

An Algorithmic Study on Free-gyro Positioning System(I)

- Measuring Nadir Angle by using the Motion Rate of a Spin Axis -

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Abstract : The authors aim to establish the theory necessary for developing free gyro positioning system and focus on measuring the nadir angle by using the motion rate of a free gyro. The azimuth of a gyro vector from the North can be given by using the property of the free gyro. The motion rate of the spin axis in the gyro frame is transformed into the platform frame and again into the NED (north-east-down) navigation frame. The nadir angle of a gyro vector is obtained by using the North components of the motion rate of the spin axis in the NED frame. The component has to be transformed into the horizontal component of the gyro by using the azimuth of the gyro vector and then has to be integrated over the sampling interval

Key words : Free gyro, Gyro vector, Nadir angle, Angular velocity of the earth's rotation, Motion rate of spin axis, Gyro frame, Platform frame, NED navigation frame

1. Introduction

A free gyro positioning system (FPS), which determines the position of a vehicle by using two free gyros, was first suggested by Park & Jeong(2004). It is originally an active positioning system like an inertial navigation system (INS) in view of obtaining a position without external source. However, a FPS is to determine its own position by using the angle between the vertical axis of local geodetic frame and the axis of free gyro (hereinafter called 'nadir angle'), while an INS is to do so by measuring its acceleration.

The errors in the FPS were investigated broadly by Jeong(2005). And the algorithmic design of a free gyroscopic compass was suggested by measuring the earth's rotation rate on the basis of a free gyroscope (Jeong and Park, 2006).

This paper is to deal with how to measure the nadir angle by using the earth's rotation rate. Firstly, the determination of the position on or near the earth is briefed. The motion rate of the spin axis caused by the earth's rotation rate is to be transformed into the platform frame and then into the local geodetic frame, i.e. the NED(north-east-down) navigation frame. Finally the nadir angle is to be obtained by using the rotation rate of the horizontal component on the NED navigation frame.

2. Determination of vehicle's position by NED navigation frame

First consider the transformation matrix C_i^n (Rogers R M, 2000) from the inertial frame to the navigation frame which is simply given by Eq. (1).

$$\begin{aligned}
 C_i^n &= C_e^n C_i^e \\
 &= \begin{bmatrix} -\sin\phi \cos\lambda & -\sin\phi \sin\lambda & \cos\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\phi \cos\lambda & -\cos\phi \sin\lambda & -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\varpi_e t & \sin\varpi_e t & 0 \\ -\sin\varpi_e t & \cos\varpi_e t & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -\sin\phi \cos(\lambda + \varpi_e t) & -\sin\phi \sin(\lambda + \varpi_e t) & \cos\phi \\ -\sin(\lambda + \varpi_e t) & \cos(\lambda + \varpi_e t) & 0 \\ -\cos\phi \cos(\lambda + \varpi_e t) & -\cos\phi \sin(\lambda + \varpi_e t) & -\sin\phi \end{bmatrix} \quad (1)
 \end{aligned}$$

Here, ω_e is the (presumably uniform) rate of earth's rotation, λ is the geodetic longitude, ϕ is the geodetic latitude and t denotes time. This transformation matrix C_i^n denotes the transformation from the unit vectors of axes in the inertial frame to those in the navigation frame. Consider an arbitrary gyro vector $g_o^i = [u_x, u_y, u_z]^T$ which is unit vector in the inertial frame. We obtain easily the gyro vector transformed in the navigation frame, $g_v^n = [N_u, E_u, D_u]^T$, as Eq. (2).

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$$g_v^n = C_i^n g_v^i = \begin{bmatrix} -\sin \phi \cos(\lambda + \varpi_e t) & -\sin \phi \sin(\lambda + \varpi_e t) & \cos \phi \\ -\sin(\lambda + \varpi_e t) & \cos(\lambda + \varpi_e t) & 0 \\ -\cos \phi \cos(\lambda + \varpi_e t) & -\cos \phi \sin(\lambda + \varpi_e t) & -\sin \phi \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} -u_x \sin \phi \cos(\lambda + \varpi_e t) - u_y \sin \phi \sin(\lambda + \varpi_e t) + u_z \cos \phi \\ -u_x \sin(\lambda + \varpi_e t) + u_y \cos(\lambda + \varpi_e t) \\ -u_x \cos \phi \cos(\lambda + \varpi_e t) - u_y \cos \phi \sin(\lambda + \varpi_e t) - u_z \sin \phi \end{bmatrix} = \begin{bmatrix} N_u \\ E_u \\ D_u \end{bmatrix} \quad (2)$$

