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HYERS-ULAM-RASSIAS STABILITY OF A CUBIC FUNCTIONAL EQUATION

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Abstract. In this paper, we will find out the general solution and investigate the generalized Hyers–Ulam–Rassias stability problem for the following cubic functional equation

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y).$$

The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms : Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [10] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [28]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [3, 4, 8, 12, 15, 17, 19, 20, 24, 25, 26, 31]. The terminology *'generalized Hyers–Ulam–Rassias stability'* originates from these historical backgrounds. These terminologies are also applied to the case of

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other functional equations. For more detailed definitions of such terminologies, we can refer to [11, 13, 29].

Quadratic functional equation was used to characterize inner product spaces [1, 2, 14]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric biadditive mapping [1, 22]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x (see [1, 22]). The biadditive mapping B is given by

$$B(x,y) = \frac{1}{4}\Big(f(x+y) - f(x-y)\Big). \tag{1.2}$$

A Hyers–Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space (see [32]). Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1). Grabiec [9] generalized these results mentioned above. Jung [21] dealt with stability problems for the quadratic functional equation of Pexider type, $f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y)$, and Jun and Lee [18] proved the generalized Hyers–Ulam–Rassias stability of the Pexiderized quadratic equation.

Jun and Kim [16] introduced the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.3).

Park and Jung [27] introduced the functional equation

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x) \tag{1.4}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.4).

It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equations (1.3) and (1.4). Thus, it is natural that (1.3) and (1.4) is called a *cubic functional equation* and every solution of the cubic functional equations (1.3) and (1.4) is said to be a *cubic mapping.*

In this paper, we introduce the following new functional equation, which is somewhat different from (1.3) and (1.4) :

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y). \tag{1.5}$$

It is easy to see that the function $f(x)=cx^3$ is a solution of the functional equation (1.5).

In this paper, we establish the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.5).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([5], [30]). Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(*i*) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.

(*ii*) $\|\lambda x\|=|\lambda|\|x\|$ for all $\lambda\in\mathbb{R}$ and all $x\in X$.

(*iii*) There is a constant $K\geq 1$ such that $\|x+y\|\leq K(\|x\|+\|y\|)$ for all $x,y\in X$.

It follows from condition (*iii*) that

$$\Big\|\sum_{i=1}^{2n}x_i\Big\|\leq K^n\sum_{i=1}^{2n}\|x_i\|,\qquad \Big\|\sum_{i=1}^{2n+1}x_i\Big\|\leq K^{n+1}\sum_{i=1}^{2n+1}\|x_i\|$$

for all integers $n\geq 1$ and all $x_1,x_2,\ldots,x_{2n+1}\in X$.

The pair $(X,\|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0<p\leq 1)$ if

$$\|x+y\|^p\leq\|x\|^p+\|y\|^p$$

for all $x,y\in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

By the Aoki–Rolewicz theorem [30] (see also [5]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Throughout this note, we will denote by $(G,+)$ an abelian group.

2. Solution of Eq. (1.5)

Let X be real vector space. We here present the general solution of (1.5).

Theorem 2.1 ([16]). *A mapping $f:G\to X$ satisfies the functional equation (1.3) if and only if there exist a mapping $B:G\times G\times G\to X$ such that $f(x)=B(x,x,x)$ for all $x\in G$, and B is symmetric for each fixed one variable and is additive for fixed two variables.*

Theorem 2.2 ([23]). *A mapping $f:G\to X$ satisfies the functional equation (1.3) if and only if $f:G\to X$ satisfies the functional equation*

$$2f(x+2y)+f(2x-y)=5f(x+y)+5f(x-y)+15f(y) \tag{2.1}$$

for all $x, y \in G$.

Theorem 2.3. *A mapping* $f : G \to X$ *satisfies the functional equation* (2.1) *if and only if* $f : G \to X$ *satisfies the functional equation* (1.5).

Proof. (*Necessity*). Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Set $y = 0$ in (2.1) to get $f(2x) = 8f(x)$ for all $x \in G$. Letting $x = 0$ in (2.1), we obtain that $f(-y) = -f(y)$ for all $y \in G$. Replacing x by $x + y$ in (2.1), we have

$$2f(x+3y) + f(2x+y) = 5f(x+2y) + 5f(x) + 15f(y) \tag{2.2}$$

for all $x, y \in G$. Since f is odd, replacing y by $y - x$ in (2.1), we get that

$$f(3x-y) - 2f(x-2y) = 5f(2x-y) - 15f(x-y) + 5f(y) \tag{2.3}$$

for all $x, y \in G$. Replacing x and y by y and $-x$ in (2.3), respectively, we obtain

