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HYERS-ULAM-RASSIAS STABILITY OF A SYSTEM OF FIRST ORDER LINEAR RECURRENCES

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Abstract. In this paper we discuss the Hyers-Ulam-Rassias stability of a system of first order linear recurrences with variable coefficients in Banach spaces. The concept of the Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300. As an application, the Hyers-Ulam-Rassias stability of a p-order linear recurrence with variable coefficients is proved.

1. Introduction

Hyers-Ulam stability is a basic sense of stability for functional equations. In 1940, S. M. Ulam proposed the following problem:

Given a group G, a metric group $(G^{'}, \cdot, \varrho)$ and a positive number ε, does there exist a constant $\delta > 0$ depending on ε such that, if $f : G \to G^{'}$ satisfies the inequality $\varrho(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G$, then there exists a homomorphism $\varphi : G \to G^{'}$ such that $\varrho(f(x), \varphi(x)) \leq \varepsilon$ for all $x \in G$?

In case of a positive answer to the previous problem, we usually say that the Cauchy functional equation $\varphi(xy) = \varphi(x)\varphi(y)$ is stable. In 1941, D. H. Hyers [5] first proved that the Cauchy equation $f(x + y) = f(x) + f(y)$ is stable in Banach spaces. Later, the Hyers-Ulam stability was studied extensively and also, the notion of the original Ulam problem was generalized (see, e.g., [4, 6, 11, 12, 14]). Most of the papers discussed the stability of the continuous functions in several variables. In 2005, the discrete case for equations in single variable was investigated by D. Popa [9, 10], concretely to say, first the Hyers-Ulam-Rassias stability of the first order linear recurrence

$$x_{n+1} = a_n x_n + b_n, \tag{1.1}$$

was studied in [9], then the obtained results have been generalized to a p-order linear recurrence with constant coefficients [10]

$$x_{n+p} = a_1 x_{n+p-1} + \cdots + a_p x_n + b_n. \tag{1.2}$$

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In this paper we discuss the Hyers-Ulam-Rassias stability of a system of first order linear recurrences with variable coefficients

(1.3)
$$\begin{cases} x_1(n+1) = a_{11}(n)x_1(n) + a_{12}(n)x_2(n) + \cdots + a_{1p}(n)x_p(n) + b_1(n), \\ x_2(n+1) = a_{21}(n)x_1(n) + a_{22}(n)x_2(n) + \cdots + a_{2p}(n)x_p(n) + b_2(n), \\ \quad\vdots \\ x_p(n+1) = a_{p1}(n)x_1(n) + a_{p2}(n)x_2(n) + \cdots + a_{pp}(n)x_p(n) + b_p(n). \end{cases}$$

As an application, the Hyers-Ulam-Rassias stability of a p-order linear recurrence with variable coefficients

$$x_{n+p} = a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n, \tag{1.4}$$

is proved.

2. Stability of the system (1.3)

In what follows we denote by K the field C of complex numbers or the field R of real numbers, N the positive integers, N_0 the nonnegative integers and $K^{p\times p}$ the vector space consisting of all $p \times p$ matrices. Let $(E, \|\cdot\|)$ be a Banach space over K, Ω is a product space of E, $X = (x_1, x_2, \ldots, x_p)^T \in \Omega$ (T denotes transposition), where $x_i \in E (i = 1, 2, \ldots, p)$. On Ω, we get the norm as $\|X\|_\infty = \max_{1\le i\le p} \|x_i\|$, then it can be easily checked that $(\Omega, \|\cdot\|_\infty)$ is also a Banach space over K.

Matrix $A = (a_{ij})_{p\times p} \in K^{p\times p}$ acting on $X \in \Omega$ can be regarded as a linear operator $A : \Omega \to \Omega$, then one can verify that $\|A\|_\infty = \max_{1\le i\le p} \sum_{j=1}^{p} |a_{ij}|$ being subject to the vector norm $\|\cdot\|_\infty$. For any $X \in \Omega$, $A \in K^{p\times p}$, $B \in K^{p\times p}$, we have $\|AX\|_\infty \le \|A\|_\infty \|X\|_\infty$, $\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$.

