW_s + \sum_j \lambda_j \int_0^t (\eta_s^j - \hat{\eta}_s^j) ds \\
&\quad + \int (\sum_j \lambda_j \hat{\xi}_s^j - \hat{\xi}_s) dW_s + \int (\sum_j \lambda_j \hat{\eta}_s^j - \hat{\eta}_s) ds \Big| \\
&\leq \sum_j \lambda_j |X_t - X_t^j| + \sum_j \lambda_j \Big| \int_0^t (\xi_s^j - \hat{\xi}_s^j) dW_s \Big| + \Big| \int_0^t (\sum_j \lambda_j \hat{\xi}_s^j - \hat{\xi}_s) dW_s \Big| \\
&\quad + \sum_j \lambda_j \int_0^t |\eta_s^j - \hat{\eta}_s^j| ds + \int_0^t |\sum_j \lambda_j \hat{\eta}_s^j - \hat{\eta}_s| ds,
\end{aligned}$$

we have

$$\begin{aligned}
&\| \sup_{0 \leq t \leq T} |X_t - x - \int_0^t \hat{\xi}_s dW_s - \int_0^t \hat{\eta}_s ds| \, \|_p \\
&\leq \sum_j \lambda_j \| \sup_{0 \leq t \leq T} |X_t - X_t^j| \, \|_p + \sum_j \lambda_j \| \sup_{0 \leq t \leq T} | \int_0^t (\xi_s^j - \hat{\xi}_s^j) dW_s| \, \|_p \\
&\quad + \| \sup_{0 \leq t \leq T} | \int_0^t (\sum_j \lambda_j \hat{\xi}_s^j - \hat{\xi}_s) dW_s| \, \|_p + \sum_j \lambda_j \| \int_0^T |\eta_s^j - \hat{\eta}_s^j| ds \|_p \\
&\quad + \| \int_0^T |\sum_j \lambda_j \hat{\eta}_s^j - \hat{\eta}_s| ds \|_p \\
&\leq \sum_j \lambda_j \| \sup_{0 \leq t \leq T} |X_t - X_t^j| \, \|_p + C_1 \sum_j \lambda_j E[\{\int_0^T |\xi_s^j - \hat{\xi}_s^j|^2 ds\}^{p/2}]^{1/p} \\
&\quad + C_1 E[\{\int_0^T |\sum_j \lambda_j \hat{\xi}_s^j - \hat{\xi}_s|^2 ds\}^{p/2}]^{1/p} + \sum_j \lambda_j E[\{\int_0^T |\eta_s^j - \hat{\eta}_s^j| ds\}^p]^{1/p} \\
&\quad + E[\{\int_0^T |\sum_j \lambda_j \hat{\eta}_s^j - \hat{\eta}_s| ds\}^p]^{1/p}
\end{aligned}$$

$$\leq \sum_j \lambda_j \| \sup_{0\leq t\leq T} |X_t - X_t^j| \|_p + C_1 T^{(\frac{1}{2}-\frac{1}{p})} \sum_j \lambda_j E[\int_0^T |\xi_s^j - \hat{\xi}_s^j|^p ds]$$
$$+ C_1 T^{(\frac{1}{2}-\frac{1}{p})} E[\int_0^T |\sum_j \lambda_j \hat{\xi}_s^j - \hat{\xi}_s|^p ds] + T^{(1-\frac{1}{p})} \sum_j \lambda_j E[\int_0^T |\eta_s^j - \hat{\eta}_s^j|^p ds]$$
$$+ T^{(1-\frac{1}{p})} E[\int_0^T |\sum_j \lambda_j \hat{\eta}_s^j - \hat{\eta}_s|^p ds].$$

Letting $n \to \infty$, the right hand side tends to 0. We can $\sum_{j=1}^{N_n} \lambda_j \hat{\xi}_t^j - \hat{\xi}_t$ a.e.$t$, a.e.$\omega$ for some subsequence. And since $\sigma(X_t)$ is convex, $\hat{\xi}_t \in \sigma(X_t)$ a.e.$t$, a.e.$\omega$. By the same way, $\hat{\eta}_t \in b(X_t)$ a.e.$t$, a.e.$\omega$. This proves that $(X_t)$ is a solution of (1.1). Thus $S^p(x)$ is closed. □

