HYERS-ULAM-RASSIAS STABILITY OF A SYSTEM OF FIRST ORDER LINEAR RECURRENTS

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ABSTRACT. In this paper we discuss the Hyers-Ulam-Rassias stability of a system of first order linear recurrences with variable coefficients in Banach spaces. The concept of the Hyers-Ulam-Rassias stability originated from Th. M. Rassias’ stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300. As an application, the Hyers-Ulam-Rassias stability of a p-order linear recurrence with variable coefficients is proved.

1. Introduction

Hyers-Ulam stability is a basic sense of stability for functional equations. In 1940, S. M. Ulam proposed the following problem:

Given a group G, a metric group \((G, \cdot, g)\) and a positive number \(\varepsilon\), does there exist a constant \(\delta > 0\) depending on \(\varepsilon\) such that, if \(f: G \to G\) satisfies the inequality \(d(f(xy), f(x)f(y)) \leq \delta\) for all \(x, y \in G\), then there exists a homomorphism \(\varphi: G \to G\) such that \(d(f(x), \varphi(x)) \leq \varepsilon\) for all \(x \in G\)?

In case of a positive answer to the previous problem, we usually say that the Cauchy functional equation \(\varphi(xy) = \varphi(x)\varphi(y)\) is stable. In 1941, D. H. Hyers [5] first proved that the Cauchy equation \(f(x + y) = f(x) + f(y)\) is stable in Banach spaces. Later, the Hyers-Ulam stability was studied extensively and also, the notion of the original Ulam problem was generalized (see, e.g., [4, 6, 11, 12, 14]). Most of the papers discussed the stability of the continuous functions in several variables. In 2005, the discrete case for equations in single variable was investigated by D. Popa [9, 10], concretely to say, first the Hyers-Ulam-Rassias stability of the first order linear recurrence

\[
x_{n+1} = a_n x_n + b_n,
\]

(1.1)

was studied in [9], then the obtained results have been generalized to a p-order linear recurrence with constant coefficients [10]

\[
x_{n+p} = a_1 x_{n+p-1} + \cdots + a_p x_n + b_n.
\]

(1.2)
In this paper we discuss the Hyers-Ulam-Rassias stability of a system of first order linear recurrences with variable coefficients

\begin{equation}
\begin{aligned}
x_1(n + 1) &= a_{11}(n)x_1(n) + a_{12}(n)x_2(n) + \cdots + a_{1p}(n)x_p(n) + b_1(n), \\
x_2(n + 1) &= a_{21}(n)x_1(n) + a_{22}(n)x_2(n) + \cdots + a_{2p}(n)x_p(n) + b_2(n), \\
&\vdots \\
x_p(n + 1) &= a_{p1}(n)x_1(n) + a_{p2}(n)x_2(n) + \cdots + a_{pp}(n)x_p(n) + b_p(n).
\end{aligned}
\end{equation}

As an application, the Hyers-Ulam-Rassias stability of a $p$-order linear recurrence with variable coefficients

\begin{equation}
x_{n+p} = a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n,
\end{equation}

is proved.

2. Stability of the system (1.3)

In what follows we denote by $K$ the field $C$ of complex numbers or the field $R$ of real numbers, $N$ the positive integers, $N_0$ the nonnegative integers and $K^{p \times p}$ the vector space consisting of all $p \times p$ matrices. Let $(E, \| \cdot \|)$ be a Banach space over $K$, $\Omega$ is a product space of $E$, $X = (x_1, x_2, \ldots, x_p)^T \in \Omega$ (T denotes transposition), where $x_i \in E (i = 1, 2, \ldots, p)$. On $\Omega$, we get the norm as $\|X\|_\infty = \max_{1 \leq i \leq p} \|x_i\|$, then it can be easily checked that $(\Omega, \| \cdot \|_\infty)$ is also a Banach space over $K$.

Matrix $A = (a_{ij})_{p \times p} \in K^{p \times p}$ acting on $X \in \Omega$ can be regarded as a linear operator $A : \Omega \to \Omega$, then one can verify that $\|A\|_\infty = \max_{1 \leq i \leq p} \sum_{j=1}^{p} |a_{ij}|$ being subject to the vector norm $\| \cdot \|_\infty$. For any $X \in \Omega, A \in K^{p \times p}, B \in K^{p \times p}$, we have $\|AX\|_\infty \leq \|A\|_\infty \|X\|_\infty, \|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$.