For each $n \in N_0$, define

$$X_n := \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_p(n) \end{pmatrix}, \quad A_n := \begin{pmatrix} a_{11}(n) & a_{12}(n) & \cdots & a_{1p}(n) \\ a_{21}(n) & a_{22}(n) & \cdots & a_{2p}(n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}(n) & a_{p2}(n) & \cdots & a_{pp}(n) \end{pmatrix},$$
$$B_n := \begin{pmatrix} b_1(n) \\ b_2(n) \\ \vdots \\ b_p(n) \end{pmatrix}.$$

Then the system (1.3) can be rewritten as

$$X_{n+1} = A_n X_n + B_n, \quad n \in N_0. \tag{2.5}$$

Lemma 1. *If the sequence $(X_n)_{n\in N_0}$ satisfies the relation (2.5), then*

$$(2.6)\quad X_n = A_{n-1}A_{n-2}\cdots A_0X_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_kB_{k-1} + B_{n-1},\ n \geq 2.$$

Proof. It can be easily proved by induction. For $n = 2$ the relation (2.6) becomes

$$X_2 = A_1A_0X_0 + A_1B_0 + B_1,$$

which is true according to the relation (2.5). Suppose that (2.6) is true for a fixed $n \geq 2$, we have to prove that

$$(2.7)\quad X_{n+1} = A_nA_{n-1}\cdots A_0X_0 + \sum_{k=1}^{n} A_nA_{n-1}\cdots A_kB_{k-1} + B_n.$$

In virtue of the relation (2.5) and (2.6), it follows that

$$\begin{aligned} &X_{n+1} \\ = \;& A_n\left(A_{n-1}A_{n-2}\cdots A_0X_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_kB_{k-1} + B_{n-1}\right) + B_n \\ = \;& A_nA_{n-1}\cdots A_0X_0 + \sum_{k=1}^{n} A_nA_{n-1}\cdots A_kB_{k-1} + B_n. \end{aligned}$$

Hence, the relation (2.7) is true for every $n \geq 2$. □

Theorem 1. *Let $A_n \in K^{p\times p}$ be a nonzero matrix for every $n \in N_0$, $(\varepsilon_n)_{n\in N_0}$ be a sequence of positive numbers with the property $\liminf \frac{\varepsilon_n}{\varepsilon_{n-1}\|A_n\|_\infty} > 1$. If the sequence $(Y_n)_{n\in N_0}$ in Ω satisfies the inequality*

$$(2.8)\qquad \|Y_{n+1} - A_nY_n - B_n\|_\infty \leq \varepsilon_n, \quad n \in N_0,$$

then there exists a sequence $(X_n)_{n\in N_0}$ in Ω given by (2.5) and a positive constant L_0 such that

$$(2.9)\qquad \|X_n - Y_n\|_\infty \leq L_0\varepsilon_{n-1}, \quad n \in N.$$

Proof. If the sequence $(Y_n)_{n\in N_0}$ in Ω satisfies the inequality (2.8), then we can denote that

$$(2.10)\qquad Y_{n+1} = A_nY_n + B_n + C_n, \quad n \in N_0,$$

where $C_n = (c_1(n), c_2(n), \ldots, c_p(n))^T \in \Omega$, $\|C_n\|_\infty \leq \varepsilon_n$ for every $n \in N_0$. From Lemma 1, the general solution of (2.10) can be obtained as
(2.11)

$$Y_n = A_{n-1}\cdots A_0Y_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_k(B_{k-1} + C_{k-1}) + (B_{n-1} + C_{n-1}).$$

Define the sequence $(X_n)_{n\in N_0}$ in Ω given by (2.5) with $X_0 := Y_0$, then

$$\begin{aligned}(2.12)\qquad &\|X_n - Y_n\|_\infty \\ = \ & \|A_{n-1}\cdots A_0X_0 + \sum_{k=1}^{n-1} A_{n-1}\cdots A_kB_{k-1} + B_{n-1} - A_{n-1}\cdots A_0Y_0 \\ & - \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_k(B_{k-1}+C_{k-1}) - (B_{n-1}+C_{n-1})\|_\infty \\ = \ & \|\sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_kC_{k-1} + C_{n-1}\|_\infty \\ \leq \ & \sum_{k=1}^{n-1} \|A_{n-1}A_{n-2}\cdots A_k\|_\infty \varepsilon_{k-1} + \varepsilon_{n-1}.\end{aligned}$$