**Theorem 3.4.** *The mapping $x \mapsto S(x)$ is Lipschitz continuous.*

*Proof.* Let $X_t \in S(x)$. Then there exist $\xi_s \in \sigma(X_s)$ and $\eta_s \in b(X_s)$ such that

$$X_t = x + \int_0^t \xi_s dW_s + \int_0^t \eta_s ds.$$

Let $\xi_t^0 = \xi_t$ and $\eta_t^0 = \eta_t$, and define $(Y_t^n), (\xi_t^n)$, and $(\eta_t^n)$ as the following by induction.

$$Y_t^n = y + \int_0^t \xi_s^n dW_s + \int_0^t \eta_s^n ds,$$
$$\xi_t^{n+1} = P_{\sigma(Y_t^n)} \xi_t^n, \ \eta_t^{n+1} = P_{b(Y_t^n)} \eta_t^n,$$

where $P_A x$ is the nearest point of $A$ from $x$ for closed convex set $A$. Then by the proof of Theorem 3.1, $(Y_t^n)$ converges to a solution $Y_t \in S(y)$.

Put $\phi(t) = \sup_{0\leq s\leq t} \|\xi_s^1 - \xi_s^0\|_p + \sup_{0\leq s\leq t} \|\eta_s^1 - \eta_s^0\|_p$. Since

$$\| \sup_{0\leq s\leq t} |\xi_s^{n+1} - \xi_s^n| \|_p \leq \frac{e^{Mt}}{2^n}\phi(t) \text{ and } \| \sup_{0\leq s\leq t} |\eta_s^{n+1} - \eta_s^n| \|_p \leq \frac{e^{Mt}}{2^n}\phi(t),$$

we have

$$\begin{aligned}
\| \sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n| \|_p &\leq C_1 \{\int_0^t \|\xi_s^{n+1} - \xi_s^n\|_p^2 ds\}^{\frac{1}{2}} + \int_0^t \|\eta_s^{n+1} - \eta_s^n\|_p ds \\
&\leq C_1 \{\int_0^t \frac{e^{2Ms}}{2^{2n}} \phi(t)^2 ds\}^{\frac{1}{2}} + \int_0^t \frac{e^{Ms}}{2^n}\phi(t) ds \\
&\leq \frac{C_1}{2^n} \frac{e^{Mt}}{2M+1}\phi(t) + \frac{e^{Mt}}{2^n M}\phi(t) \\
&= \frac{C_1 M + 2M + 1}{M(2M+1)} \frac{e^{Mt}}{2^n}\phi(t).
\end{aligned}$$

Thus

$$\| \sup_{0\le t\le T} |Y_t - X_t| \|_p \le \sum_{n=0}^{\infty} \| \sup_{0\le t\le T} |Y_t^{n+1} - Y_t^n| \|_p + \| \sup_{0\le t\le T} |Y_t^0 - X_t| \|_p$$
$$\le \frac{(C_1+2)M+1}{M(2M+1)} 2e^{MT}\phi(t) + |x-y|.$$

By $|\xi_t^1 - \xi_t^0| \le d_H(\sigma(X_t), \sigma(Y_t^0)) \le L|X_t - Y_t^0| = L|x-y|$ and $|\eta_t^1 - \eta_t^0| \le L|x-y|$, $\phi(t) \le 2L|x-y|$. Thus

$$\| \sup_{0\le t\le T} |Y_t - X_t| \|_p \le \{\frac{4L((C_1+2)M+1)}{M(2M+1)} e^{MT} + 1\}|x-y|.$$

Therefore $d_H(S^p(x), S^p(y)) \le C|x-y|$. Hence the mapping $x \mapsto S^p(x)$ is Lipschitz continuous. □

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