For each $n \in N_0$, define

\[
X_n := \begin{pmatrix} x_1(n) \\
x_2(n) \\
\vdots \\
x_p(n) \end{pmatrix}, \quad A_n := \begin{pmatrix} a_{11}(n) & a_{12}(n) & \cdots & a_{1p}(n) \\
a_{21}(n) & a_{22}(n) & \cdots & a_{2p}(n) \\
\vdots & \vdots & \ddots & \vdots \\
a_{p1}(n) & a_{p2}(n) & \cdots & a_{pp}(n) \\
b_1(n) \\
b_2(n) \\
\vdots \\
b_p(n) \end{pmatrix}, \quad B_n := \begin{pmatrix} b_1(n) \\
b_2(n) \\
\vdots \\
b_p(n) \end{pmatrix}.
\]

Then the system (1.3) can be rewritten as

\begin{equation}
X_{n+1} = A_nX_n + B_n, \quad n \in N_0.
\end{equation}
Lemma 1. If the sequence \((X_n)_{n \in \mathbb{N}_0}\) satisfies the relation (2.5), then
\[
(2.6) \quad X_n = A_{n-1}A_{n-2} \cdots A_0X_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2} \cdots A_k B_{k-1} + B_{n-1}, \quad n \geq 2.
\]

Proof. It can be easily proved by induction. For \(n = 2\) the relation (2.6) becomes
\[
X_2 = A_1A_0X_0 + A_1B_0 + B_1,
\]
which is true according to the relation (2.5). Suppose that (2.6) is true for a fixed \(n \geq 2\), we have to prove that
\[
(2.7) \quad X_{n+1} = A_nA_{n-1} \cdots A_0X_0 + \sum_{k=1}^{n} A_nA_{n-1} \cdots A_k B_{k-1} + B_n.
\]

In virtue of the relation (2.5) and (2.6), it follows that
\[
\begin{align*}
X_{n+1} &= A_n \left( A_{n-1}A_{n-2} \cdots A_0X_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2} \cdots A_k B_{k-1} + B_{n-1} \right) + B_n \\
&= A_nA_{n-1} \cdots A_0X_0 + \sum_{k=1}^{n} A_nA_{n-1} \cdots A_k B_{k-1} + B_n.
\end{align*}
\]

Hence, the relation (2.7) is true for every \(n \geq 2\). \(\square\)

Theorem 1. Let \(A_n \in \mathbb{K}^{p \times p}\) be a nonzero matrix for every \(n \in \mathbb{N}_0\), \((\varepsilon_n)_{n \in \mathbb{N}_0}\) be a sequence of positive numbers with the property \(\lim\inf_{n \to \infty} \frac{\varepsilon_n}{\|A_n\|_{\infty}} > 1\). If the sequence \((Y_n)_{n \in \mathbb{N}_0}\) in \(\Omega\) satisfies the inequality
\[
(2.8) \quad \|Y_{n+1} - A_nY_n - B_n\|_{\infty} \leq \varepsilon_n, \quad n \in \mathbb{N}_0,
\]
then there exists a sequence \((X_n)_{n \in \mathbb{N}_0}\) in \(\Omega\) given by (2.5) and a positive constant \(L_0\) such that
\[
(2.9) \quad \|X_n - Y_n\|_{\infty} \leq L_0\varepsilon_{n-1}, \quad n \in \mathbb{N}.
\]

Proof. If the sequence \((Y_n)_{n \in \mathbb{N}_0}\) in \(\Omega\) satisfies the inequality (2.8), then we can denote that
\[
(2.10) \quad Y_{n+1} = A_nY_n + B_n + C_n, \quad n \in \mathbb{N}_0,
\]
where \(C_n = (c_1(n), c_2(n), \ldots, c_p(n))^T \in \Omega\), \(\|C_n\|_{\infty} \leq \varepsilon_n\) for every \(n \in \mathbb{N}_0\). From Lemma 1, the general solution of (2.10) can be obtained as
\[
(2.11) \quad Y_n = A_{n-1} \cdots A_0Y_0 + \sum_{k=1}^{n-1} A_{n-1}A_{n-2} \cdots A_k (B_{k-1} + C_{k-1}) + (B_{n-1} + C_{n-1}).
\]
Define the sequence \((X_n)_{n \in \mathbb{N}}\) in \(\Omega\) given by (2.5) with \(X_0 := Y_0\), then

