Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 861–870

NEW ITERATIVE PROCESS FOR THE EQUATION INVOLVING STRONGLY ACCRETIVE OPERATORS IN BANACH SPACES

Ling-yan Zeng, Jun Li, and Jong Kyu Kim

Abstract. In this paper, under suitable conditions, we show that the new class of iterative process with errors introduced by Li et al converges strongly to the unique solution of the equation involving strongly accretive operators in real Banach spaces. Furthermore, we prove that it is equivalent to the classical Ishikawa iterative sequence with errors.

1. Introduction and preliminaries

Throughout this paper, let X be a real Banach space with dual space X^*, and $\langle .,. \rangle$ denote the pairing of X and X^*. Let $J : X \to 2^{X^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

Definition 1.1. Let D be a nonempty subset of X and $T : D \to D$ be a mapping,

(1) T is said to be accretive if for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0.$$

(2) T is said to be strongly accretive if for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2,$$

where k is the strongly accretive constant of T.

(3) T is said to be strictly pseudocontractive if $I - T$ (I denotes the identity operator) is strongly accretive.

Let $S : X \to X$ be a mapping. In [13], Li et al introduced the following new iterative process with errors: For any given $u_i \in X (i = 0, 1, \ldots, p)$, where

Received May 1, 2007.
2000 *Mathematics Subject Classification.* 47H05, 47H10, 49J40.
Key words and phrases. convergence, iterative process, strongly accretive operators.

$p \in N$ is a fixed number. The sequence $\{u_n\}_{n=p+1}^{\infty}$ in X is defined by

$$(1.1)\quad \begin{cases} v_n = (1-\beta_n)u_n + \beta_n S u_n + \eta_n, & n = 0, 1, \ldots, \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n S v_{n-p} + \xi_n, & n = p, p+1, \ldots, \end{cases}$$

where $\{\xi_n\}, \{\eta_n\}$ are arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying some conditions.

If $\eta_n = \xi_n = 0$ for all $n \geq 0$, then the iterative process $\{u_n\}_{n=p+1}^{\infty}$ defined by (1.1) reduces to

$$\begin{cases} v_n = (1-\beta_n)u_n + \beta_n S u_n, & n = 0, 1, \ldots, \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n S v_{n-p}, & n = p, p+1, \ldots, \end{cases}$$

which has been investigated by Li [12].

If $\beta_n \equiv 0$ for all $n \geq 0$, then the iterative sequence $\{u_n\}_{n=p+1}^{\infty}$ defined by (1.1) reduces to

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n S v_{n-p} + \xi_n, \quad n = p, p+1, \ldots.$$

As is well known, for any given $x_0 \in X$, the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ with errors [7] is defined by

$$(1.2)\quad \begin{cases} y_n = (1-\beta_n)x_n + \beta_n S x_n + \eta_n, & n = 0, 1, \ldots, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + \xi_n, & n = 0, 1, \ldots, \end{cases}$$

where $\{\xi_n\}, \{\eta_n\}$, $\{\alpha_n\}, \{\beta_n\}$ are the same as in (1.1).

If $\beta_n \equiv 0$, then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given in (1.2) reduces to

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + \xi_n, \quad n = 0, 1, \ldots,$$

which is the Mann iterative sequence with errors.

The stability and the convergence problems of Ishikawa and Mann iterative sequences have been studied by many authors for approximating the fixed points of some nonlinear mapping and for approximating solutions of some nonlinear operator equations in Banach spaces (see, for example, [1-15, 17]). Recently, Li [12] introduced a new kind of iterative procedure involving contractive mapping, gave a proof of the convergence for it, and under certain conditions, he showed that it is equivalent to the convergence of Ishikawa iterative sequence. In 2005, the convergence of this class of iterative sequence involving quasi-contractive mapping with errors has been further studied by Li [13] et al.

In this paper, under suitable conditions, we prove that the iterative process with errors defined by (1.1) converges strongly to the unique solution of the equation involving strongly accretive operators in real Banach spaces. Moreover, we establish the equivalence between (1.1) and (1.2).

