EQUIVARIANT SEMIALGEBRAIC LOCAL-TRIVIALITY

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Abstract. We prove the equivariant version of the semialgebraic local-triviality of semialgebraic maps.

1. Introduction

In this paper we generalize the semialgebraic local-triviality of semialgebraic maps.

A semialgebraic set is a subset of some \( \mathbb{R}^n \) defined by finite number of polynomial equations and inequalities, and a semialgebraic map between semialgebraic sets is a continuous map whose graph is a semialgebraic set. In this paper we only consider the semialgebraic sets in \( \mathbb{R}^n \) for some \( n \) equipped with the subspace topology induced by the usual topology of \( \mathbb{R}^n \), and all semialgebraic maps are continuous.


Proposition 1.1 ([5], [1, Theorem 9.3.2]). Let \( M, N \) be two semialgebraic sets and \( f : M \to N \) a semialgebraic map. Then there exists a finite decomposition of \( N \) into semialgebraic subsets \( \{B_i\} \) such that for each \( B_i \) there exists a semialgebraic homeomorphism \( \varphi_i : f^{-1}(B_i) \to B_i \times f^{-1}(b_i) \) such that \( f|_{f^{-1}(B_i)} = p_i \circ \varphi_i \), where \( b_i \in B_i \) and \( p_i : B_i \times f^{-1}(b_i) \to B_i \) is the projection.

The purpose of this paper is to prove the equivariant version of Proposition 1.1. For this we need some basic definitions. A semialgebraic set \( G \) in some \( \mathbb{R}^m \) is called a semialgebraic group if it is a topological group whose multiplication and inversion are semialgebraic maps. A semialgebraic \( G \)-set means

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a semialgebraic set $M$ in some $\mathbb{R}^k$ with a semialgebraic action $\theta: G \times M \to M$ of $G$. A map $f: M \to N$ between semialgebraic $G$-sets is said to be a semialgebraic $G$-map if it is a continuous $G$-map and a semialgebraic map between ordinary semialgebraic sets $M$ and $N$, i.e., its graph is a semialgebraic subset of $M \times N$.

The main result of this paper is as follows.

**Theorem 1.2.** Let $G$ be a compact semialgebraic group. Let $M$, $N$ be semialgebraic $G$-sets and $f: M \to N$ a semialgebraic $G$-map. Then there exists a finite decomposition of $N$ into semialgebraic $G$-subsets $\{T_i\}$ such that for each $T_i$, there exist semialgebraic $G$-homeomorphisms $\psi_i: T_i \to B_i \times G(y_i)$ and $\varphi_i: f^{-1}(T_i) \to B_i \times f^{-1}(G(y_i))$ such that $\psi_i \circ f \big|_{f^{-1}(T_i)} = (\text{id}_{B_i} \times f \big|_{f^{-1}(G(y_i))}) \circ \varphi_i$, where $y_i \in T_i$ and $B_i$ is a semialgebraic set with the trivial $G$-action.

Note $f^{-1}(G(y_i)) = G(f^{-1}(y_i))$. In case $G$ is trivial, Theorem 1.2 is same to Proposition 1.1 with the identification $T_i = B_i \times \{y_i\} = B_i$ by $\psi_i$.

To prove Theorem 1.2 we need the following result which is the equivariant semialgebraic local-triviality of a semialgebraic $G$-invariant map.

**Theorem 1.3.** Let $G$ be a compact semialgebraic group and $M$ a semialgebraic $G$-set. Let $N$ be a semialgebraic set and $f: M \to N$ a semialgebraic $G$-invariant map. Then there exists a finite decomposition of $N$ into semialgebraic subsets $\{B_i\}$ such that for each $B_i$, there exists a semialgebraic $G$-homeomorphism $\varphi_i: f^{-1}(B_i) \to B_i \times f^{-1}(b_i)$ such that $f \big|_{f^{-1}(B_i)} = p_i \circ \varphi_i$, where $b_i \in B_i$ and $p_i: B_i \times f^{-1}(b_i) \to B_i$ is the projection.