$$f(3y+x) - 2f(2x+y) = 5f(x+2y) - 15f(x+y) - 5f(x) \tag{2.4}$$

for all $x, y \in G$. Adding (2.2) and (2.4), we get that

$$3f(x+3y) - f(2x+y) = 10f(x+2y) - 15f(x+y) + 15f(y) \tag{2.5}$$

for all $x, y \in G$. Once again adding (2.5) and (2.3), we obtain

$$\begin{aligned} 3f(x+3y) + f(3x-y) = {} & [2f(x-2y) + f(2x+y)] \\ & + 5[2f(x+2y) + f(2x-y)] \\ & - 15f(x-y) - 15f(x+y) + 20f(y) \end{aligned} \tag{2.6}$$

for all $x, y \in G$. Therefore (1.5) follows from (2.1) and (2.6).

(*Sufficiency*). Putting $x = y = 0$ in (1.5), we get $f(0) = 0$. Set $y = 0$ in (1.5) to get $f(3x) = 27f(x)$ for all $x \in G$. Letting $x = 0$ in (1.5), we obtain that $f(-y) = -f(y)$ for all $y \in G$. Replacing x by $x - y$ in (1.5), we have

$$3f(x+2y) + f(3x-4y) = 15f(x-2y) + 15f(x) + 80f(y) \tag{2.7}$$

for all $x, y \in G$. Since f is odd, replacing y by $x + y$ in (1.5), we get that

$$3f(4x+3y) + f(2x-y) = 15f(2x+y) + 80f(x+y) - 15f(y) \tag{2.8}$$

for all $x, y \in G$. Replacing x and y by y and $-x$ in (2.8), respectively, and multiplying both sides of (2.8) to (-1), we obtain

$$3f(3x-4y) - f(x+2y) = 15f(x-2y) + 80f(x-y) - 15f(x) \tag{2.9}$$

for all $x, y \in G$. Adding (2.7) and (2.9), we have

$$2f(3x-4y) + f(x+2y) = 15f(x-2y) + 40f(x-y) + 40f(y) \tag{2.10}$$

for all $x, y \in G$. Therefore we infer from (2.7) and (2.10) that

$$f(x+2y) - 3f(x-2y) = 6f(x) + 24f(y) - 8f(x-y) \tag{2.11}$$

for all $x, y \in G$. Replacing y by $-y$ in (2.11), we obtain

$$f(x-2y) - 3f(x+2y) = 6f(x) - 24f(y) - 8f(x+y) \tag{2.12}$$

for all $x, y \in G$. It follows from (2.11) and (2.12) that

$$f(x+2y) = 3f(x+y) + f(x-y) + 6f(y) - 3f(x) \tag{2.13}$$

for all $x, y \in G$. Replacing x and y by $-y$ and x in (2.13), respectively, we get

$$f(2x-y) = 3f(x-y) - f(x+y) + 6f(x) + 3f(y) \tag{2.14}$$

for all $x, y \in G$. Hence (2.1) follows from (2.13) and (2.14). □

Corollary 2.4. *A mapping $f : G \to X$ satisfies the functional equation* (1.3) *if and only if $f : G \to X$ satisfies the functional equation* (1.5). *Therefore A mapping $f : G \to X$ satisfies the functional equation* (1.5) *if and only if there exists a mapping $B : G \times G \times G \to X$ such that $f(x) = B(x, x, x)$ for all $x \in G$, and B is symmetric for each fixed one variable and is additive for fixed two variables.*

3. Generalized Hyers–Ulam–Rassias stability of Eq. (1.5)

From now on, let E be a normed real linear space with norm $\|\cdot\|_E$ and X be a real p-Banach space with norm $\|\cdot\|_X$. In this section, using an idea of Găvruta [8] we prove the stability of Eq. (1.5) in the spirit of Hyers, Ulam, and Rassias. Thus we find the condition that there exists a true cubic mapping near a approximately cubic mapping. For convenience, we use the following abbreviation for a given mapping $f : G \to X$

$$Df(x,y) := 3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y)$$

for all $x, y \in G$.

Theorem 3.1. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{27^{np}} \varphi^p(3^n x, 0) < \infty \tag{3.1}$$

and

$$\lim_{n\to\infty} \frac{1}{27^n} \varphi(3^n x, 3^n y) = 0 \tag{3.2}$$

for all $x, y \in G$. Suppose that a mapping $f : G \to X$ satisfies the inequality

$$\|Df(x,y)\|_X \le \varphi(x,y) \tag{3.3}$$

for all $x, y \in G$. Then there exists a unique cubic mapping $T : G \to X$ which satisfies Eq. (1.5) *and the inequality*

$$\left\| T(x) - f(x) - \frac{40}{13} f(0) \right\|_X \le \frac{1}{27} \left[\widetilde{\varphi}(x) \right]^{\frac{1}{p}} \tag{3.4}$$

for all $x \in G$. The mapping $T : G \to X$ is given by

$$T(x) = \lim_{n\to\infty} \frac{1}{27^n} f(3^n x) \tag{3.5}$$

for all $x \in G$.