Lemma 1.1 ([3]). *For any given $x, y \in X$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Lemma 1.2 ([2]). *Let $T : X \to X$ be a strongly accretive operator. For any given $f \in X$, define $S : X \to X$ by $Sx = f + x - Tx$, $\forall x \in X$. Then for any given $x, y \in X$, there is $j(x-y) \in J(x-y)$ such that*

$$\langle Sx - Sy, j(x-y)\rangle \leq (1-k)\|x-y\|^2,$$

where $k \in (0,1)$ is the strongly accretive constant of T.

Lemma 1.3 ([14]). *Suppose that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are nonnegative real sequences and there exists a positive integer $n_0 \in N$ such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \qquad \forall n \geq n_0,$$

where $\{t_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4. *Let $T : X \to X$ be a strongly accretive operator. For any given $f \in X$, define $S : X \to X$ by $Sx = f + x - Tx$, $\forall x \in X$. Then S has a unique fixed point. Conversely, if x^* is a fixed point of S, then x^* is a solution of the equation $Tx = f$.*

Proof. From the result of Martin [16], we know that the equation $Tx = f$ has a solution. Let x^* be a solution of the equation, that is, $Tx^* = f$. Then $Sx^* = f + x^* - Tx^* = f + x^* - f = x^*$, that is to say, x^* is a fixed point of S. Suppose that y^* is also a fixed point of S. Since T is a strongly accretive operator, it follows from Lemma 1.2 that

$$\begin{aligned}(1-k)\|x^* - y^*\|^2 &\geq \langle Sx^* - Sy^*, j(x^* - y^*)\rangle \\ &= \langle x^* - y^*, j(x^* - y^*)\rangle = \|x^* - y^*\|^2,\end{aligned}$$

which implies that $x^* = y^*$ since $k \in (0,1)$, and thus x^* is a unique fixed point of S. Conversely, let x^* be a fixed point of S. Then, one has $Sx^* = f + x^* - Tx^* = x^*$, that is, $Tx^* = f$. This completes the proof. □

2. Convergence results

Theorem 2.1. *Let $T : X \to X$ be a uniformly continuous strongly accretive operator. For any given $f \in X$, define $S : X \to X$ by $Sx = f + x - Tx$, $\forall x \in X$. Denote by $F(S)$ the set of fixed points of S. Suppose that $R(I-T)$ is bounded, where $R(I-T)$ denotes the range of $I - T$. Suppose that $\{\xi_n\}, \{\eta_n\}$ are sequences in X and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ *and* $\beta_n \to 0 (n \to \infty)$;
(ii) $\sum_{n=0}^{\infty} \|\xi_n\| < \infty, \|\eta_n\| \to 0 (n \to \infty)$.

Then for any given $u_i \in X (i = 0, 1, \ldots, p)$, the iterative sequence $\{u_n\}_{n=p+1}^{\infty}$ defined by (1.1) converges strongly to x^* in $F(S)$. Furthermore, x^* is a solution of the equation $Tx = f$.

To prove Theorem 2.1, we first give the following lemma.

Lemma 2.1. *Suppose that all assumptions in Theorem* 2.1 *are satisfied. Then the following statements are true:*

(1) $M_1 = \sup\{\|Sx - x^*\| : x \in X\} + \sum_{i=0}^{p} \|u_i - x^*\| < \infty$;

(2) $\|u_{n+1} - x^*\| \le M_1 + \sum_{i=0}^{n} \|\xi_i\| \le M, \quad \forall n \ge p$, *where* $M = M_1 + \sum_{i=0}^{\infty} \|\xi_i\|$;

(3) $M_1^* = \sup\{\|Sx - x^*\| : x \in X\} + \|x_0 - x^*\| < \infty$;

(4) $\|x_{n+1} - x^*\| \le M_1^* + \sum_{i=0}^{n} \|\xi_i\| \le M^*, \quad \forall n \ge 0$, *where* $M^* = M_1^* + \sum_{i=0}^{\infty} \|\xi_i\|$.

Proof. Since $R(I - T)$ is bounded, so is the range of S. Thus there exist constants M_1 and M_1^* such that

$$M_1 = \sup\{\|Sx - x^*\| : x \in X\} + \sum_{i=0}^{p} \|u_i - x^*\| < \infty$$

and

$$M_1^* = \sup\{\|Sx - x^*\| : x \in X\} + \|x_0 - x^*\| < \infty,$$

which imply that conclusions (1) and (3) hold.