This paper is organized as follows. In Section 2 we discuss some background materials on semialgebraic $G$-sets. In Section 3 we prove Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.2.

### 2. Some background materials on semialgebraic $G$-sets

In this section we discuss some background materials on semialgebraic $G$-sets. It is easy to see that the composition of two semialgebraic maps is also semialgebraic. Moreover, the image and the preimage of a semialgebraic subset
by a semialgebraic map are semialgebraic. See [1] for more detailed arguments on semialgebraic sets and maps. We state the following elementary proposition because it will be used several times in this paper.

**Proposition 2.1 ([8, Lemma 2.4]).** Let $A$, $B$, and $C$ be semialgebraic sets, and let $f: A \to B$ and $g: A \to C$ be semialgebraic. Assume $f$ is surjective. If $h: B \to C$ is a continuous map such that $h \circ f = g$, then $h$ is a semialgebraic map.

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \\
B & \xrightarrow{h} & C
\end{array}
\]

If $f: M \to N$ is a semialgebraic map which is a homeomorphism, then Proposition 2.1 implies that the inverse $f^{-1}$ is also semialgebraic.

H. Hironaka [6] proved the existence of semialgebraic triangulation for semialgebraic sets as follows: Let $M$ be a semialgebraic set and $M_1, \ldots, M_k$ semialgebraic subsets of $M$. Then there exist a finite open simplicial complex $K$ and a semialgebraic homeomorphism $\tau: |K| \to M$ such that each $M_j$ is a finite union of some of the $\tau(\sigma)$, where $\sigma$ is an open simplex of $K$. In this case, set

\[\{B_i\} = \{\tau(\sigma) \mid \sigma \text{ is an open simplex of } K\}.\]

Then we obtain the following proposition.

**Proposition 2.2.** Let $M$ be a semialgebraic set and $M_1, \ldots, M_k$ semialgebraic subsets of $M$. Then there exists a finite decomposition of $M$ into semialgebraic subsets $B_1, B_2, \ldots, B_n$ such that

1. each $M_j$ is a finite union of some $B_i$;
2. $M = B_1 \cup B_2 \cup \cdots \cup B_n$;
3. $B_i \cap B_{i'} = \emptyset$ if $i \neq i'$.

In this case $\{B_i\}$ is called compatible with $\{M_j\}$.

Now we study some elementary theory of semialgebraic transformation groups. The following is one of the fundamental facts in the theory of semialgebraic transformation groups.

**Proposition 2.3 ([3]).** Let $G$ be a compact semialgebraic group and $M$ a semialgebraic $G$-set. Then the orbit space $M/G$ exists as a semialgebraic set such that the orbit map $\pi: M \to M/G$ is semialgebraic.

As an immediate consequence of Proposition 2.3, if $G$ is a semialgebraic group and $H$ a compact semialgebraic subgroup of $G$, the homogeneous space $G/H$ is a semialgebraic $G$-set. On the other hand, for a semialgebraic $G$-set $M$ the orbit $G(x)$ of $x \in M$ is clearly a semialgebraic $G$-set. Moreover, the isotropy subgroup $G_x$ is also a closed semialgebraic subgroup of $G$ for all $x \in M$. When
$G_x$ is compact, as in the theory of Lie group actions, by Proposition 2.1, we have the natural semialgebraic $G$-homeomorphism:

$$\alpha_x: G/G_x \to G(x), \quad (gG_x \mapsto gx)$$

Note that every semialgebraic group has a Lie group structure [7].

**Proposition 2.4 ([4, 9]).** Let $G$ be a compact semialgebraic group. Then every semialgebraic $G$-set has only finitely many orbit types.