Proof. Putting $y = 0$ in (3.3) and dividing both sides of (3.3) by 27, we have

$$\Big\|\frac{f(3x)}{27} - f(x) - \frac{80}{27}f(0)\Big\|_X \le \frac{1}{27}\varphi(x,0) \tag{3.6}$$

for all $x \in G$. Replacing x by $3^n x$ in (3.6) and dividing both sides of (3.6) by 27^n, we get

$$\Big\|\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n} - \frac{80}{27^{n+1}}f(0)\Big\|_X \le \frac{1}{27^{n+1}}\varphi(3^n x,0) \tag{3.7}$$

for all $x \in G$ and all integers $n \ge 0$. Since X is a p-Banach space, we have

$$\begin{aligned}&\Big\|\sum_{i=k}^{n}\Big[\frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i} - \frac{80}{27^{i+1}}f(0)\Big]\Big\|_X^p\\ &\le \sum_{i=k}^{n}\Big\|\frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i} - \frac{80}{27^{i+1}}f(0)\Big\|_X^p\\ &\le \sum_{i=k}^{n}\frac{1}{27^{(i+1)p}}\varphi^p(3^i x,0)\end{aligned}$$

for all $x \in G$ and all integers $n \ge k \ge 0$. Hence

$$\Big\|\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^k x)}{27^k} - \sum_{i=k}^{n}\frac{80}{27^{i+1}}f(0)\Big\|_X^p \le \sum_{i=k}^{n}\frac{1}{27^{(i+1)p}}\varphi^p(3^i x,0) \tag{3.8}$$

for all $x \in G$ and all integers $n \ge k \ge 0$. Since $\sum_{i=0}^{\infty}\frac{1}{27^i}$ is convergent, it follows from (3.1) and (3.8) that the sequence $\{\frac{f(3^n x)}{27^n}\}$ is a Cauchy sequence in X for all $x \in G$. Since X is a p-Banach space, it follows that the sequence $\{\frac{f(3^n x)}{27^n}\}$ converges for all $x \in G$. We define $T : G \to X$ by

$$T(x) = \lim_{n\to\infty}\frac{f(3^n x)}{27^n}$$

for all $x \in G$. So it follows from (3.2) and (3.3) that

$$\|DT(x,y)\|_X = \lim_{n\to\infty}\frac{1}{27^n}\|Df(3^n x, 3^n y)\|_X \le \lim_{n\to\infty}\frac{1}{27^n}\varphi(3^n x, 3^n y) = 0$$

for all $x, y \in G$. Hence by Corollary 2.4, T is cubic.

One can obtain (3.4) by putting $k = 0$ and letting $n \to \infty$ in (3.8).

It remains to show that T is unique. Suppose that there exists another cubic mapping $Q : G \to X$ which satisfies (1.5) and (3.4). Since $Q(3^n x) = 27^n Q(x)$

for all $x \in G$ and all $n \in \mathbb{N}$, we conclude that

$$\begin{aligned}\|Q(x) - T(x)\|_X^p &= \lim_{n\to\infty} \frac{1}{27^{np}} \|Q(3^n x) - f(3^n x)\|_X^p \\ &\le \lim_{n\to\infty} \frac{1}{27^{np}} \Big\|Q(3^n x) - f(3^n x) - \frac{40}{13} f(0)\Big\|_X^p \\ &\le \lim_{n\to\infty} \frac{1}{27^{(n+1)p}} \widetilde{\varphi}(3^n x) \\ &= \lim_{n\to\infty} \frac{1}{27^p} \sum_{m=n}^{\infty} \frac{1}{27^{mp}} \varphi^p(3^m x, 0) = 0\end{aligned}$$

for all $x \in G$. Hence we have $Q(x) = T(x)$ for all $x \in G$ which gives the uniqueness of T. □

Corollary 3.2. *Let θ be non-negative real number and let $f : E \to X$ be mapping satisfying*

$$\|Df(x,y)\|_X \le \theta \tag{3.9}$$

for all $x, y \in E$. Then there exists a unique cubic mapping $T : E \to X$ which satisfies Eq. (1.5) *and the inequality*

$$\Big\|T(x) - f(x)\Big\|_X \le K\theta\Big\{\frac{1}{(27^p - 1)^{\frac{1}{p}}} + \frac{20}{689}\Big\}$$

for all $x \in E$ where K is the modulus of concavity of $\|\cdot\|_X$.