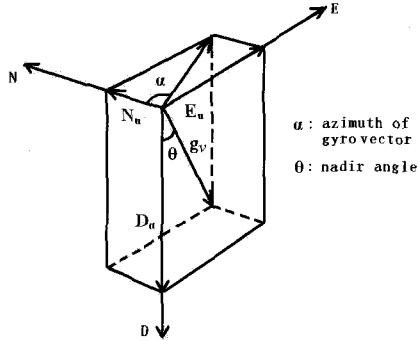


Fig. 1 Measurement quantities in the navigation frame

As shown in Fig. 1, the azimuth angle α and the nadir angle θ can be obtained respectively as Eq. (3) and Eq. (4), noting $\|g_v\| = 1$.

$$\cos \theta = \frac{D_u}{\|g_v\|} = \frac{-u_x \cos \phi \cos(\lambda + \varpi_e t) - u_y \cos \phi \sin(\lambda + \varpi_e t) - u_z \sin \phi}{\|g_v\|} \quad (3)$$

$$\tan \alpha = \frac{E_u}{N_u} = \frac{-u_x \sin(\lambda + \varpi_e t) + u_y \cos(\lambda + \varpi_e t)}{-u_x \sin \phi \cos(\lambda + \varpi_e t) - u_y \sin \phi \sin(\lambda + \varpi_e t) + u_z \cos \phi} \quad (4)$$

If we use two free gyros whose gyro vectors in Eq. (3) are $g_{va}^i = [u_{ax}, u_{ay}, u_{az}]^T$ and $g_{vb}^i = [u_{bx}, u_{by}, u_{bz}]^T$ respectively, we can determine the position (ϕ, λ) of a vehicle at the given nadir angles θ_a, θ_b . Once determining the position, we can also obtain the azimuth of a gyro vector by using Eq. (4). Park and Jeong(2004) already suggested the algorithm of how to determine a position.

3. Ship's heading, azimuth and nadir angles of gyro vector

3.1 Relation between ship's heading and azimuth of gyro vector

As Jeong and Park(2006) mentioned, let's consider that the earth's rate is ω_e and its north component is $\omega_e \cos \phi$, where ϕ is the geodetic latitude of the point concerned. Fig. 2 shows that the angular velocities of the fore and aft and the athwartship components are given by Eq.(5) (Titterson et al., 2004), where ψ is ship's heading.

$$\begin{aligned} \omega_x &= \omega_e \cos \phi \cos \psi \\ \omega_y &= -\omega_e \cos \phi \sin \psi \end{aligned} \quad (5)$$

By taking the ratio of the two independent gyroscopic measurement, the latitude dependent terms cancel, allowing the heading, ψ , to be computed by Eq. (6).

$$\begin{aligned} \tan \psi &= \frac{\omega_y}{\omega_x} = -\frac{\omega_e \cos \phi \sin \psi}{\omega_e \cos \phi \cos \psi} \\ \psi &= \arctan(\omega_y/\omega_x) \end{aligned} \quad (6)$$

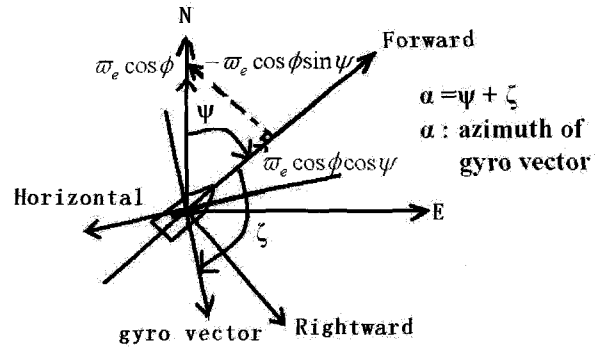


Fig. 2 Relation between ship's heading and azimuth of a gyro vector

Meanwhile assuming that a gyro vector is ζ away from ship's head, its azimuth from North is represented by Eq. (7). Therefore the angular velocity of the horizontal axis of a gyro (hereinafter called ' ω_H ') is given by Eq. (8) on the navigation frame or local geodetic frame.

$$\alpha = \psi + \zeta \quad (7)$$

$$\omega_H = -\omega_e \cos \phi \sin \alpha \quad (8)$$

Eq.(8) shows that if the North component of the earth's rotation rate can be known on the navigation frame, the nadir angle of a gyro vector, θ , is obtained by Eq.(9), by integrating Eq. (8) incrementally over a time interval.