Let $G$ be a compact semialgebraic group and $M$ a semialgebraic $G$-set. Then the set

$$M^G = \{ x \in M \mid gx = x \text{ for all } g \in G \}$$

is a closed semialgebraic subset of $M$. Moreover, for a subgroup $H$ of $G$, let $M_{(H)}$ denote the subspace of points on orbits of type $G/H$, i.e.,

$$M_{(H)} = \{ x \in M \mid G_x = gHg^{-1} \text{ for some } g \in G \}.$$  

By the same way as in the proof of Lemma 3.3 in [8], we obtain that, for any subgroup $H$ of $G$, $M_{(H)}$ is a semialgebraic $G$-subset of $M$. In particular, if $H$ is not a closed semialgebraic subgroup of $G$ then $M_{(H)} = \emptyset$ because the isotropy subgroup $G_x$ is a closed semialgebraic subgroup of $G$ for each $x \in M$.

Furthermore, let $H$ be a closed semialgebraic subgroup of a compact semialgebraic group $G$, then we can easily show that the normalizer $N(H)$ of $H$ is also a closed semialgebraic subgroup of $G$ as follows; since $N(H)$ is a closed subgroup of $G$, thus it remains to show that it is a semialgebraic subset of $G$. We define $c: G \times H \to G$ by $c(g, h) = ghg^{-1}$, then $c$ is a semialgebraic map. Moreover, the set $c^{-1}(G - H)$ is a semialgebraic subset of $G \times H$. Then $N(H) = G - p(c^{-1}(G - H))$ is also semialgebraic, where $p: G \times H \to G$ is the projection given by $p(g, h) = g$. Therefore $N(H)$ is a closed semialgebraic subgroup of $G$.

We conclude this section with the following observation for semialgebraic $G$-sets with only one orbit type.

**Proposition 2.5.** Let $G$ be a compact semialgebraic group, and $M$ a semialgebraic $G$-set with only one orbit type $G/H$. Then we have the following semialgebraic $G$-homeomorphisms:

1. $\alpha: G \times N M^H \cong M, \quad [g, x] \mapsto g(x)$ where $N$ is the normalizer of $H$ in $G$.
2. The map $\beta: M^H / N \cong M / G$ induced from the inclusion $M^H \hookrightarrow M$.
3. $\gamma: (G/H) \times K M^H \cong M, \quad [gH, x] \mapsto g(x)$ where $K = N/H$.

**Proof.** These maps are well-known to be $G$-homeomorphisms, see e.g. [2, Chater II]. That these maps are semialgebraic follows easily from Propositions 2.1 and 2.3.

1. The map $\alpha$ is a continuous homeomorphism. Thus we only need to show that it is semialgebraic. For this, we consider the following commutative
diagram:

\[
\begin{array}{ccc}
G \times M^H & \xrightarrow{\theta|} & M \\
\pi' \downarrow & & \downarrow \\
G \times_N M^H & \xrightarrow{\alpha} & M
\end{array}
\]

where \( \pi' \) is the semialgebraic orbit map and \( \theta| \) is the restriction of the semialgebraic \( G \)-action \( \theta \) on \( M \). Since \( \pi' \) is surjective, \( \alpha \) is semialgebraic by Proposition 2.1.

(2) We only need to show that \( \beta \) is semialgebraic. For this, we consider the following commutative diagram:

\[
\begin{array}{ccc}
M^H & \xrightarrow{i} & M \\
\pi' \downarrow & & \downarrow \pi \\
M^H/N & \xrightarrow{\beta} & M/G
\end{array}
\]

where \( \pi', \pi \) are semialgebraic orbit maps and \( i \) is the inclusion. Since \( \pi' \) is surjective, \( \beta \) is semialgebraic by Proposition 2.1.

(3) We only need to show that \( \gamma \) is semialgebraic. For this, we consider the following commutative diagram:

\[
\begin{array}{ccc}
G \times M^H & \xrightarrow{\theta|} & M \\
\pi' \times \text{id} \downarrow & & \downarrow \\
G/H \times M^H & \xrightarrow{\pi''} & (G/H) \times_K M^H \\
\downarrow & & \downarrow \gamma \\
& & M
\end{array}
\]

where \( \pi', \pi'' \) are semialgebraic orbit maps and \( \theta| \) is the restriction of the semialgebraic \( G \)-action \( \theta \) on \( M^H \). Since \( \pi'' \circ (\pi' \times \text{id}) \) is surjective and semialgebraic, \( \gamma \) is semialgebraic by Proposition 2.1.

\[\square\]

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. For this we need the equivariant semialgebraic local-triviality of the orbit map \( \pi: M \rightarrow M/G \) for a semialgebraic \( G \)-set.