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3 T(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. Define $\varphi : E \times E \to [0, \infty)$ by $\varphi(x,y) = \theta$ for all $x, y \in E$. Letting $x = y = 0$ in (3.9), we get that $\|f(0)\| \le \theta/106$. By Theorem 3.1 there exists a unique cubic mapping $T : E \to X$ such that

$$\begin{aligned}\Big\|T(x) - f(x)\Big\|_X &\le K\Big\|T(x) - f(x) - \frac{40}{13} f(0)\Big\|_X + K\Big\|\frac{40}{13} f(0)\Big\|_X \\ &\le K\theta\Big\{\frac{1}{(27^p - 1)^{\frac{1}{p}}} + \frac{20}{689}\Big\}\end{aligned}$$

for all $x \in E$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, by the same reasoning as in the proof of [28], the cubic mapping $T : E \to X$ satisfies $T(tx) = t^3 T(x)$ for all $x \in E$ and all $t \in \mathbb{R}$. □

Corollary 3.3. *Let θ be non-negative real number and let $\alpha, \beta \in (0, 3)$. Suppose that a mapping $f : E \to X$ satisfies*

$$\|Df(x,y)\|_X \le \theta(\|x\|_E^\alpha + \|y\|_E^\beta)$$

for all $x, y \in E$. Then there exists a unique cubic mapping $T : E \to X$ which satisfies Eq. (1.5) *and the inequality*

$$\Big\|T(x) - f(x)\Big\|_X \le \frac{\theta}{(27^p - 3^{\alpha p})^{\frac{1}{p}}} \|x\|_E^\alpha$$

for all $x \in E$.

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3T(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. Define $\varphi : E \times E \to [0, \infty)$ by

$$\varphi(x, y) = \theta(||x||_E^\alpha + ||y||_E^\beta)$$

for all $x, y \in E$. Since $\varphi(0,0) = 0$, then $f(0) = 0$. By Theorem 3.1 there exists a unique cubic mapping $T : E \to X$ satisfying in the requirement inequality. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, by the same reasoning as in the proof of [28], the cubic mapping $T : E \to X$ satisfies $T(tx) = t^3T(x)$ for all $t \in \mathbb{R}$. □

Remark 3.4. If a mapping $f : E \to X$ satisfies (1.3), it is easy to show that $f(nx) = n^3 f(x)$ for all $x \in E$ and all $n \in \mathbb{Z}$. So we can conclude that $f(rx) = r^3 f(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$. Hence by Corollary 2.4, if a mapping $f : E \to X$ satisfies (1.5), then $f(rx) = r^3 f(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

Corollary 3.5. *Let θ be non-negative real number and let $\beta \in (0,3)$. Suppose that a mapping $f : E \to X$ satisfies*

$$||Df(x, y)||_X \leq \theta ||y||_E^\beta \tag{3.10}$$

for all $x, y \in E$. Then the mapping $f : E \to X$ is cubic.

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $f(tx) = t^3 f(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. Letting $x = y = 0$ in (3.10), we get that $f(0) = 0$. So by letting $y = 0$ in (3.10), we get $f(3x) = 27f(x)$ for all $x \in E$. Hence by using induction we have

$$f(3^n x) = 27^n f(x) \tag{3.11}$$

$x \in E$ and all $n \in \mathbb{Z}$. By Theorem 3.1 the mapping $T : E \to X$ defined by

$$T(x) = \lim_{n\to\infty} \frac{1}{27^n} f(3^n x)$$

is cubic. Therefore it follows from (3.11) that $f = T$. So the mapping $f : E \to X$ is cubic. The rest of the proof is obvious by Remark 3.4. □

Corollary 3.6. *Let θ be non-negative real number and let $\alpha \in (0,3)$. Suppose that a mapping $f : E \to X$ satisfies*

$$||Df(x, y)||_X \leq \theta ||x||_E^\alpha \tag{3.12}$$

for all $x, y \in E$. Then the mapping $f : E \to X$ is cubic.

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $f(tx) = t^3 f(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. Letting $x = y = 0$ in (3.12), we get that $f(0) = 0$. So by letting $x = 0$ in (3.12), we get

$$3f(3y) - 95f(y) - 14f(-y) = 0 \tag{3.13}$$

for all $y \in E$. We decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in E$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in E$. It follows from (3.13) that

$$[3f_e(3y) - 109f_e(y)] + [3f_o(3y) - 81f_o(y)] = 0 \tag{3.14}$$

for all $y \in E$. Replacing y by $-y$ in (3.14), we get

$$[3f_e(3y) - 109f_e(y)] - [3f_o(3y) - 81f_o(y)] = 0 \tag{3.15}$$

for all $y \in E$. Therefore it follows from (3.14) and (3.15) that

$$f_e(3y) = \frac{109}{3} f_e(y), \qquad f_o(3y) = 27f_o(y)$$

for all $y \in E$. Hence by using induction we have

$$f_e(3^n y) = \left(\frac{109}{3}\right)^n f_e(y), \qquad f_o(3^n y) = 27^n f_o(y)$$