$$\theta = \int_{t_1}^{t_2} \omega_H dt \quad (9)$$

3.2 Representation of the motion rate of the spin axis in the frame

1) The motion rate of the spin axis

Let the motion rate of spin axis in the gyro frame $\omega_{i/g}^g = [0 \ \omega_{gy} \ \omega_{gz}]^T$, where we denote: $\omega_{i/g}^g$ = the motion rate of the gyro frame(g) relative to the inertial frame(i), with coordinates in the gyro frame(g), and hereafter the same notation of the angular velocity is applied. In fact this

angular velocity is all you can get from a free gyro and has to be transformed into the local geodetic frame through the platform frame or the body frame (Jeong and Park, 2006).

First, the motion rate of the spin axis in the gyro frame, $\omega_{i/g}^g$, is transformed into that in the platform frame, $\omega_{i/g}^p$, as follows.

$$\omega_{i/g}^p = C_g^p \omega_{i/g}^g \quad (10)$$

Next, the motion rate of the spin axis in the platform frame, $\omega_{i/g}^p$, is also transformed into that in the navigation frame, $\omega_{i/g}^n$, as Eq. (11).

$$\omega_{i/g}^n = C_p^n \omega_{i/g}^p \quad (11)$$

By the way assuming that there is no error in the free gyro and rate sensors, the motion rate of the spin axis in the local geodetic frame, $\omega_{i/g}^n$ (hereinafter called ' ω_L '), is equal to the angular velocity on the navigation frame, $\omega_{i/n}^n$, which is composed of "earth" and "transport" rates and rewritten in Eq. (13) (Rogers, 2003). Considering that the earth's rotation rate, $\omega_{i/e}^n$, is shown in Eq. (14), it is represented by Eq. (13). Therefore the north component of the earth's rotation rate is computed by using the measured horizontal components, i.e. the fore-aft and athwartship ones.

$$\omega_L = \omega_{i/g}^n = \omega_{i/n}^n = \begin{bmatrix} \omega_{Lx} \\ \omega_{Ly} \\ \omega_{Lz} \end{bmatrix} \quad (12)$$

$$\omega_{i/n}^n = \omega_{i/e}^n + \omega_{e/n}^n$$

$$\omega_N = \omega_{i/e}^n = \omega_{i/g}^n - \omega_{e/n}^n = \begin{bmatrix} \omega_{Nx} \\ \omega_{Ny} \\ \omega_{Nz} \end{bmatrix} \quad (13)$$

$$\omega_{i/e}^n = \begin{bmatrix} \omega_e \cos \lambda \\ 0 \\ -\omega_e \sin \lambda \end{bmatrix} \quad (14)$$

The transport rate, $\omega_{e/n}^n$, is shown in Eq. (15). In Eq. (15) $\dot{\lambda}$ denotes the time rate of change of the longitude while $\dot{\phi}$ is the time rate of change of the latitude. And v_E is the east velocity, v_N is the north velocity, R is the radius of the earth and h is the height above ground.

$$\omega_{e/n}^n = \begin{bmatrix} \dot{\lambda} \cos \phi \\ -\dot{\phi} \\ -\dot{\lambda} \sin \phi \end{bmatrix} = \begin{bmatrix} \frac{v_E}{R+h} \\ -\frac{v_N}{R+h} \\ -\frac{v_E}{R+h} \tan \phi \end{bmatrix} \quad (15)$$

Meanwhile the ship's heading, ψ , can be computed by using Eq. (13) and given by Eq. (16). Of course Eq. (15) has to be transformed by multiplying the transformation matrix, C_n^H , which changes from the NE frame to the fore-aft and rightward one.

$$\psi = \arctan \frac{\omega_{Ny}}{\omega_{Nx}} \quad (16)$$

The azimuth of the gyro vector from the ship's head, ζ , can be obtained by integrating the vertical component of the motion rate of the spin axis, i.e. Eq. (12) and given by Eq. (17).

$$\zeta = \int_1^2 \omega_{Lz} dt \quad (17)$$

The northward angular velocity of the local geodetic frame, ω_{LH} , which is determined by the sum of the rates the earth's rotation and the ship's transport, is represented by Eq. (18).