Lemma 3.1. Let \( G \) be a compact semialgebraic group, \( M \) a semialgebraic \( G \)-set and let \( \pi: M \rightarrow M/G \) be the semialgebraic orbit map. Then there exists a finite decomposition of \( M/G \) into semialgebraic subsets \( B_1, \ldots, B_k \) such that for each \( B_i \) there exists a semialgebraic \( G \)-homeomorphism \( \varphi_i: \pi^{-1}(B_i) \rightarrow B_i \times \pi^{-1}(b_i) \).
such that $\pi|_{\pi^{-1}(B_i)} = p_i \circ \varphi_i$, where $b_i \in B_i$ and $p_i : B_i \times \pi^{-1}(b_i) \to B_i$ is the projection.

Proof. We first prove the case when $M$ has only one orbit type, say $G/H$. By Proposition 2.5, we have semialgebraic $G$-homeomorphisms $\alpha : G \times N M^H \to M$ and $\beta : (M^H)/N \to M/G$ where $N$ is the normalizer of $H$. Let $\pi_* : M^H \to M^H/N$ be the semialgebraic orbit map. Apply Proposition 1.1 to $\pi_*$, so that there exists a finite decomposition of $(M^H)/N$ into semialgebraic subsets $\{A_1, \ldots, A_k\}$ such that for each $A_i$ there exists a semialgebraic homeomorphism $\phi_i : \pi_*^{-1}(A_i) \to A_i \times N/H$ such that $\pi_*|_{\pi_*^{-1}(A_i)} = p_i \circ \phi_i$ where $p_i : A_i \times N/H \to A_i$ is the projection. Note $N/H \cong \pi_*^{-1}(a_i)$ for $a_i \in A_i$. Set $C_i = \phi_i^{-1}(A_i \times \{eH\}) \subset \pi_*^{-1}(A_i)$.

Then it is easy to see that $NC_i = \pi_*^{-1}(A_i)$. The subgroup $N$ acts on $A_i \times N/H$ and $\pi_*^{-1}(A_i)$ but the homeomorphism $\phi_i : \pi_*^{-1}(A_i) \to A_i \times N/H$ is not necessarily $N$-equivariant. Therefore we need to define a new map $\gamma_i : N/H \times A_i \to NC_i = \pi_*^{-1}(A_i)$ by $\gamma_i(gH, x) = g\psi_i(x)$, where $\psi_i : A_i \to C_i$ is a semialgebraic homeomorphism defined by $\psi_i(x) = \phi_i^{-1}(x, eH)$. We claim that $\gamma_i$ is a semialgebraic $N$-homeomorphism. Consider the following commutative diagram

where $\pi'$ is the quotient map and $\theta|_i$ is the restriction of the action map $\theta : G \times M \to M$. Since all other maps in the above diagram are surjective and semialgebraic, $\gamma_i$ is surjective and semialgebraic by Proposition 2.1. Suppose $\gamma_i(gH, x) = \gamma_i(g'H, x')$ for $(gH, x), (g'H, x') \in N/H \times A_i$. Then $g\psi_i(x) = g'\psi_i(x')$ implies that $\psi_i(x) = g^{-1}g'\psi_i(x')$. Hence $\psi_i(x)$ and $\psi_i(x')$ are contained in the same $N$-orbit in $M^H$, which implies that $x = x'$ in $A_i$. Therefore $\psi_i(x) = g^{-1}g'\psi_i(x)$ and thus $g^{-1}g' \in N_{\psi_i(x)} = H$. Hence $gH = g'H$.
which implies that \( \gamma_i \) is injective. This completes the proof of the claim. Clearly \( \gamma_i \) induces a semialgebraic \( N \)-homeomorphism \( \gamma'_i : \pi^{-1}(A_i) = NC_i \to A_i \times N/H \) by \( \gamma'_i = c \circ \gamma_i^{-1} \), where \( c : N/H \times A_i \to A_i \times N/H \) is a semialgebraic map defined by \( c(gH, x) = (x, gH) \). And the following diagram commutes.