for all $y \in E$ and all $n \in \mathbb{Z}$. So we have

$$f(3^n y) = \left(\frac{109}{3}\right)^n f_e(y) + 27^n f_o(y)$$

for all $y \in E$ and all $n \in \mathbb{Z}$. So

$$\frac{f(3^n y)}{27^n} = \left(\frac{109}{81}\right)^n f_e(y) + f_o(y) \tag{3.16}$$

for all $y \in E$ and all $n \in \mathbb{Z}$. By Theorem 3.1 the sequence $\{\frac{f(3^n y)}{27^n}\}$ is Cauchy for all $y \in E$ and the mapping $T : E \to X$ defined by

$$T(y) = \lim_{n\to\infty} \frac{1}{27^n} f(3^n y)$$

is cubic. Therefore it follows from (3.16) that $f_e(y) = 0$ for all $y \in E$. Therefore $f = f_o$ and we conclude that $f = T$. So the mapping $f : E \to X$ is cubic.

The rest of the proof is obvious by Remark 3.4. □

Corollary 3.7. *Let θ, α, β be non-negative real numbers such that $\alpha + \beta \in (0, 3)$. Suppose that a mapping $f : E \to X$ satisfies*

$$\|Df(x, y)\|_X \le \theta \|x\|_E^\alpha \|y\|_E^\beta \tag{3.17}$$

for all $x, y \in E$ (by putting $\|\cdot\|_E = 0$). Then the mapping $f : E \to X$ is cubic.

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3 T(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. If $\alpha = 0$ or $\beta = 0$, the result follows from Corollaries 3.5 and 3.6. Therefore we may assume that $\alpha, \beta > 0$. Letting $x = y = 0$ in (3.17), we get that $f(0) = 0$. So by letting $y = 0$ in (3.17), we get $f(3x) = 27f(x)$ for all $x \in E$. Hence by using induction we have

$$f(3^n x) = 27^n f(x)$$

$x \in E$ and all $n \in \mathbb{Z}$. The rest of the proof is similar to the proof of Corollary 3.5. □

Theorem 3.8. *Let $\Phi : E \times E \to [0, \infty)$ be a function such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 27^{np} \Phi^p\Big(\frac{x}{3^n}, 0\Big) < \infty \tag{3.18}$$

and

$$\lim_{n\to\infty} 27^n \Phi\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) = 0 \tag{3.19}$$

for all $x, y \in E$. Suppose that a mapping $f : E \to X$ satisfies the inequality

$$\|Df(x,y)\|_X \le \Phi(x,y) \tag{3.20}$$

for all $x, y \in E$. Then there exists a unique cubic mapping $T : E \to X$ which satisfies Eq. (1.5) *and the inequality*

$$\|T(x) - f(x)\|_X \le \frac{1}{27}\Big[\widetilde{\Phi}(x)\Big]^{\frac{1}{p}} \tag{3.21}$$

for all $x \in E$. The mapping $T : E \to X$ is given by

$$T(x) = \lim_{n\to\infty} 27^n f\Big(\frac{x}{3^n}\Big) \tag{3.22}$$

for all $x \in E$.

Proof. It follows from (3.18) that $\Phi(0,0) = 0$, and therefore (3.20) implies that $f(0) = 0$.

Putting $y = 0$ in (3.20), we have

$$\|f(3x) - 27f(x)\|_X \le \Phi(x, 0) \tag{3.23}$$

for all $x \in E$. Replacing x by $\frac{x}{3^{n+1}}$ in (3.23) and multiplying both sides of (3.23) to 27^n, we get

$$\Big\|27^{n+1} f\Big(\frac{x}{3^{n+1}}\Big) - 27^n f\Big(\frac{x}{3^n}\Big)\Big\|_X \le 27^n \Phi\Big(\frac{x}{3^{n+1}}, 0\Big) \tag{3.24}$$

for all $x \in E$ and all integers $n \ge 0$. Since X is a p-Banach space, we have

$$\begin{aligned}\Big\|\sum_{i=k}^{n}\Big[27^{i+1} f\Big(\frac{x}{3^{i+1}}\Big) - 27^i f\Big(\frac{x}{3^i}\Big)\Big]\Big\|_X^p &\le \sum_{i=k}^{n}\Big\|27^{i+1} f\Big(\frac{x}{3^{i+1}}\Big) - 27^i f\Big(\frac{x}{3^i}\Big)\Big\|_X^p \\ &\le \sum_{i=k}^{n} 27^{ip}\Phi^p\Big(\frac{x}{3^{i+1}}, 0\Big)\end{aligned}$$