Taking account of the condition $\liminf \frac{\varepsilon_n}{\varepsilon_{n-1}\|A_n\|_\infty} > 1$, there exists a constant q_0 and $n_0 \in N$ such that

$$\frac{\varepsilon_n}{\varepsilon_{n-1}\|A_n\|_\infty} \geq q_0 > 1 \quad \text{for all } n \geq n_0, \tag{2.13}$$

which implies that

$$\begin{aligned}\frac{\varepsilon_{n-1}}{\varepsilon_{k-1}\|A_{n-1}A_{n-2}\cdots A_k\|_\infty} &\geq \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}\|A_{n-1}\|_\infty}\frac{\varepsilon_{n-2}}{\varepsilon_{n-3}\|A_{n-2}\|_\infty}\cdots\frac{\varepsilon_k}{\varepsilon_{k-1}\|A_k\|_\infty} \\ (2.14)\qquad &\geq q_0^{n-k}, \quad n > k \geq n_0.\end{aligned}$$

Hence for every $n > n_0$, we have

$$\begin{aligned}& \sum_{k=1}^{n-1} \|A_{n-1}A_{n-2}\cdots A_k\|_\infty\varepsilon_{k-1} + \varepsilon_{n-1} \\ = \ & \sum_{k=1}^{n_0-1} \|A_{n-1}\cdots A_k\|_\infty\varepsilon_{k-1} + \sum_{k=n_0}^{n-1} \|A_{n-1}\cdots A_k\|_\infty\varepsilon_{k-1} + \varepsilon_{n-1} \\ \leq \ & \|A_{n-1}A_{n-2}\cdots A_{n_0}\|_\infty \sum_{k=1}^{n_0-1} \|A_{n_0-1}A_{n_0-2}\cdots A_k\|_\infty\varepsilon_{k-1} \\ & + \sum_{k=n_0}^{n-1} \|A_{n-1}A_{n-2}\cdots A_k\|_\infty\varepsilon_{k-1} + \varepsilon_{n-1} \\ \leq \ & \frac{\varepsilon_{n-1}}{\varepsilon_{n_0-1}q_0^{n-n_0}} \sum_{k=1}^{n_0-1} \|A_{n_0-1}A_{n_0-2}\cdots A_k\|_\infty\varepsilon_{k-1} + \sum_{k=n_0}^{n-1} \frac{\varepsilon_{n-1}}{q_0^{n-k}} + \varepsilon_{n-1} \\ < \ & (\frac{1}{\varepsilon_{n_0-1}} \sum_{k=1}^{n_0-1} \|A_{n_0-1}A_{n_0-2}\cdots A_k\|_\infty\varepsilon_{k-1} + \frac{1}{q_0-1} + 1)\varepsilon_{n-1}.\end{aligned}$$

In view of (2.12), it is enough to take

$$L_0 \geq \frac{1}{\varepsilon_{n_0-1}} \sum_{k=1}^{n_0-1} ||A_{n_0-1}A_{n_0-2}\cdots A_k||_\infty \varepsilon_{k-1} + \frac{1}{q_0-1} + 1,$$

and such that $||X_n - Y_n||_\infty \leq L_0\varepsilon_{n-1}$ for $1 \leq n \leq n_0$. □

Theorem 2. *Let $A_n \in K^{p\times p}$ be an invertible matrix for every $n \in N_0$, $(\varepsilon_n)_{n\in N_0}$ be a sequence of positive numbers with the property*

$$\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} ||A_n^{-1}||_\infty < 1.$$

If the sequence $(Y_n)_{n\in N_0}$ in Ω satisfies the inequality

$$||Y_{n+1} - A_nY_n - B_n||_\infty \leq \varepsilon_n, \quad n \in N_0, \tag{2.15}$$

then there exists a sequence $(X_n)_{n\in N_0}$ in Ω given by (2.5) and a positive constant L_1 such that

$$||X_n - Y_n||_\infty \leq L_1\varepsilon_{n-1}, \quad n \in N. \tag{2.16}$$

Proof. If the sequence $(Y_n)_{n\in N_0}$ in Ω satisfies the inequality (2.15), then we can also denote that $(Y_n)_{n\in N_0}$ satisfies the relation (2.10). Then we discuss the convergence of the series $\sum_{n=1}^{+\infty} A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}C_{n-1}$. The series

$$\sum_{n=1}^{+\infty} ||A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}||_\infty \varepsilon_{n-1}$$

is convergent in virtue of D'Alembert test since

$$\limsup \frac{||A_0^{-1}A_1^{-1}\cdots A_n^{-1}||_\infty \varepsilon_n}{||A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}||_\infty \varepsilon_{n-1}} \leq \limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} ||A_n^{-1}||_\infty < 1.$$