\[
\begin{array}{ccc}
\pi^{-1}(A_i) = NC_i & \xrightarrow{\gamma'_i} & A_i \times N/H \\
\downarrow{\pi} & & \downarrow{p_i} \\
A_i & & 
\end{array}
\]

Now let us continue our original proof. Let \( B_i = \beta(A_i) \subset M/G \). Then \( \{B_i\} \) is a finite semialgebraic decomposition of \( M/G \) and \( \pi^{-1}(A_i) = (\pi^{-1}(B_i))^H \). Hence we have a semialgebraic \( G \)-homeomorphism

\[
\varphi_i : \pi^{-1}(B_i) \cong G \times_N (\pi^{-1}(B_i))^H \quad (\because \alpha^{-1})
\]

\[
= G \times_N \pi^{-1}(A_i)
\]

\[
\cong G \times_N (N/H \times A_i) \quad (\because \text{id} \times_N \gamma'_i)
\]

\[
\cong G \times_N (N/H \times B_i) \quad (\because \text{id} \times_N (\text{id} \times \beta))
\]

\[
\cong (G \times_N N/H) \times B_i
\]

\[
\cong G/H \times B_i \cong B_i \times G/H
\]

such that \( \pi|_{\pi^{-1}(B_i)} = p_i \circ \varphi_i \) where \( p_i : B_i \times G/H \to B_i \) is the projection. This completes the proof of the case when \( M \) has only one orbit type.

We now prove the general case. By Proposition 2.4, \( M \) has finite orbit types, say \( G/H_1, \ldots, G/H_l \). Then for each \( i = 1, \ldots, l \), \( M(H_i) \) has only one orbit type. Hence, by the previous case, the restriction \( \pi| : M(H_i) \to M(H_i)/G \) has the equivariant semialgebraic local-triviality. Since \( M \) (resp. \( M/G \)) is the disjoint union of \( M(H_i) \) (resp. \( M(H_i)/G \)), \( \pi : M \to M/G \) has obviously the equivariant semialgebraic local-triviality. \( \square \)

As an application of Lemma 3.1, we prove Theorem 1.3 as follows.

**Proof of Theorem 1.3.** By Lemma 3.1, there exists a finite decomposition of \( M/G \) into semialgebraic subsets \( A_1, \ldots, A_l \) such that for each \( A_j \) there exists a semialgebraic \( G \)-homeomorphism \( \psi_j : \pi^{-1}(A_j) \to A_j \times \pi^{-1}(a_j) \) such that \( \pi|_{\pi^{-1}(A_j)} = q_j \circ \psi_j \) where \( a_j \in A_j \), \( \pi : M \to M/G \) is the semialgebraic orbit map and \( q_j : A_j \times \pi^{-1}(a_j) \to A_j \) is the projection.

On the other hand, since \( f : M \to N \) is a semialgebraic \( G \)-invariant map, it induces a semialgebraic map \( \bar{f} : M/G \to N \) by Proposition 2.1. Apply Proposition 1.1 to \( \bar{f} \), then there exists a finite decomposition of \( N \) into semialgebraic subsets \( C_1, \ldots, C_m \) such that for each \( C_k \) there exists a semialgebraic homeomorphism \( \phi_k : \bar{f}^{-1}(C_k) \to C_k \times \bar{f}^{-1}(c_k) \) such that \( \bar{f}|_{\bar{f}^{-1}(C_k)} = r_k \circ \phi_k \) where \( c_k \in C_k \) and \( r_k : C_k \times \bar{f}^{-1}(c_k) \to C_k \) is the projection.