for all $x \in E$ and all integers $n \geq k \geq 0$. Hence

$$\left\|27^{n+1}f\left(\frac{x}{3^{n+1}}\right) - 27^k f\left(\frac{x}{3^k}\right)\right\|_X^p \leq \sum_{i=k}^{n} 27^{ip}\Phi^p\left(\frac{x}{3^{i+1}}, 0\right) \tag{3.25}$$

for all $x \in E$ and all integers $n \geq k \geq 0$. It follows from (3.18) and (3.25) that the sequence $\{27^n f(\frac{x}{3^n})\}$ is a Cauchy sequence in X for all $x \in E$. Since X is a p-Banach space, it follows that the sequence $\{27^n f(\frac{x}{3^n})\}$ converges for all $x \in E$. We define $T : E \to X$ by

$$T(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right)$$

for all $x \in E$. Putting $k = 0$ and letting $n \to \infty$ in (3.25), we get (3.21).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.9. *Let θ be a non-negative real number and $\alpha, \beta \in (3, \infty)$. Suppose that a mapping $f : E \to X$ satisfies*

$$\|Df(x,y)\|_X \leq \theta(\|x\|_E^\alpha + \|y\|_E^\beta)$$

for all $x, y \in E$. Then there exists a unique cubic mapping $T : E \to X$ which satisfies Eq. (1.5) *and the inequality*

$$\left\|T(x) - f(x)\right\|_X \leq \frac{\theta}{(3^{\alpha p} - 27^p)^{\frac{1}{p}}}\|x\|_E^\alpha$$

for all $x \in E$. Also, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3T(x)$ for all $x \in E$ and all $t \in \mathbb{R}$.

Proof. Define $\Phi : E \times E \to [0, \infty)$ by $\Phi(x, y) = \theta(\|x\|_E^\alpha + \|y\|_E^\beta)$. Now, apply Theorem 3.8. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, by the same reasoning as in the proof of [28], the cubic mapping $T : E \to X$ satisfies $T(tx) = t^3T(x)$ for all $x \in E$ and all $t \in \mathbb{R}$. □

Corollary 3.10. *Let θ be non-negative real number and let $\beta \in (3, \infty)$. Suppose that a mapping $f : E \to X$ satisfies* (3.10) *for all $x, y \in E$. Then the mapping $f : E \to X$ is cubic. Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $f(tx) = t^3f(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.*

Proof. By the same reasoning as in the proof of Corollary 3.5, we get (3.11). By Theorem 3.8 the mapping $T : E \to X$ defined by

$$T(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right)$$

is cubic. Therefore we have from (3.11) that $f = T$. So the mapping $f : E \to X$ is cubic. The rest of the proof is obvious by Remark 3.4. □

Corollary 3.11. *Let θ be non-negative real number and let $\alpha \in (3, \infty)$ with $\alpha + 1 \neq \log_3 109$. Suppose that a mapping $f : E \to X$ satisfies* (3.12) *for all $x, y \in E$. Then the mapping $f : E \to X$ is cubic. Moreover, if for each fixed*

$x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $f(tx) = t^3 f(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. By the same reasoning as in the proof of Corollary 3.6, we get

$$f_e\Big(\frac{x}{3^n}\Big) = \Big(\frac{3}{109}\Big)^n f_e(x), \qquad f_o\Big(\frac{x}{3^n}\Big) = \frac{1}{27^n} f_o(x)$$

and

$$27^n f\Big(\frac{x}{3^n}\Big) = \Big(\frac{81}{109}\Big)^n f_e(x) + f_o(x) \tag{3.26}$$

for all $x \in E$ and all $n \in \mathbb{Z}$. By Theorem 3.8 the mapping $T : E \to X$ defined by

$$T(x) = \lim_{n\to\infty} 27^n f\Big(\frac{x}{3^n}\Big)$$

is cubic. Therefore we have from (3.26) that $f_o = T$. Hence (3.21) implies that

$$\|f_e(x)\|_X \le M \|x\|_E^\alpha \tag{3.27}$$

for all $x \in E$ where $M = \dfrac{\theta}{(3^{\alpha p} - 27^p)^{\frac{1}{p}}}$. Let $\alpha + 1 < \log_3 109$. Replacing x by $3^n x$ in (3.27), we get

$$\|f_e(x)\|_X \le M \Big(\frac{3^{1+\alpha}}{109}\Big)^n \|x\|_E^\alpha \tag{3.28}$$

for all $x \in E$ and all integers $n \ge 1$. Letting $n \to \infty$ in (3.28), we get that $f_e(x) = 0$ for all $x \in E$. Similarly, we get $f_e(x) = 0$ for all $x \in E$ when $\alpha + 1 > \log_3 109$. So $f = f_o = T$. Therefore the mapping $f : E \to X$ is cubic.