\[
(2.12) \quad \|X_n - Y_n\|_\infty = \|A_{n-1} \cdots A_0 X_0 + \sum_{k=1}^{n-1} A_{n-1} \cdots A_k B_{k-1} + B_{n-1} - A_{n-1} \cdots A_0 Y_0
\]

\[
- \sum_{k=1}^{n-1} A_{n-1} A_{n-2} \cdots A_k (B_{k-1} + C_{k-1}) - (B_{n-1} + C_{n-1})\|_\infty
\]

\[
= \|\sum_{k=1}^{n-1} A_{n-1} A_{n-2} \cdots A_k C_{k-1} + C_{n-1}\|_\infty
\]

\[
\leq \sum_{k=1}^{n-1} \|A_{n-1} A_{n-2} \cdots A_k\|_\infty \epsilon_{k-1} + \epsilon_{n-1}.
\]

Taking account of the condition \(\liminf_{\epsilon_{n-1} \to 0} \frac{\epsilon_n}{\|A_n\|_\infty} > 1\), there exists a constant \(q_0 \) and \(n_0 \in \mathbb{N}\) such that

\[
(2.13) \quad \frac{\epsilon_n}{\epsilon_{n-1} \|A_n\|_\infty} \geq q_0 > 1 \quad \text{for all } n \geq n_0,
\]

which implies that

\[
\frac{\epsilon_{n-1}}{\epsilon_{k-1} \|A_{n-1} A_{n-2} \cdots A_k\|_\infty} \geq \frac{\epsilon_{n-1}}{\epsilon_{n-2} \|A_{n-1}\|_\infty \epsilon_{n-3} \|A_{n-2}\|_\infty \cdots \epsilon_{k-1} \|A_k\|_\infty}
\]

\[
\geq \frac{q_0^{n-k}}{q_0^n}, \quad n > k \geq n_0.
\]

(2.14)

Hence for every \(n > n_0\), we have

\[
\sum_{k=1}^{n-1} \|A_{n-1} A_{n-2} \cdots A_k\|_\infty \epsilon_{k-1} + \epsilon_{n-1}
\]

\[
= \sum_{k=1}^{n_0-1} \|A_{n-1} \cdots A_k\|_\infty \epsilon_{k-1} + \sum_{k=n_0}^{n-1} \|A_{n-1} \cdots A_k\|_\infty \epsilon_{k-1} + \epsilon_{n-1}
\]

\[
\leq \|A_{n-1} A_{n-2} \cdots A_{n_0}\|_\infty \sum_{k=1}^{n_0-1} \|A_{n_0-1} A_{n_0-2} \cdots A_k\|_\infty \epsilon_{k-1}
\]

\[
+ \sum_{k=n_0}^{n-1} \|A_{n-1} A_{n-2} \cdots A_k\|_\infty \epsilon_{k-1} + \epsilon_{n-1}
\]

\[
\leq \frac{\epsilon_{n-1}}{\epsilon_{n_0-1} q_0^{n-n_0}} \sum_{k=1}^{n_0-1} \|A_{n_0-1} A_{n_0-2} \cdots A_k\|_\infty \epsilon_{k-1} + \sum_{k=n_0}^{n-1} \frac{\epsilon_{n_0-1}}{q_0^{n-k}} + \epsilon_{n-1}
\]

\[
< \left(\frac{1}{\epsilon_{n_0-1}} \sum_{k=1}^{n_0-1} \|A_{n_0-1} A_{n_0-2} \cdots A_k\|_\infty \epsilon_{k-1} + \frac{1}{q_0 - 1} + 1\right) \epsilon_{n-1}.
\]
In view of (2.12), it is enough to take

$$L_0 \geq \frac{1}{\varepsilon_{n_0-1}} \sum_{k=1}^{n_0-1} \|A_{n_0-1}A_{n_0-2} \cdots A_k\|_{\infty} \varepsilon_{k-1} + \frac{1}{q_0 - 1} + 1,$$

and such that $\|X_n - Y_n\|_{\infty} \leq L_0 \varepsilon_{n-1}$ for $1 \leq n \leq n_0$. 

\[\square\]

**Theorem 2.** Let $A_n \in K^{p \times p}$ be an invertible matrix for every $n \in N_0$, $(\varepsilon_n)_{n \in N_0}$ be a sequence of positive numbers with the property

$$\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} \|A_n^{-1}\|_{\infty} < 1.$$