$$\omega_{LH} = \sqrt{\omega_{Lx}^2 + \omega_{Ly}^2} \quad (18)$$

And the horizontal component of the motion rate of the free gyro, ω_H , can be obtained by Eq. (19). It is evident that the azimuth of the gyro vector, α , is given by the sum of the ship's heading ψ and the azimuth of the gyro vector from the ship's head ζ .

$$\omega_H = -\omega_{LH} \sin \alpha$$

$$\alpha = \psi + \zeta \quad (19)$$

Therefore the nadir angle of the gyro vector, θ , can be obtained by integrating Eq. (19).

$$\theta = \int_1^2 \omega_H dt \quad (20)$$

2) Coordinate transformation from gyro frame to platform frame

In Fig. 3 the gyro frame refers to free gyro itself on the platform, whose axes are defined along the spin(x_g),

horizontal(y_g), and downward(z_g) directions. The platform frame refers to the vehicle to be navigated, whose axes are defined along the forward(x_p), right(y_p), and through-the-floor(z_p) directions.

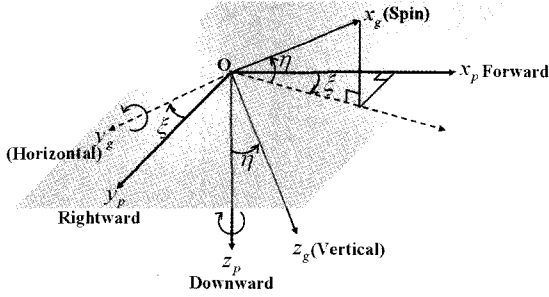


Fig. 3 Gyro & platform frames

The angle ξ is a rotation angle about the spin axis z_p and is positive in the counterclockwise sense as viewed along the axis toward the origin O , while the angle η is a rotation angle about the horizontal axis(y_g) and is positive in the same manner as above. Here the transformation matrix C_g^p from the gyro frame to platform frame is given by Eq.(21), using Euler angles and direction cosines.

$$C_g^p = \begin{pmatrix} \cos \xi \cos \eta & -\sin \xi \cos \xi \sin \eta \\ \sin \xi \cos \eta & \cos \xi \sin \xi \sin \eta \\ -\sin \eta & 0 & \cos \eta \end{pmatrix} \quad (21)$$

3) Coordinate transformation into the NED navigation frame

With respect to the NED navigation frame whose axes are defined as the first axis points the north, the second axis points east and the third axis is aligned with the ellipsoidal normal at a point, in the downward direction. Let's consider the platform frame axes point forward(x_p), to the right(y_p), and down(z_p) as shown in the above. Euler angles define the transformation, that is, they are the roll (R), pitch(P), and yaw(Y) relative to the NED axes as shown in Fig. 4. Then the transformation matrix C_p^m is given by Eq. (22).

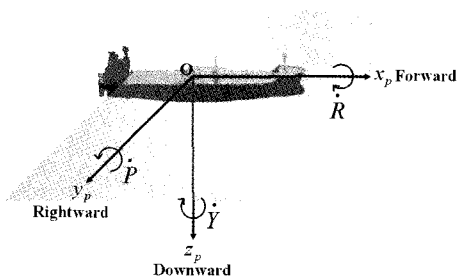


Fig. 4 Platform frame relative to NED frame

$$C_p^m = \begin{pmatrix} \cos P & \sin R \sin P & \cos R \sin P \\ 0 & \cos R & -\sin R \\ -\sin P & \sin R \cos P & \cos R \cos P \end{pmatrix} \quad (22)$$

4) Determination of transformation matrices

First, for transformation we have to know the rotation angles of the gyro frame, ξ and η . They are obtained by integrating the respective components of the spin motion rate. In doing so we can get the transformation matrix by solving the following first order linear differential equation (23) as Jeong and Park(2006) suggested.

$$\frac{dC_p^g}{dt} = -\Omega_{p/g}^g C_p^g \quad (23)$$

Here $\Omega_{p/g}^g$ is a skew-symmetric matrix and we assume it is constant over the sampling interval. The solution is given by Eq. (24).

$$\begin{aligned} C_p^g(t) &= \Psi(t, t_0) C_p^g(t_0) \\ \Psi(t, t_0) &= \exp\left(\int_{t_0}^t (-\Omega_{p/g}^g) d\tau\right) \\ &= I + \frac{\sin(|a|)}{|a|} A + \frac{1 - \cos(|a|)}{|a|^2} A^2 \end{aligned} \quad (24)$$

where,

$$\begin{aligned} A &= \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} = \int_{t_0}^t (-\Omega_{p/g}^g) d\tau \\ &= -\Omega_{p/g}^g \Delta t \\ a_i &= -\int_{t_0}^t \omega_{p/g}^g(i) d\tau \\ |a| &= \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned}$$