Next, we prove the conclusion (2) holds. In fact, from (1.1), we have for $n = p$,

$$\begin{aligned}
\|u_{p+1} - x^*\| &= \|(1 - \alpha_p)u_p + \alpha_p S v_0 + \xi_p - x^*\| \\
&\le \|u_p - x^*\| + \|Sv_0 - x^*\| + \|\xi_p\| \\
&\le \sup\{\|Sx - x^*\| : x \in X\} + \sum_{i=0}^{p} \|u_i - x^*\| + \sum_{i=0}^{p} \|\xi_i\| \\
&\le M_1 + \sum_{i=0}^{p} \|\xi_i\| \le M,
\end{aligned}$$

where $M = M_1 + \sum_{i=0}^{\infty} \|\xi_i\|$. Suppose that conclusion (2) holds for $n = k - 1 (k > p + 1)$. For $n = k$, it follows from (1.1) that

$$\begin{aligned}
\|u_{k+1} - x^*\| &= \|(1 - \alpha_k)(u_k - x^*) + \alpha_k (S v_{k-p} - x^*) + \xi_k\| \\
&\le (1 - \alpha_k)\|u_k - x^*\| + \alpha_k \|S v_{k-p} - x^*\| + \|\xi_k\| \\
&\le (1 - \alpha_k)(M_1 + \sum_{i=0}^{k-1} \|\xi_i\|) + \alpha_k M_1 + \|\xi_k\| \\
&\le M_1 + \sum_{i=0}^{k} \|\xi_i\| \le M,
\end{aligned}$$

which implies that conclusion (2) holds for $n = k$. Thus conclusion (2) holds for all $n \ge p$.

Finally, we show that conclusion (4) is true. If $n = 0$, then from (1.2) we have

$$\|x_1 - x^*\| = \|(1 - \alpha_0)(x_0 - x^*) + \alpha_0 (S y_0 - x^*) + \xi_0\|$$

$$\begin{aligned} &\leq \|x_0 - x^*\| + \|Sy_0 - x^*\| + \|\xi_0\| \\ &\leq M_1^* + \|\xi_0\| \leq M^*, \end{aligned}$$

where $M^* = M_1^* + \sum_{i=0}^{\infty} \|\xi_i\|$. Suppose that conclusion (4) holds for $n = k-1 > 0$. For $n = k$. we have from (1.2),

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|(1-\alpha_k)(x_k - x^*) + \alpha_k(Sy_k - x^*) + \xi_k\| \\ &\leq (1-\alpha_k)\|x_k - x^*\| + \alpha_k\|Sy_k - x^*\| + \|\xi_k\| \\ &\leq (1-\alpha_k)(M_1^* + \sum_{i=0}^{k-1} \|\xi_i\|) + \alpha_k M_1^* + \|\xi_k\| \\ &\leq M_1^* + \sum_{i=0}^{k} \|\xi_i\| \leq M^*, \end{aligned}$$

which proves that conclusion (4) holds for $n = k$, and thus conclusion (4) is true for all $n \geq 0$. This completes the proof. □

The Proof of Theorem 2.1. From (1.1), Lemmas 1.1 and 1.2, and conclusion (2) of Lemma 2.1, for any $n \geq n_0$, there exists $j(u_{n+1} - x^*) \in J(u_{n+1} - x^*)$ such that

$$\begin{aligned} &\|u_{n+1} - x^*\|^2 \\ = &\|(1-\alpha_n)(u_n - x^*) + \alpha_n(Sv_{n-p} - x^*) + \xi_n\|^2 \\ \leq &(1-\alpha_n)^2\|u_n - x^*\|^2 + 2\langle \alpha_n(Sv_{n-p} - x^*) + \xi_n, j(u_{n+1} - x^*)\rangle \\ \leq &(1-\alpha_n)^2\|u_n - x^*\|^2 + 2\alpha_n\langle Sv_{n-p} - Su_{n+1}, j(u_{n+1} - x^*)\rangle \\ &+2\alpha_n\langle Su_{n+1} - x^*, j(u_{n+1} - x^*)\rangle + 2\langle \xi_n, j(u_{n+1} - x^*)\rangle \\ \leq &(1-\alpha_n)^2\|u_n - x^*\|^2 + 2\alpha_n\|Sv_{n-p} - Su_{n+1}\|\|u_{n+1} - x^*\| \\ &+2\alpha_n(1-k)\|u_{n+1} - x^*\|^2 + 2\|\xi_n\|M, \end{aligned} \tag{2.3}$$

where $k \in (0,1)$ is the strongly accretive constant of T, M_1 and M are the same as in Lemma 2.1.