If the sequence $(Y_n)_{n \in N_0}$ in $\Omega$ satisfies the inequality

$$\|Y_{n+1} - A_n Y_n - B_n\|_{\infty} \leq \varepsilon_n, \quad n \in N_0,$$

then there exists a sequence $(X_n)_{n \in N_0}$ in $\Omega$ given by (2.5) and a positive constant $L_1$ such that

$$\|X_n - Y_n\|_{\infty} \leq L_1 \varepsilon_{n-1}, \quad n \in N.$$

**Proof.** If the sequence $(Y_n)_{n \in N_0}$ in $\Omega$ satisfies the inequality (2.15), then we can also denote that $(Y_n)_{n \in N_0}$ satisfies the relation (2.10). Then we discuss the convergence of the series $\sum_{n=1}^{+\infty} A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1} C_{n-1}$. The series

$$\sum_{n=1}^{+\infty} \|A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1}\|_{\infty} \varepsilon_{n-1}$$

is convergent in virtue of D’Alembert test since

$$\limsup \frac{\|A_0^{-1} A_1^{-1} \cdots A_n^{-1}\|_{\infty} \varepsilon_n}{\|A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1}\|_{\infty} \varepsilon_{n-1}} \leq \limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} \|A_n^{-1}\|_{\infty} < 1.$$

So, the series $\sum_{n=1}^{+\infty} A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1} C_{n-1}$ is also convergent by considering that

$$\|A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1} C_{n-1}\|_{\infty} \leq \|A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1}\|_{\infty} \varepsilon_{n-1}.$$

Define $S := \sum_{n=1}^{+\infty} A_0^{-1} A_1^{-1} \cdots A_{n-1}^{-1} C_{n-1} \in \Omega$, and $(X_n)_{n \in N_0}$ in $\Omega$ given by (2.5) with $X_0 := Y_0 + S$, then

$$X_n = A_{n-1} A_{n-2} \cdots A_0 (Y_0 + S) + \sum_{k=1}^{n-1} A_{n-1} A_{n-2} \cdots A_k B_{k-1} + B_{n-1}. \quad (2.17)$$
By (2.11) and (2.17), we have

(2.18) \[ \|X_n - Y_n\|_\infty \]

\[ = \|A_{n-1}A_{n-2} \cdots A_0S - \sum_{k=1}^{n-1} A_{n-1}A_{n-2} \cdots A_kC_{k-1} - C_{n-1}\|_\infty \]

\[ = \|A_{n-1} \cdots A_0 \left( \sum_{k=1}^{+\infty} A_{n-k}^{-1}C_{n-k} - \sum_{k=1}^{n} A_0^{-1} \cdots A_{k-1}^{-1}C_{n-k} \right)\|_\infty \]

\[ = \|A_{n-1}A_{n-2} \cdots A_0 \sum_{k=n+1}^{+\infty} A_0^{-1}A_1^{-1} \cdots A_{k-1}^{-1}C_{n-k}\|_\infty \]

\[ = \sum_{k=0}^{+\infty} \|A_n^{-1}A_{n+1}^{-1} \cdots A_{n+k}^{-1}C_{n+k}\|_\infty \]

\[ \leq \sum_{k=0}^{+\infty} \|A_n^{-1}\|_\infty \|A_{n+1}^{-1}\|_\infty \cdots \|A_{n+k}^{-1}\|_\infty \varepsilon_{n+k}. \]

From the condition \( \limsup_{\varepsilon_n \to 0} \|A_n^{-1}\|_\infty < 1 \), there exists \( n_1 \in \mathbb{N} \) and a constant \( q_1 < 1 \) for all \( n \geq n_1 \), we have

(2.19) \[ \frac{\varepsilon_n}{\varepsilon_{n-1}} \|A_n^{-1}\|_\infty \leq q_1 < 1, \]

which implies that

(2.20) \[ \|A_n^{-1}\|_\infty \cdots \|A_{n+k}^{-1}\|_\infty \varepsilon_{n+k} \leq q_1 \|A_n^{-1}\|_\infty \cdots \|A_{n+k-1}^{-1}\|_\infty \varepsilon_{n+k-1} \]

\[ \leq q_1^2 \|A_n^{-1}\|_\infty \cdots \|A_{n+k-2}^{-1}\|_\infty \varepsilon_{n+k-2} \]

\[ \vdots \]

\[ \leq q_1^{k+1} \varepsilon_{n-1}, \quad k \geq 0. \]