Here $\Delta t = t - t_0$ and t_0 is the initial time and a_i is each component of rotation angle. Once the transformation matrix C_p^g is obtained, the inverse matrix of it, C_g^p , is immediately calculated by doing the transpose of it since it is an orthogonal matrix. The relation between them is given by Eq. (25).

$$C_g^p = (C_p^g)^{-1} = (C_p^g)^T \quad (25)$$

Secondly, for transformation we have to know the rotation angles of the platform frame Y , P , and R . They are obtained by integrating the respective components of the motion rate of the spin axis as shown in the above. We can also get the transformation matrix by solving the following first order linear differential equation (26).

$$\frac{dC_n^p}{dt} = -\Omega_{n/p}^p C_n^p \quad (26)$$

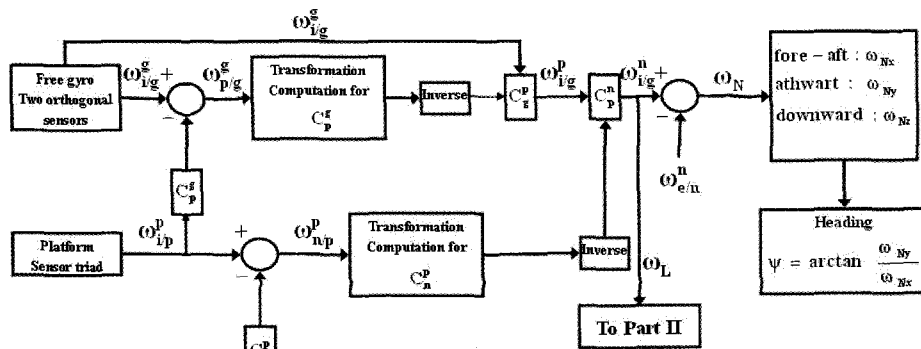
Here $\Omega_{n/p}^p$ is also a skew-symmetric matrix too and we assume it is constant over the sampling interval. The solution is given by Eq. (27).

$$\begin{aligned} C_n^p(t) &= \Gamma(t, t_0) C_n^p(t_0) \\ \Gamma(t, t_0) &= \exp\left(\int_{t_0}^t (-\Omega_{n/p}^p) d\tau\right) \\ &= I + \frac{\sin(|b|)}{|b|} B + \frac{1 - \cos(|b|)}{|b|^2} B^2 \end{aligned} \quad (27)$$

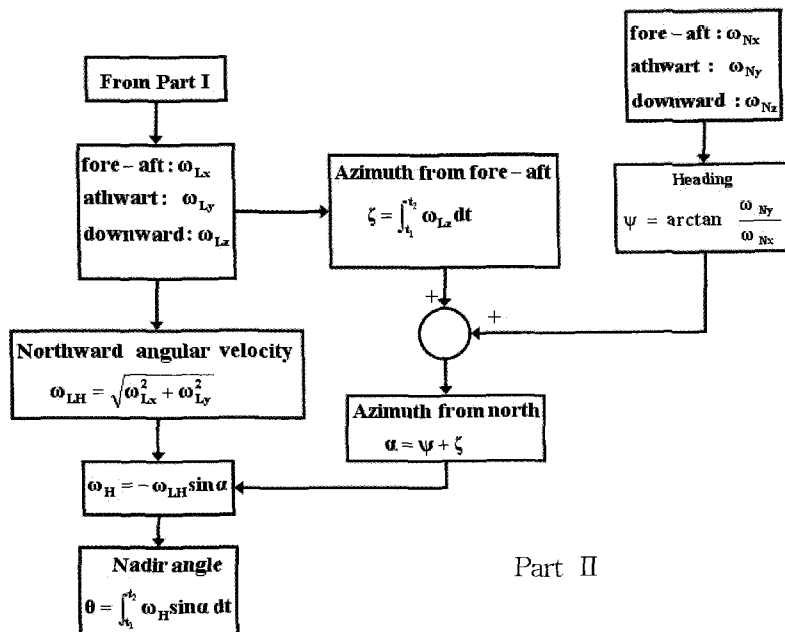
where,

$$\begin{aligned} B &= \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} = \int_{t_0}^t (-\Omega_{n/p}^p) d\tau \\ &= -\Omega_{n/p}^p \Delta t \\ b_i &= -\int_{t_0}^t \omega_{n/p}^p(i) d\tau \\ |b| &= \sqrt{b_1^2 + b_2^2 + b_3^2} \end{aligned}$$

Here $\Delta t = t - t_0$ and t_0 is the initial time and b_i is each component of rotation angle. Once the transformation matrix C_n^p is obtained, the inverse matrix of it, C_p^n , is immediately calculated by doing the transpose of it since it is an orthogonal matrix.