Set

$$d_n = \|Sv_{n-p} - Su_{n+1}\|\|u_{n+1} - x^*\|.$$

We show that $d_n \to 0 (n \to \infty)$. In fact, by virtue of (1.1), we have

$$\begin{aligned} &\|v_{n-p} - u_{n+1}\| \\ = &\|(1-\beta_{n-p})u_{n-p} + \beta_{n-p}Su_{n-p} + \eta_{n-p} - (1-\alpha_n)u_n - \alpha_n Sv_{n-p} - \xi_n\| \\ \leq &\|(1-\beta_{n-p})u_{n-p} - (1-\alpha_n)u_n\| + \beta_{n-p}\|Su_{n-p}\| + \alpha_n\|Sv_{n-p}\| \\ &+\|\eta_{n-p}\| + \|\xi_n\| \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
(2.5)\quad & \|(1-\beta_{n-p})u_{n-p}-(1-\alpha_n)u_n\| \\
= \; & \|(1-\alpha_n)u_n-(1-\beta_{n-p})u_{n-p}\| \\
= \; & \|(1-\alpha_n)[(1-\alpha_{n-1})u_{n-1}+\alpha_{n-1}Sv_{n-1-p}+\xi_{n-1}]-(1-\beta_{n-p})u_{n-p}\| \\
= \; & \|(1-\alpha_n)(1-\alpha_{n-1})u_{n-1}+(1-\alpha_n)\alpha_{n-1}Sv_{n-1-p}+(1-\alpha_n)\xi_{n-1} \\
& -(1-\beta_{n-p})u_{n-p}\| \\
= \; & \| \prod_{i=n-p}^{n}(1-\alpha_i)u_{n-p}+\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}(1-\alpha_i)\alpha_{j-1}Sv_{j-1-p} \\
& +\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}(1-\alpha_i)\xi_{j-1}-(1-\beta_{n-p})u_{n-p}\| \\
= \; & \|[\prod_{i=n-p}^{n}(1-\alpha_i)-1+\beta_{n-p}]u_{n-p}+\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}(1-\alpha_i)\alpha_{j-1}Sv_{j-1-p} \\
& +\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}(1-\alpha_i)\xi_{j-1}\| \\
= \; & [\prod_{i=n-p}^{n}(1-\alpha_i)-1+\beta_{n-p}]\|u_{n-p}\|+\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}\alpha_{j-1}\|(1-\alpha_i)Sv_{j-1-p}\| \\
& +\sum_{j=n-p+1}^{n}\prod_{i=j}^{n}(1-\alpha_i)\|\xi_{j-1}\|.
\end{aligned}
$$

From Lemma 2.1, $\{u_n\},\{Su_n\},\{Sv_n\}$ are bounded sequences in X. Again since $\sum_{n=0}^{\infty}\|\xi_n\|<\infty$, $\alpha_n\to 0,\beta_n\to 0,\|\eta_n\|\to 0(n\to\infty)$, it follows from (2.3) that $\|(1-\beta_{n-p})u_{n-p}-(1-\alpha_n)u_n\|\to 0(n\to\infty)$, and hence from (2.2), $\|v_{n-p}-u_{n+1}\|\to 0(n\to\infty)$. Since T is uniformly continuous, so is S. Therefore,

$$
(2.6)\qquad \|Sv_{n-p}-Su_{n+1}\|\to 0(n\to\infty).
$$

From (4) of Lemma 2.1, we know that $\{\|u_{n+1}-x^*\|\}$ is bounded. It follows from (2.4) that $d_n\to 0(n\to\infty)$.