By Proposition 2.2, there exists a finite decomposition of \( N \) into semialgebraic subsets \( \{B_i\} \) which is compatible with \( \{C_k\} \cup \{\bar{f}(A_j)\} \). We claim that
\{B_i\} is the desired finite decomposition of \(N\). Notice that each \(B_i\) is either \(B_i \cap \bar{f}(M/G) = \emptyset\) or \(B_i \subset \bar{f}(M/G) = f(M)\) by the compatibility of \(\{B_i\}\).

In case \(B_i \cap \bar{f}(M/G) = \emptyset\), \(f^{-1}(B_i) = f^{-1}(b_i) = \emptyset\), and hence \(f^{-1}(B_i) = \emptyset = B_i \times f^{-1}(b_i)\).

In case \(B_i \subset \bar{f}(M/G)\), there exist \(C_{k(i)}\) and \(A_{j(i)}\) such that \(B_i \subset C_{k(i)}\), \(B_i \subset \bar{f}(A_{j(i)})\) by the compatibility of \(\{B_i\}\). Thus we obtain a semialgebraic \(G\)-homeomorphism
\[
\varphi_i: \quad f^{-1}(B_i) = \pi^{-1}(\bar{f}^{-1}(B_i)) \cong \bar{f}^{-1}(B_i) \times \pi^{-1}(a_j) \\
\cong \phi_{k(i) \times \id} B_i \times \bar{f}^{-1}(b_i) \times \pi^{-1}(a_j) \cong \id \times h B_i \times f^{-1}(b_i)
\]
where \(b_i \in B_i\), \(a_j \in \bar{f}^{-1}(b_i)\) and \(h: \bar{f}^{-1}(b_i) \times \pi^{-1}(a_j) \to f^{-1}(b_i)\) is the semi-algebraic \(G\)-homeomorphism which is the restriction of \(\psi^{-1}_{j(i)}\). Note \(f^{-1}(b_i) = \pi^{-1}(\bar{f}^{-1}(b_i))\). It is easy to check that the diagram
\[
\begin{array}{ccc}
B_i & \xrightarrow{\varphi_i} & B_i \times f^{-1}(b_i) \\
\downarrow{f} & & \downarrow{p_1} \\
B_i & & \\
\end{array}
\]

commutes where \(p_1\) is the projection. This completes the proof of Theorem 1.3.

\[\square\]

4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2.

Let \(\pi_M: M \to M/G\) and \(\pi_N: N \to N/G\) denote the semialgebraic orbit maps. Apply Theorem 1.3 to \(\pi_M\), then we have a finite decomposition of \(M/G\) into semialgebraic subsets \(\{A_j\}\) such that for each \(A_j\) there exists a semialgebraic \(G\)-homeomorphism \(\phi_j: \pi_M^{-1}(A_j) \to A_j \times \pi_M^{-1}(a_j)\) such that \(\pi_M|_{\pi_M^{-1}(A_j)} = a_j \circ \phi_j\) where \(a_j \in A_j\) and \(\phi_j: A_j \times \pi_M^{-1}(a_j) \to A_j\) is the projection.

Similarly, there exists a finite decomposition of \(N/G\) into semialgebraic subsets \(\{C_k\}\) such that for each \(C_k\) there exists a semialgebraic \(G\)-homeomorphism \(\psi_k: \pi_N^{-1}(C_k) \to C_k \times \pi_N^{-1}(c_k)\) such that \(\pi_N|_{\pi_N^{-1}(C_k)} = c_k \circ \psi_k\) where \(c_k \in C_k\) and \(\psi_k: C_k \times \pi_N^{-1}(c_k) \to C_k\) is the projection.

Moreover, since \(f: M \to N\) is a semialgebraic \(G\)-map, it induces a semialgebraic map \(\bar{f}: M/G \to N/G\). By Proposition 1.1, there exists a finite decomposition of \(N/G\) into semialgebraic subsets \(\{D_l\}\) such that for each \(D_l\) there exists a semialgebraic homeomorphism \(\chi_l: \bar{f}^{-1}(D_l) \to D_l \times \bar{f}^{-1}(d_l)\) such that \(\bar{f}|_{\bar{f}^{-1}(D_l)} = s_l \circ \chi_l\) where \(d_l \in D_l\) and \(s_l: D_l \times \bar{f}^{-1}(d_l) \to D_l\) is the projection.