Part I



Part II

Fig. 5 Free gyro positioning system mechanization

In addition, the other methods to solve the differential equations (23) and (26) are also represented by the integration of four quaternions or three rotation vectors, the integration of three Euler angle equations, and etc. Such equations suggested in chapter 3 are developed by referring to and using Farrell et al.(1999), Jekeli(2001), and Rogers (2003).

4. Algorithmic design of free gyro positioning system

Fig. 5 shows the algorithmic design of free gyros positioning system mechanization. First, let's look into the ship's heading. In this mechanization two sensors for sensing the motion rate of the spin axis are mounted in the free gyro. Three sensors for sensing the motion rate of the platform are mounted in orthogonal triad. From the sensors in the gyro frame, the spin motion rate, $\omega_{i/g}^g$, is obtained and from the ones in the platform frame, $\omega_{i/p}^p$, is detected. By using the sum, $\omega_{p/g}^g$, of the rates from the free gyro and the ones detected from the platform sensors, the transformation matrix C_p^g is calculated and its inverse is determined. Therefore the spin motion rate, $\omega_{i/g}^g$, sensed from the free gyro is transformed into $\omega_{i/g}^p$ by using the inverse matrix, C_p^g .

Meanwhile the rate of the earth's rotation $\omega_{i/e}^n$ and the rate of the vehicle movement $\omega_{e/n}^n$ are summed and transformed into $\omega_{i/n}^n$. It is subtracted from the sensed rate from the platform, $\omega_{i/p}^p$. As a result, $\omega_{i/p}^n$ is generated. By using this, the transformation matrix, C_n^p , is calculated and the inverse of it, C_p^n , is obtained. And the rate $\omega_{i/g}^p$ is transformed into $\omega_{i/g}^n$ by using the transformation matrix, C_p^n . By using Eq. (13), the spin motion rate in the NED frame, ω_N , is obtained from the rate, $\omega_{i/g}^n$. Finally, the ship's heading is calculated by using the components of the spin motion rate according to Eq. (13) and Eq. (16). Next let's look into the nadir angle. Because the motion rate of the spin axis in the local geodetic frame, $\omega_{i/g}^n$ is represented by ω_L , The azimuth of the gyro vector from the ship's head, ζ , can be obtained by using Eq. (17). Then the azimuth of the gyro vector from the North, α , can be easily taken by Eq. (19).

The northward angular velocity of the local geodetic frame, ω_{LH} , is represented by Eq. (18). And the horizontal

component of the motion rate of the free gyro, ω_H , can be obtained by Eq. (19). As a result the nadir angle of the gyro vector, θ , can be obtained by Eq. (20).

5. Results and discussions

This paper investigated and developed the algorithm regarding free gyro positioning system theoretically and analytically. As a result conclusions are the following.

- ① Once the spin motion rate of free gyro is known, the ship's heading is determined by using Eq. (16).
- ② The azimuth of the gyro vector from the ship's head, ζ , can be obtained by Eq. (17). And the northward angular velocity of the local geodetic frame, ω_{LH} , can be given by Eq. (18).
- ③ The horizontal component of the motion rate of the free gyro, ω_H , can be obtained by Eq. (19). Finally the nadir angle of the gyro vector, θ , can be obtained by Eq. (20).
- ④ In order to transform the spin motion rate of the gyro frame into the one of the NED navigation frame, the differential equations of Eq. (23) and Eq. (26) are solved by using Eq. (24) and Eq. (27) and the transformation matrices are obtained respectively.

This paper ascertained the feasibility to set a stepping stone to the development of the free gyro positioning system. However, several problems remain unsolved in the aspect of the following. Firstly a two-degree-of-freedom gyro is very expensive and is commercially disadvantageous in practice. Secondly the inherent errors caused by many elements complicated. Errors caused by free gyro itself, sensors of the platform, sensors of the free gyro, sampling time and etc. will be dealt with in the next study.

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