From (2.1), we have

$$
\begin{aligned}
(2.7)\quad & \|u_{n+1}-x^*\|^2 \\
\leq \; & (1-\alpha_n)^2\|u_n-x^*\|^2+2\alpha_n d_n+2\alpha_n(1-k)\|u_{n+1}-x^*\|^2+2\|\xi_n\|M.
\end{aligned}
$$

Since $\alpha_n\to 0(n\to\infty)$, there exists a positive integer n_1 such that for any $n\geq n_1$, $2\alpha_n<1$, and so $2\alpha_n(1-k)<1-k$, i.e., $1-2\alpha_n(1-k)>k$. It follows from (2.5) that for any $n\geq n_1$,

$$
(2.8)\qquad \|u_{n+1}-x^*\|^2
$$

$$\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n(1-k)}\|u_n - x^*\|^2 + \frac{2\alpha_n}{1-2\alpha_n(1-k)}d_n + \frac{2}{1-2\alpha_n(1-k)}\|\xi_n\|M$$
$$\leq [1 - \frac{2k-\alpha_n}{1-2\alpha_n(1-k)}\alpha_n]\|u_n - x^*\|^2 + \frac{2}{k}\alpha_n d_n + \frac{2}{k}\|\xi_n\|M.$$

Since $\frac{2k-\alpha_n}{1-2\alpha_n(1-k)} \to 2k(n \to \infty)$, there exists a positive integer $n_0(> n_1)$ such that

$$\frac{2k-\alpha_n}{1-2\alpha_n(1-k)} > k, \quad \forall n \geq n_0,$$

and thus (2.6) allows that

$$\text{(2.9)} \quad \|u_{n+1} - x^*\|^2 \leq (1-k\alpha_n)\|u_n - x^*\|^2 + \frac{2\alpha_n}{k}d_n + \frac{2}{k}\|\xi_n\|M, \quad \forall n \geq n_0.$$

Set $a_n = \|u_n - x^*\|^2, b_n = \frac{2\alpha_n d_n}{k}, c_n = \frac{2}{k}\|\xi_n\|M$ and $t_n = k\alpha_n$. It follows from (2.7) that

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n$$

for all $n \geq n_0$. Since a_n, b_n, c_n, and t_n satisfy all conditions of Lemma 1.3, we obtain that $\lim_{n\to\infty} a_n = 0$, and so $\lim_{n\to\infty} u_n = x^*$. By virtue of Lemma 1.4, x^* is a solution of equation $Tx = f$. The proof is complete. □

Theorem 2.2. *Assume that all assumptions in Theorem* 2.1 *hold. If* $R(I-T)$ *is bounded, then the following statements are equivalent:*

(1) $\{u_n\}_{n=p+1}^{\infty}$ *defined by* (1.1) *converges strongly to* x^*.
(2) $\{x_n\}_{n=0}^{\infty}$ *defined by* (1.2) *converges strongly to* x^*.

Similarly, we first show the following lemma.

Lemma 2.2. *Assume that all the assumptions in Theorem* 2.2 *hold. Then* $\|u_n - x_n\| \to 0(n \to \infty)$.

Proof. From (1.1), (1.2), Lemmas 1.1 and 1.2, we have that for any $n \geq 0$, there exists $j(u_{n+1} - x_{n+1}) \in J(u_{n+1} - x_{n+1})$ such that

$$\begin{aligned}
\text{(2.10)} \qquad & \|u_{n+1} - x_{n+1}\|^2 \\
= \; & \|(1-\alpha_n)u_n + \alpha_n S v_{n-p} + \xi_n - (1-\alpha_n)x_n - \alpha_n S y_n - \xi_n\|^2 \\
= \; & \|(1-\alpha_n)(u_n - x_n) + \alpha_n(S v_{n-p} - S y_n)\|^2 \\
\leq \; & (1-\alpha_n)^2\|u_n - x_n\|^2 + 2\alpha_n\langle S v_{n-p} - S y_n, j(u_{n+1} - x_{n+1})\rangle \\
= \; & (1-\alpha_n)^2\|u_n - x_n\|^2 \\
& +2\alpha_n\langle S v_{n-p} - S y_n - (S u_{n+1} - S x_{n+1}), j(u_{n+1} - x_{n+1})\rangle \\
& +2\alpha_n\langle S u_{n+1} - S x_{n+1}, j(u_{n+1} - x_{n+1})\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2\|u_n - x_n\|^2 \\
&\quad +2\alpha_n[\langle Sv_{n-p} - Su_{n+1}, j(u_{n+1}-x_{n+1})\rangle \\
&\quad +\langle Sx_{n+1} - Sy_n, j(u_{n+1}-x_{n+1})\rangle] \\
&\quad +2\alpha_n(1-k)\|u_{n+1}-x_{n+1}\|^2 \\
&\leq (1-\alpha_n)^2\|u_n - x_n\|^2 + 2\alpha_n(d_n^- + e_n^-) + 2\alpha_n(1-k)\|u_{n+1}-x_{n+1}\|^2,
\end{aligned}
$$