So, the series $\sum_{n=1}^{+\infty} A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}C_{n-1}$ is also convergent by considering that

$$||A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}C_{n-1}||_\infty \leq ||A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}||_\infty \varepsilon_{n-1}.$$

Define $S := \sum_{n=1}^{+\infty} A_0^{-1}A_1^{-1}\cdots A_{n-1}^{-1}C_{n-1} \in \Omega$, and $(X_n)_{n\in N_0}$ in Ω given by (2.5) with $X_0 := Y_0 + S$, then

$$X_n = A_{n-1}A_{n-2}\cdots A_0(Y_0 + S) + \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_kB_{k-1} + B_{n-1}. \tag{2.17}$$

By (2.11) and (2.17), we have

$$(2.18)\quad \begin{aligned} & \|X_n - Y_n\|_\infty \\ = & \|A_{n-1}A_{n-2}\cdots A_0 S - \sum_{k=1}^{n-1} A_{n-1}A_{n-2}\cdots A_k C_{k-1} - C_{n-1}\|_\infty \\ = & \|A_{n-1}\cdots A_0(\sum_{k=1}^{+\infty} A_0^{-1}\cdots A_{k-1}^{-1}C_{k-1} - \sum_{k=1}^{n} A_0^{-1}\cdots A_{k-1}^{-1}C_{k-1})\|_\infty \\ = & \|A_{n-1}A_{n-2}\cdots A_0 \sum_{k=n+1}^{+\infty} A_0^{-1}A_1^{-1}\cdots A_{k-1}^{-1}C_{k-1}\|_\infty \\ = & \|\sum_{k=0}^{+\infty} A_n^{-1}A_{n+1}^{-1}\cdots A_{n+k}^{-1}C_{n+k}\|_\infty \\ \le & \sum_{k=0}^{+\infty} \|A_n^{-1}\|_\infty \|A_{n+1}^{-1}\|_\infty \cdots \|A_{n+k}^{-1}\|_\infty \varepsilon_{n+k}. \end{aligned}$$

From the condition $\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}}\|A_n^{-1}\|_\infty < 1$, there exists $n_1 \in N$ and a constant $q_1 < 1$ for all $n \ge n_1$, we have

$$(2.19)\qquad \frac{\varepsilon_n}{\varepsilon_{n-1}}\|A_n^{-1}\|_\infty \le q_1 < 1,$$

which implies that

$$(2.20)\quad \begin{aligned} \|A_n^{-1}\|_\infty \cdots \|A_{n+k}^{-1}\|_\infty \varepsilon_{n+k} & \le q_1\|A_n^{-1}\|_\infty \cdots \|A_{n+k-1}^{-1}\|_\infty \varepsilon_{n+k-1} \\ & \le q_1^2\|A_n^{-1}\|_\infty \cdots \|A_{n+k-2}^{-1}\|_\infty \varepsilon_{n+k-2} \\ & \;\;\vdots \\ & \le q_1^{k+1}\varepsilon_{n-1}, \quad k \ge 0. \end{aligned}$$

Therefore, by (2.18) and (2.20), we have

$$\|X_n - Y_n\|_\infty \le \varepsilon_{n-1}\sum_{k=0}^{+\infty} q_1^{k+1} = \frac{q_1}{1-q_1}\varepsilon_{n-1}, \quad n \ge n_1.$$

So it is enough to take $L_1 \ge \frac{q_1}{1-q_1}$ and such that $\|X_n - Y_n\|_\infty \le L_1\varepsilon_{n-1}$ for all $1 \le n < n_1$. □

3. Stability of the recurrence (1.4)

In what follows we give applications of Theorem 1 and 2 to the Hyers-Ulam-Rassias stability of the p-order recurrence (1.4).