The rest of the proof is obvious by Remark 3.4. □

Corollary 3.12. *Let θ be non-negative real number and let $\alpha = \log_3 109 - 1$. Suppose that a mapping $f : E \to X$ satisfies (3.12) for all $x, y \in E$. Then there exists a unique cubic mapping $T : E \to X$ which satisfies Eq. (1.5) and the inequality*

$$\Big\|T(x) - f(x)\Big\|_X \le \frac{3\theta}{28} \|x\|_E^\alpha \tag{3.29}$$

for all $x \in E$. Also, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3 T(x)$ for all $x \in E$ and all $t \in \mathbb{R}$.

Proof. In the proof of Corollary 3.11, we showed that $f_o = T$ and

$$f_e(3x) = \frac{109}{3} f_e(x), \qquad f_o(3x) = 27 f_o(x)$$

for all $x \in E$. Hence by letting $y = 0$ in (3.12), we get

$$\|f_e(x)\|_X \le \frac{3\theta}{28} \|x\|_E^\alpha$$

for all $x \in E$. Since $f_e = f - f_o = f - T$, then the requirement inequality is proved. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, by the same reasoning as in the proof of [28], the cubic mapping $T : E \to X$ satisfies $T(tx) = t^3T(x)$ for all $x \in E$ and all $t \in \mathbb{R}$. $\square$

Remark 3.13. If we apply Theorem 3.8 in the proof of Corollary 3.12, we know that there exists a unique cubic mapping $T : E \to X$ satisfies Eq. (1.5) and the inequality

$$(\diamondsuit) \qquad \left\| T(x) - f(x) \right\|_X \le \frac{3\theta}{(109^p - 81^p)^{\frac{1}{p}}} \|x\|_E^{\alpha}$$

for all $x \in E$ where $\alpha = \log_3 109 - 1$. But we have a better possible upper bound (3.29) than that of the inequality ($\diamondsuit$).

Corollary 3.14. *Let θ, α, β be non-negative real numbers such that $\alpha + \beta \in (3, \infty)$. Suppose that a mapping $f : E \to X$ satisfies (3.17) for all $x, y \in E$. Then the mapping $f : E \to X$ is cubic.*

Moreover, if for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $T(tx) = t^3T(x)$ for all $t \in \mathbb{R}$ and all $x \in E$.

Proof. The result follows by the same reasoning as in the proof of Corollary 3.10. $\square$

4. Stability in quasi-Banach B-modules

In this section, let B be a unital p-Banach space with norm $|.|$ and $B_1 := \{ u \in B : |u| = 1 \}$, and let $\mathbb{X}$ be a quasi left B-module with norm $\|\cdot\|_{\mathbb{X}}$ and $\mathbb{Y}$ be a p-Banach left B-module with norm $\|\cdot\|_{\mathbb{Y}}$. A cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ is called B-cubic if $T(ax) = a^3T(x)$ for all $a \in B$ and all $x \in \mathbb{X}$.

Theorem 4.1. *Suppose that a mapping $f : \mathbb{X} \to \mathbb{Y}$ satisfies*

$$(4.1) \qquad \begin{aligned} &\Big\| 3f(ax + 3ay) + f(3ax - ay) \\ &\quad - 15a^3 f(x + y) - 15a^3 f(x - y) - 80a^3 f(y) \Big\|_{\mathbb{Y}} \le \varphi(x, y) \end{aligned}$$

and $\varphi : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is a mapping satisfying the conditions

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{27^{np}} \varphi^p(3^n x, 0) < \infty, \qquad \lim_{n\to\infty} \frac{1}{27^n} \varphi(3^n x, 3^n y) = 0$$

for all $a \in B_1$ and all $x, y \in \mathbb{X}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then there exists a unique B-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ which satisfies Eq. (1.5) and the inequality

$$(4.2) \qquad \left\| T(x) - f(x) - \frac{40}{13} f(0) \right\|_{\mathbb{Y}} \le \frac{1}{27} \Big[\widetilde{\varphi}(x) \Big]^{\frac{1}{p}}$$

for all $x \in \mathbb{X}$.

Proof. By Theorem 3.1, it follows from the inequality (4.1) for $a = 1 \in B_1$ that there exists a unique cubic mapping $T : \mathbb{X} \to \mathbb{Y}$, defined by $T(x) = \lim_{n\to\infty} \frac{1}{27^n} f(3^n x)$, which satisfies Eq. (1.5) and the inequality (4.2) for all $x \in \mathbb{X}$. It follows by definition of T and (4.1) that

$$3T(ax+3ay)+T(3ax-ay) = 15a^3T(x+y)+15a^3T(x-y)+80a^3T(y) \tag{4.3}$$

for all $x, y \in \mathbb{X}$ and all $a \in B_1$. Since T is cubic, setting $y = 0$ in (4.3), we get $T(ax) = a^3T(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1$. The last relation is also true for $a = 0$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, by the same reasoning as in the proof of [28], the cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ satisfies $T(tx) = t^3T(x)$ for all $t \in \mathbb{R}$. That is, T is $\mathbb{R}$-cubic.