Therefore, by (2.18) and (2.20), we have

\[ \|X_n - Y_n\|_\infty \leq \varepsilon_{n-1} \sum_{k=0}^{+\infty} q_1^{k+1} = \frac{q_1}{1 - q_1} \varepsilon_{n-1}, \quad n \geq n_1. \]

So it is enough to take \( L_1 \geq \frac{q_1}{1 - q_1} \) and such that \( \|X_n - Y_n\|_\infty \leq L_1 \varepsilon_{n-1} \) for all \( 1 \leq n < n_1. \)

\[ \square \]

3. Stability of the recurrence (1.4)

In what follows we give applications of Theorem 1 and 2 to the Hyers-Ulam-Rassias stability of the \( p \)-order recurrence (1.4).

Corollary 1. Let \( E \) be a Banach space over \( K \), \( (b_n)_{n \in \mathbb{N}_0} \) be a sequence in \( E \), and \( (\varepsilon_n)_{n \in \mathbb{N}_0} \) be a sequence of positive numbers with the property \( \liminf_{n} \frac{\varepsilon_n}{\varepsilon_{n-1}} > 1 \)
where $f_n = \max\{1, |a_0(n)| + \cdots + |a_{p-1}(n)|\}$. If the sequence $(y_n)_{n \in \mathbb{N}_0}$ in $E$ satisfies the inequality
\begin{equation}
\|y_{n+p} - (a_{p-1}(n)y_{n+p-1} + \cdots + a_0(n)y_n + b_n)\| \leq \varepsilon_n, \quad n \in \mathbb{N}_0,
\end{equation}
then there exists a sequence $(x_n)_{n \in \mathbb{N}_0}$ in $E$ given by (1.4) and a positive constant $L_2$ such that
\begin{equation}
\|x_n - y_n\| \leq L_2\varepsilon_{n-1}, \quad n \in N.
\end{equation}

**Proof.** Let $x_1(n) := x_{n+p-1}, x_2(n) := x_{n+p-2}, \ldots, x_p(n) := x_n$, then the recurrence (1.4) is equivalent to the system (2.5) with $B_n := (b_n, 0, \ldots, 0)^T$ and
\[
A_n := \begin{pmatrix}
a_{p-1}(n) & a_{p-2}(n) & a_{p-3}(n) & \cdots & a_1(n) & a_0(n) \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

It is easy to see that the property $\liminf_{\varepsilon_n \to 0} \frac{\varepsilon_n}{\varepsilon_{n-1} f_n} > 1$ is equivalent to the condition $\lim\inf_{\varepsilon_n \to 0} \frac{\varepsilon_n}{\varepsilon_{n-1} |A_n|_{\infty}} > 1$ and the inequality (3.21) is equivalent to the inequality (2.8) with $Y_n := (y_{n+p-1}, y_{n+p-2}, \ldots, y_n)^T$. So, according to Theorem 1, there exists a sequence $(X_n)_{n \in \mathbb{N}_0}$ given by (2.5) and a positive constant $L_2$ such that $\|X_n - Y_n\|_{\infty} \leq L_2\varepsilon_{n-1}, n \in N$.

Let $x_n := p_1(X_n)$ for $X_n \in \Omega$ and $n \in \mathbb{N}_0$, where $p_1 : \Omega \to E$ is given by: $p_1(z_1, z_2, \ldots, z_p)^T := z_p$. Clearly $x_n$ is a solution of (1.4) and (3.22) holds. □

**Corollary 2.** Let $E$ be a Banach space over $K$, $(b_n)_{n \in \mathbb{N}_0}$ be a sequence in $E$, and $(\varepsilon_n)_{n \in \mathbb{N}_0}$ be a sequence of positive numbers with the property $\limsup_{\varepsilon_n \to 0} \frac{\varepsilon_n}{\varepsilon_{n-1} e_n} < 1$ where $e_n = \max\{1, \frac{1+|a_1(n)|+\cdots+|a_{p-1}(n)|}{|a_0(n)|}\}$. If the sequence $(y_n)_{n \in \mathbb{N}_0}$ in $E$ satisfies the inequality
\begin{equation}
\|y_{n+p} - (a_{p-1}(n)y_{n+p-1} + \cdots + a_0(n)y_n + b_n)\| \leq \varepsilon_n, \quad n \in \mathbb{N}_0,
\end{equation}
then there exists a sequence $(x_n)_{n \in \mathbb{N}_0}$ in $E$ given by (1.4) and a positive constant $L_3$ such that
\begin{equation}
\|x_n - y_n\| \leq L_3\varepsilon_{n-1}, \quad n \in N.
\end{equation}