Corollary 1. *Let E be a Banach space over K, $(b_n)_{n\in N_0}$ be a sequence in E, and $(\varepsilon_n)_{n\in N_0}$ be a sequence of positive numbers with the property* $\liminf \frac{\varepsilon_n}{\varepsilon_{n-1}f_n} > 1$

where $f_n = \max\{1, |a_0(n)| + \cdots + |a_{p-1}(n)|\}$. *If the sequence* $(y_n)_{n\in N_0}$ *in* E *satisfies the inequality*

$$\|y_{n+p} - (a_{p-1}(n)y_{n+p-1} + \cdots + a_0(n)y_n + b_n)\| \le \varepsilon_n, \quad n \in N_0, \tag{3.21}$$

then there exists a sequence $(x_n)_{n\in N_0}$ *in* E *given by* (1.4) *and a positive constant* L_2 *such that*

$$\|x_n - y_n\| \le L_2\varepsilon_{n-1}, \quad n \in N. \tag{3.22}$$

Proof. Let $x_1(n) := x_{n+p-1}, x_2(n) := x_{n+p-2}, \ldots, x_p(n) := x_n$, then the recurrence (1.4) is equivalent to the system (2.5) with $B_n := (b_n, 0, \ldots, 0)^T$ and

$$A_n := \begin{pmatrix} a_{p-1}(n) & a_{p-2}(n) & a_{p-3}(n) & \cdots & a_1(n) & a_0(n) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

It is easy to see that the property $\liminf \frac{\varepsilon_n}{\varepsilon_{n-1}f_n} > 1$ is equivalent to the condition $\liminf \frac{\varepsilon_n}{\varepsilon_{n-1}\|A_n\|_\infty} > 1$ and the inequality (3.21) is equivalent to the inequality (2.8) with $Y_n := (y_{n+p-1}, y_{n+p-2}, \ldots, y_n)^T$. So, according to Theorem 1, there exists a sequence $(X_n)_{n\in N_0}$ given by (2.5) and a positive constant L_2 such that $\|X_n - Y_n\|_\infty \le L_2\varepsilon_{n-1}, n \in N$.

Let $x_n := p_1(X_n)$ for $X_n \in \Omega$ and $n \in N_0$, where $p_1 : \Omega \to E$ is given by: $p_1(z_1, z_2, \ldots, z_p)^T := z_p$. Clearly x_n is a solution of (1.4) and (3.22) holds. □

Corollary 2. *Let* E *be a Banach space over* K, $(b_n)_{n\in N_0}$ *be a sequence in* E, *and* $(\varepsilon_n)_{n\in N_0}$ *be a sequence of positive numbers with the property* $\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}}e_n < 1$ *where* $e_n = \max\{1, \frac{1+|a_1(n)|+\cdots+|a_{p-1}(n)|}{|a_0(n)|}\}$. *If the sequence* $(y_n)_{n\in N_0}$ *in* E *satisfies the inequality*

$$\|y_{n+p} - (a_{p-1}(n)y_{n+p-1} + \cdots + a_0(n)y_n + b_n)\| \le \varepsilon_n, \quad n \in N_0, \tag{3.23}$$

then there exists a sequence $(x_n)_{n\in N_0}$ *in* E *given by* (1.4) *and a positive constant* L_3 *such that*

$$\|x_n - y_n\| \le L_3\varepsilon_{n-1}, \quad n \in N.$$

Proof. The recurrence (1.4) is equivalent to the system (2.5) with X_n, B_n, A_n defined as in Corollary 1 and then

$$A_n^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{1}{a_0(n)} & \frac{-a_{p-1}(n)}{a_0(n)} & \frac{-a_{p-2}(n)}{a_0(n)} & \cdots & \frac{-a_1(n)}{a_0(n)} \end{pmatrix}.$$

It is easy to see that the property $\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} e_n < 1$ is equivalent to the condition $\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} \|A_n^{-1}\|_\infty < 1$. Thus our result can be obtained by applying Theorem 2. □

Example 1. Let $(\varepsilon_n)_{n\in N_0}$ be a sequence of positive numbers with $\varepsilon_n = \frac{1}{3^n}$ and consider the system

$$\begin{cases} x_1(n+1) = 2nx_1(n) + (2n+1)x_2(n) + b_1(n), \\ x_2(n+1) = (n-1)x_1(n) + (n+1)x_2(n) + b_2(n). \end{cases} \tag{3.24}$$