For each element $a \in B\,(a \neq 0)$, $a = |a| \cdot \frac{a}{|a|}$. Since T is $\mathbb{R}$-cubic and $T(ax) = a^3T(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1$, then we have

$$T(ax) = T\Big(|a| \cdot \frac{a}{|a|}x\Big) = |a|^3 T\Big(\frac{a}{|a|}x\Big) = |a|^3 \cdot \frac{a^3}{|a|^3} \cdot T(x) = a^3T(x)$$

for all $x \in \mathbb{X}$ and all $a \in B\,(a \neq 0)$. So the unique $\mathbb{R}$-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ is also B-cubic. This completes the proof of the theorem. □

The following theorem is an alternative result of Theorem 4.1.

Theorem 4.2. *Suppose that a mapping* $f : \mathbb{X} \to \mathbb{Y}$ *satisfies*

$$\begin{aligned}\Big\| 3a^3f(x+3y) &+ a^3f(3x-y)\\ &- 15f(ax+ay) - 15f(ax-ay) - 80f(ay)\Big\|_{\mathbb{Y}} \le \varphi(x,y)\end{aligned} \tag{4.4}$$

and $\varphi : \mathbb{X} \times \mathbb{X} \to [0,\infty)$ *is a mapping satisfying the conditions*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{27^{np}} \varphi^p(3^n x, 0) < \infty, \qquad \lim_{n\to\infty} \frac{1}{27^n}\varphi(3^n x, 3^n y) = 0$$

for all $a \in B_1$ *and all* $x, y \in \mathbb{X}$*. If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathbb{X}$*, then there exists a unique* B*-cubic mapping* $T : \mathbb{X} \to \mathbb{Y}$ *which satisfies Eq.* (1.5) *and the inequality* (4.2).

Theorem 4.3. *Suppose that a mapping* $f : \mathbb{X} \to \mathbb{Y}$ *satisfies* (4.1) *and* $\Phi : \mathbb{X} \times \mathbb{X} \to [0,\infty)$ *is a mapping satisfying the conditions*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 27^{np}\Phi^p\Big(\frac{x}{3^n}, 0\Big) < \infty, \qquad \lim_{n\to\infty} 27^n\Phi\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) = 0$$

for all $a \in B_1$ *and all* $x, y \in \mathbb{X}$*. If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathbb{X}$*, then there exists a unique* B*-cubic mapping* $T : \mathbb{X} \to \mathbb{Y}$ *which satisfies Eq.* (1.5) *and the inequality*

$$\|T(x) - f(x)\|_{\mathbb{Y}} \le \frac{1}{27}\Big[\widetilde{\Phi}(x)\Big]^{\frac{1}{p}} \tag{4.5}$$

for all $x \in \mathbb{X}$.

Proof. By Theorem 3.8, it follows from the inequality (4.1) for $a = 1 \in B_1$ that there exists a unique cubic mapping $T : \mathbb{X} \to \mathbb{Y}$, defined by $T(x) = \lim_{n\to\infty} 27^n f(\frac{x}{3^n})$, which satisfies Eq. (1.5) and the inequality (4.5) for all $x \in \mathbb{X}$. It follows by definition of T and (4.1) that T satisfies (4.3) for all $x, y \in \mathbb{X}$ and all $a \in B_1$. Since T is cubic, setting $y = 0$ in (4.3), we get $T(ax) = a^3 T(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1$. The last relation is also true for $a = 0$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, by the same reasoning as in the proof of [28], the cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ satisfies $T(tx) = t^3 T(x)$ for all $t \in \mathbb{R}$. That is, T is $\mathbb{R}$-cubic.

The rest of the proof is similar to the proof of Theorem 4.1. □

The following theorem is an alternative result of Theorem 4.3.

Theorem 4.4. *Suppose that a mapping $f : \mathbb{X} \to \mathbb{Y}$ satisfies* (4.4) *and $\Phi : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is a mapping satisfying the conditions*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 27^{np} \Phi^p\Big(\frac{x}{3^n}, 0\Big) < \infty, \qquad \lim_{n\to\infty} 27^n \Phi\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) = 0$$

for all $a \in B_1$ and all $x, y \in \mathbb{X}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then there exists a unique B-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ which satisfies Eq. (1.5) *and the inequality* (4.5).

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