where $d_n^- = \|Sv_{n-p} - Su_{n+1}\|\|u_{n+1} - x_{n+1}\|$ and $e_n^- = \|Sx_{n+1} - Sy_n\|\|u_{n+1} - x_{n+1}\|$.

Note that (2.4) implies that $\|Sv_{n-p} - Su_{n+1}\| \to 0 (n \to \infty)$. Again from (2) and (4) of Lemma 2.1, we have

$$
\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - x^*\| + \|x_{n+1} - x^*\| \leq M + M^* < \infty. \tag{2.11}
$$

It follows that $d_n^- \to 0(n \to \infty)$. Next, we prove $e_n^- \to 0(n \to \infty)$. From (1.2), we have

$$
\begin{aligned}
\|y_n - x_{n+1}\| &= \|(\alpha_n - \beta_n)x_n + \beta_n Sx_n - \alpha_n Sy_n + \eta_n - \xi_n\| \\
&\leq (\alpha_n - \beta_n)\|x_n\| + \beta_n\|Sx_n\| + \alpha_n\|Sy_n\| + \|\eta_n\| + \|\xi_n\|. \qquad (2.12)
\end{aligned}
$$

From (3) and (4) of Lemma 2.1, we know that $\{x_n\}, \{Sx_n\}$ and $\{Sy_n\}$ are bounded sequences in X. Since $\alpha_n \to 0, \beta_n \to 0, \|\xi_n\| \to 0, \|\eta_n\| \to 0(n \to \infty)$, it follows from (2.10) that $\|y_n - x_{n+1}\| \to 0(n \to \infty)$. Since T is uniformly continuous, so is S, and so $\|Sx_{n+1} - Sy_n\| = \|Sy_n - Sx_{n+1}\| \to 0(n \to \infty)$. Then inequality (2.9) allows that $e_n^- \to 0(n \to \infty)$, and thus $d_n^- + e_n^- \to 0(n \to \infty)$.

Since $\alpha_n \to 0(n \to \infty)$, there is a positive integer n_1 such that $2\alpha_n < 1$ for any $n \geq n_1$, and hence $2\alpha_n(1-k) < 1-k$, that is, $1 - 2\alpha_n(1-k) > k$. From (2.8), we have for any $n \geq n_1$,

$$
\begin{aligned}
\|u_{n+1} - x_{n+1}\|^2 &\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n(1-k)}\|u_n - x_n\|^2 + \frac{2\alpha_n}{1-2\alpha_n(1-k)}(d_n^- + e_n^-) \\
&\leq [1 - \frac{2k-\alpha_n}{1-2\alpha_n(1-k)}\alpha_n]\|u_n - x_n\|^2 + \frac{2}{k}\alpha_n(d_n^- + e_n^-). \qquad (2.13)
\end{aligned}
$$

Again since $\frac{2k-\alpha_n}{1-2\alpha_n(1-k)} \to 2k(n \to \infty)$, there exists a positive integer $n_0(> n_1)$ such that

$$
\frac{2k-\alpha_n}{1-2\alpha_n(1-k)} > k, \quad \forall n \geq n_0,
$$

and it follows from (2.11) that

$$
\|u_{n+1} - x_{n+1}\|^2 \leq (1 - k\alpha_n)\|u_n - x_n\|^2 + \frac{2\alpha_n}{k}(d_n^- + e_n^-), \quad \forall n \geq n_0. \tag{2.14}
$$

Setting $a_n = \|u_n - x_n\|^2, b_n = \frac{2\alpha_n}{k}(d_n^- + e_n^-), c_n = 0$ and $t_n = k\alpha_n$. Then from (2.12), we have

$$
a_{n+1} \leq (1-t_n)a_n + b_n + c_n
$$

for all $n \geq n_0$. Since a_n, b_n, c_n, and t_n satisfy all conditions of Lemma 1.3, $\lim_{n\to\infty} a_n = 0$, and so $\lim_{n\to\infty} \|u_n - x_n\| = 0$. This completes the proof. □

The Proof of Theorem 2.2. From Lemma 2.2, we know that $\|u_n - x_n\| \to 0 (n \to \infty)$.