Suppose that $Y_n = (y_1(n), y_2(n))^T$ in Ω is an approximate solution of system (3.24) satisfying

$$\|Y_{n+1} - A_nY_n - B_n\|_\infty \le \varepsilon_n, \quad n \in N_0,$$

where

$$A_n = \begin{pmatrix} 2n & 2n+1 \\ n-1 & n+1 \end{pmatrix}, \quad B_n = \begin{pmatrix} b_1(n) \\ b_2(n) \end{pmatrix}.$$

Then there exists a solution $X_n = (x_1(n), x_2(n))^T$ of (3.24) such that

$$\|X_n - Y_n\|_\infty \le 2\varepsilon_{n-1}, \quad n \in N.$$

Proof. For all $n \in N$, we have $\frac{\varepsilon_n}{\varepsilon_{n-1}} \|A_n^{-1}\|_\infty = \frac{1}{3}\frac{3n+2}{3n+1} \le \frac{2}{3} < 1$. By Theorem 2, it follows that $\|X_n - Y_n\|_\infty \le \frac{\frac{2}{3}}{1-\frac{2}{3}}\varepsilon_{n-1} = 2\varepsilon_{n-1}, n \in N$. □

Example 2. Let $(\varepsilon_n)_{n\in N_0}$ be a sequence of positive numbers with $\varepsilon_n = \frac{1}{2^n}$ and consider the recurrence

$$x_{n+2} = \frac{2n+3}{n+2}x_{n+1} + \frac{2n+1}{n+1}x_n + b_n. \tag{3.25}$$

Suppose that $(y_n)_{n\in N_0}$ is a sequence in E satisfying the inequality

$$\left\| y_{n+2} - \frac{2n+3}{n+2}y_{n+1} - \frac{2n+1}{n+1}y_n - b_n \right\| \le \varepsilon_n, \quad n \in N_0.$$

Then there exists a solution $(x_n)_{n\in N_0}$ of (3.25) such that

$$\|x_n - y_n\| \le 8\varepsilon_{n-1}, \quad n \in N.$$

Proof. For all $n \in N$, we have

$$e_n = \max\left\{1, \frac{1 + |\frac{2n+3}{n+2}|}{|\frac{2n+1}{n+1}|}\right\} = \max\left\{1, \frac{(3n+5)(n+1)}{(2n+1)(n+2)}\right\} \le \frac{16}{9}.$$

which implies that $\frac{\varepsilon_n}{\varepsilon_{n-1}} e_n \le \frac{8}{9} < 1$ for all $n \in N$. By Corollary 2, it follows that $\|x_n - y_n\|_\infty \le \frac{\frac{8}{9}}{1-\frac{8}{9}}\varepsilon_{n-1} = 8\varepsilon_{n-1}$, $n \in N$. □

References

[1] R. P. Agarwal, B. Xu, and W. Zhang, *Stability of functional equations in single variable*, J. Math. Anal. Appl. **288** (2003), no. 2, 852–869.

[2] G. L. Forty, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.

[3] ———, *Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl. **295** (2004), no. 1, 127–133.

[4] R. Ger, *Superstability is not natural*, In Report of the twenty-sixth International Symposium on Functional Equations, Aequationes Math. **37** (1989), 68.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[6] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), no. 2-3, 125–153.

[7] Y. H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), no. 1, 305–315.

[8] K. Nikodem, *The stability of the Pexider equation*, Ann. Math. Sil. No. **5** (1991), 91–93.

[9] D. Popa, *Hyers-Ulam-Rassias stability of a linear recurrence*, J. Math. Anal. Appl. **309** (2005), no. 2, 591–597.

[10] ———, *Hyers-Ulam stability of the linear recurrence with constant coefficients*, Adv. Difference Equ. **2005** (2005), no. 2, 101–107.

[11] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[12] L. Székelyhidi, *The stability of the sine and cosine functional equations*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 109–115.

[13] J. Tabor, *On functions behaving like additive functions*, Aequationes Math. **35** (1988), no. 2-3, 164–185.

[14] T. Trif, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), no. 1-2, 47–61.

[15] S. M. Ulam, *Problems in modern mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

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