By Proposition 2.2, there exists a finite decomposition of \(N/G\) into semialgebraic subsets \(\{B_i\}\) which is compatible with \(\{\bar{f}(A_j)\} \cup \{C_k\} \cup \{D_l\}\). Notice that each \(B_i\) is either \(B_i \cap \bar{f}(M/G) = \emptyset\) or \(B_i \subset \bar{f}(M/G) = \pi_N(f(M))\) by the compatibility of \(\{B_i\}\).
In case $B_i \cap \tilde{f}(M/G) = \emptyset$, $\pi_N^{-1}(b_i) = \emptyset$ for all $b_i \in B_i$. Set $T_i = \pi_N^{-1}(B_i)$, then $f^{-1}(T_i) = \emptyset$ and $f^{-1}(G(y_i)) = \emptyset$ for all $y_i \in T_i$. Hence $f^{-1}(T_i) = \emptyset = B_i \times f^{-1}(G(y_i))$.

In case $B_i \subset \tilde{f}(M/G)$, there exist $A_j(i)$, $C_k(i)$ and $D_l(i)$ such that $B_i \subset \tilde{f}(A_j(i))$, $B_i \subset C_k(i)$ and $B_i \subset D_l(i)$ by the compatibility of $\{B_i\}$. Put $T_i = \pi_N^{-1}(B_i)$, then $T_i$ is a semialgebraic $G$-subset of $N$ which is semialgebraically $G$-homeomorphic to $B_i \times \pi_N^{-1}(b_i)$ by $\psi'_k(i)$. Put

$$\psi_i = \psi'_k(i)|: T_i = \pi_N^{-1}(B_i) \xrightarrow{\approx} B_i \times \pi_N^{-1}(b_i)$$

where $\psi'_k(i)|$ denotes the restriction of $\psi'_k(i)$.

On the other hand, $f^{-1}(T_i) = \pi_M^{-1}(\tilde{f}^{-1}(B_i))$ is semialgebraically $G$-homeomorphic to $\tilde{f}^{-1}(B_i) \times \pi_M^{-1}(a_j(i))$ by $\phi_{j(i)}$ where $a_j(i) \in \tilde{f}^{-1}(b_i) \subset A_j(i)$. Thus we have a semialgebraic $G$-homeomorphism

$$\varphi_i: f^{-1}(T_i) = \pi_M^{-1}(\tilde{f}^{-1}(B_i)) \xrightarrow{\approx} \tilde{f}^{-1}(B_i) \times \pi_M^{-1}(a_j(i))$$

$$\xrightarrow{\phi_{j(i)}} B_i \times f^{-1}(b_i) \times \pi_M^{-1}(a_j(i)) \xrightarrow{id_B \times h} B_i \times f^{-1}(\pi_N^{-1}(b_i))$$

where $h: \tilde{f}^{-1}(b_i) \times \pi_M^{-1}(a_j(i)) \to \pi_M^{-1}(\tilde{f}^{-1}(b_i)) = f^{-1}(\pi_N^{-1}(b_i))$ is a semialgebraic $G$-homeomorphism which is the restriction of $\phi_{j(i)}^{-1}$.

It is easy to check that the diagram

$$
\begin{array}{ccc}
T_i & \xrightarrow{\psi_i} & B_i \times \pi_N^{-1}(b_i) \\
\downarrow f & & \downarrow \text{id}_B \times f \\
B_i \times f^{-1}(\pi_N^{-1}(b_i)) & \xrightarrow{\varphi_i} & B_i \times f^{-1}(\pi_N^{-1}(b_i))
\end{array}
$$

commutes where $f|: f^{-1}(\pi_N^{-1}(b_i)) \to \pi_N^{-1}(b_i)$ is the restriction of $f$. Note $\pi_N^{-1}(b_i) = G(y_i)$ for all $y_i \in \pi_N^{-1}(b_i) \subset T_i$. This completes the proof of Theorem 1.2.

References


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