**Proof.** The recurrence (1.4) is equivalent to the system (2.5) with $X_n, B_n, A_n$ defined as in Corollary 1 and then
\[
A_n^{-1} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\frac{1}{a_0(n)} & -a_{p-1}(n) & -a_{p-2}(n) & \cdots & -a_1(n) & a_0(n)
\end{pmatrix}.
\]
It is easy to see that the property \( \limsup_{n \to \infty} \varepsilon_n < 1 \) is equivalent to the condition \( \sup_{n \geq 1} \varepsilon_n \leq \frac{1}{3} \). Thus our result can be obtained by applying Theorem 2.

**Example 1.** Let \((\varepsilon_n)_{n \in \mathbb{N}_0}\) be a sequence of positive numbers with \(\varepsilon_n = \frac{1}{3^n}\) and consider the system

\[
\begin{align*}
  x_{1}(n+1) &= 2nx_{1}(n) + (2n+1)x_{2}(n) + b_{1}(n), \\
  x_{2}(n+1) &= (n-1)x_{1}(n) + (n+1)x_{2}(n) + b_{2}(n).
\end{align*}
\]

Suppose that \(Y_n = (y_1(n), y_2(n))^T\) in \(\Omega\) is an approximate solution of system (3.24) satisfying

\[
||Y_{n+1} - A_nY_n - B_n||_\infty \leq \varepsilon_n, \quad n \in \mathbb{N}_0,
\]

where

\[
A_n = \begin{pmatrix} 2n & 2n+1 \\ n-1 & n+1 \end{pmatrix}, \quad B_n = \begin{pmatrix} b_1(n) \\ b_2(n) \end{pmatrix}.
\]

Then there exists a solution \(X_n = (x_1(n), x_2(n))^T\) of (3.24) such that

\[
||X_n - Y_n||_\infty \leq 2\varepsilon_{n-1}, \quad n \in \mathbb{N}.
\]

**Proof.** For all \(n \in \mathbb{N}\), we have \(\frac{\varepsilon_n}{\varepsilon_{n-1}} \leq \frac{3n+2}{3n+1} \leq \frac{3}{2} < 1\). By Theorem 2, it follows that \(||X_n - Y_n||_\infty \leq \frac{3}{1-\frac{3}{2}}\varepsilon_{n-1} = 2\varepsilon_{n-1}, n \in \mathbb{N}\).

**Example 2.** Let \((\varepsilon_n)_{n \in \mathbb{N}_0}\) be a sequence of positive numbers with \(\varepsilon_n = \frac{1}{2^n}\) and consider the recurrence

\[
x_{n+2} = \frac{2n+3}{n+2}x_{n+1} + \frac{2n+1}{n+1}x_n + b_n.
\]

Suppose that \((y_n)_{n \in \mathbb{N}_0}\) is a sequence in \(E\) satisfying the inequality

\[
\left| y_{n+2} - \frac{2n+3}{n+2}y_{n+1} - \frac{2n+1}{n+1}y_n - b_n \right| \leq \varepsilon_n, \quad n \in \mathbb{N}_0.
\]

Then there exists a solution \((x_n)_{n \in \mathbb{N}_0}\) of (3.25) such that

\[
||x_n - y_n|| \leq 8\varepsilon_{n-1}, \quad n \in \mathbb{N}.
\]

**Proof.** For all \(n \in \mathbb{N}\), we have

\[
\varepsilon_n = \max \left\{ 1, \frac{\frac{2n+3}{n+2}}{\frac{2n+1}{n+1}} \right\} = \max \left\{ 1, \frac{(3n+5)(n+1)}{(2n+1)(n+2)} \right\} \leq \frac{16}{9}.
\]

which implies that \(\frac{\varepsilon_n}{\varepsilon_{n-1}} \leq \frac{8}{9} < 1\) for all \(n \in \mathbb{N}\). By Corollary 2, it follows that \(||x_n - y_n||_\infty \leq \frac{8}{1-\frac{8}{9}}\varepsilon_{n-1} = 8\varepsilon_{n-1}, n \in \mathbb{N}\). □
References


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