(1)$\Longrightarrow$ (2). Since $\lim_{n\to\infty} u_n = x^*$, we have $\|x_n - x^*\| \leq \|u_n - x^*\| + \|x_n - u_n\| \to 0(n \to \infty)$, that is, $\lim_{n\to\infty} x_n = x^*$.

(2)$\Longrightarrow$ (1). Since $\lim_{n\to\infty} x_n = x^*$, we obtain $\|u_n - x^*\| \leq \|u_n - x_n\| + \|x_n - x^*\| \to 0(n \to \infty)$, and so $\lim_{n\to\infty} u_n = x^*$. The proof is complete. □

Acknowledgement. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund)(KRF-2006-311-C00201).

References

[1] R. P. Agarwal, N. J. Huang, and Y. J. Cho, *Stability of iterative processes with errors for nonlinear equations of ϕ-strongly accretive type operators*, Numer. Funct. Anal. Optim. **22** (2001), no. 5-6, 471–485.

[2] S. S. Chang, *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. 30 (1997), no. 7, 4197–4208.

[3] ———, *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces*, J. Math. Anal. Appl. **216** (1997), no. 1, 94–111.

[4] C. E. Chidume, *An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces*, J. Math. Anal. Appl. **151** (1990), no. 2, 453–461.

[5] Y. J. Cho, H. Zhou, and J. K. Kim, *Iterative approximations of zeroes for accretive operators in Banach spaces*, Commun. Korean Math. Soc. **21** (2006), no. 2, 237–251.

[6] N. J. Huang, Y. J. Cho, B. S. Lee, and J. S. Jung, *Convergence of iterative processes with errors for set-valued pseudocontractive and accretive type mappings in Banach spaces*, Comput. Math. Appl. **40** (2000), no. 10-11, 1127–1139.

[7] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[8] J. K. Kim, *Convergence of Ishikawa iterative sequences for accretive Lipschitzian mappings in Banach spaces*, Taiwanese J. Math. **10** (2006), no. 2, 553–561.

[9] J. K. Kim, S. M. Jang, and Z. Liu, *Convergence theorems and stability problems of Ishikawa iterative sequences for nonlinear operator equations of the accretive and strongly accretive operators*, Comm. Appl. Nonlinear Anal. **10** (2003), no. 3, 85–98.

[10] J. K. Kim, K. S. Kim, and Y. M. Nam, *Convergence and stability of iterative processes for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces*, J. Comput. Anal. Appl. **9** (2007), no. 2, 159–172.

[11] J. K. Kim and Y. M. Nam, *Modified Ishikawa iterative sequences with errors for asymptotically set-valued pseudocontractive mappings in Banach spaces*, Bull. Korean Math. Soc. **43** (2006), no. 4, 847–860.

[12] J. Li, *Inequalities on sequences with applications*, Acta Math. Sinica (Chin. Ser.) **47** (2004), no. 2, 273–278.

[13] J. Li, N. J. Huang, H. J. Hwang, and Y. J. Cho, *Stability of iterative procedures with errors for approximating common fixed points of quasi-contractive mappings*, Appl. Anal. **84** (2005), no. 3, 253–267.

[14] L. S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), no. 1, 114–125.

[15] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[16] R. H. Martin, *A global existence theorem for autonomous differential equations in a Banach space*, Proc. Amer. Math. Soc. **26** (1970), 307–314.

[17] Y. G. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), no. 1, 91–101.

Ling-yan Zeng
School of Mathematics and Information
China West Normal University
Nanchong, Sichuan 637002, P. R. China
E-mail address: `junli1026@163.com`

Jun Li
School of Mathematics and Information
China West Normal University
Nanchong, Sichuan 637002, P. R. China

Jong Kyu Kim
Department of Mathematics Education
Kyungnam University
Kyungnam 631-701, Korea
E-mail address: `jongkyuk@kyungnam.ac.kr`